Extended Local Convergence for High Order Schemes Under ω-Continuity Conditions

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Abstract: There is a plethora of schemes of the same convergence order for generating a sequence approximating a solution of an equation involving Banach space operators. But the set of convergence criteria is not the same in general. This makes the comparison between them challenging and only numerically. Moreover, the convergence is established using Taylor series and by assuming the existence of high order derivatives that do not even appear on these schemes. Furthermore, no computable error estimates, uniqueness for the solution results or a ball of convergence is given. We address all these problems by using only the first derivative that actually appears on these schemes and under the same set of convergence conditions. Our technique is so general that it can be used to extend the applicability of other schemes along the same lines.

Keywords: sixth convergence order, ball of convergence, convergence criteria

MSC: 65H05, 65J15, 49M15

1. Introduction

There is a plethora of high convergence order iterative schemes (IS) producing a sequence approaching a solution \( x^* \) of equation

\[
F(x) = 0. \tag{1}
\]

Here \( F: D \subset X \rightarrow Y \) with \( X, Y \) denoting Banach space; \( D \) a nonempty, open convex set and operator \( F \) continuously differentiable according to Fréchet. One needs to use IS, since the preferred closed form of \( x^* \) is obtainable in special situations. In particular, when \( X = Y = \mathbb{R}^i \) (i a natural number) Chun and Neta¹⁴ presented a unified way of dealing with the local convergence of the three step IS given for \( x_0 \in D \) by

\[
\begin{align*}
y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n) \\
y_n &= x_n - \phi_1(x_n, y_n) F'(x_n)^{-1} F(x_n) \tag{2} \\
x_{n+1} &= z_n - \phi_2(x_n, y_n) F'(x_n)^{-1} F(x_n),
\end{align*}
\]

where \( \alpha \in \mathbb{R} \) or \( \alpha \in \mathbb{C} \), and \( \phi_1: D \times D \rightarrow L(X, Y), \phi_2: D \times D \rightarrow L(X, Y) \) with \( L(X, Y) \) being the space of continuous linear operators mapping \( X \) into \( Y \).

The weights \( \alpha, \phi_1 \) and \( \phi_2 \) were chosen as

\[
\alpha = \frac{2}{3}, \tag{3}
\]
\[ \varphi_1(x_n, y_n) = \alpha_1 l + \alpha_2 u_n + \alpha_3 v_n + \alpha_4 u_n^2 + \alpha_5 v_n^2 + \alpha_6 v_n^3, \quad \varphi_2(x_n, y_n) = \beta_1 l + \beta_2 u_n + \beta_3 v_n + \beta_4 u_n^2 + \beta_5 v_n^2 + \beta_6 v_n^3 \]

provided that \( u_n = F'(y_n)^{-1} F(x_n) \) and \( v_n = F'(x_n)^{-1} F(y_n) \) and \( \alpha_i, \beta_j, f = 1, 2, \ldots, 6 \) are scalars. The sixth order of convergence was established in [1] provided that the derivatives exist up to the seventh order

\[ \alpha_i = -\frac{1}{2} + 3\alpha_i + 3\alpha_i + 8\alpha_i, \]

\[ \alpha_2 = -\frac{9}{8} - 3\alpha_2 - \alpha_2 - 6\alpha_2, \]  

\[ \alpha_3 = -\frac{3}{8} - 3\alpha_3 - 3\alpha_3 - 6\alpha_3, \]

\[ \beta_i = -\frac{1}{2} - 2\beta_i + 3\beta_i \]

and \( \beta_2 = \frac{1}{2} + \beta_2, 2\beta_2, \) where \( \beta_4, \alpha_5, \alpha_6, \beta_3, \beta_4, \beta_5 \) are free scalars. But these derivatives do not appear on (2) and also limit applicability.

For example: Let \( X = Y = \mathbb{R} \) and \( D = [-\frac{1}{2}, \frac{3}{2}] \). Define \( f \) on \( D \) by

\[ f(t) = \begin{cases} t^3 \log t^2 + t^3 - t^4 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases} \]

Then, we have \( x_0 = 1 \),

\[ f'(t) = 3t^2 \log t^2 + 5t^4 - 4t^4 + 2t^2, \]

\[ f''(t) = 6t \log t^2 + 20t^4 - 12t^2 + 10t, \]

and

\[ f'''(t) = 6 \log t + 60t^2 - 24t^2 + 22. \]

Obviously \( f''''(t) \) is not bounded on \( D \). So, the convergence of schemes (2) and (3) are not guaranteed to converge by the analysis in [1].

Other concerns include the facts that no computable estimates on \( \| x_n - x^* \| \) or uniqueness of \( x^* \) results are given either.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \varphi_1(x_n, y_n) )</th>
<th>( \varphi_2(x_n, y_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 2(F(x_n) + F(y_n))F(x_n) )</td>
<td>( u_n )</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>( (1 + \frac{1}{2\gamma}) - \frac{1}{2\gamma}v_n )</td>
<td>( (1 + \frac{1}{\gamma}) - \frac{1}{\gamma}v_n )</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>( -\frac{2}{3}l + \frac{2}{3}u_n + \frac{2}{3}v_n )</td>
<td>( \frac{2}{3}l - \frac{5}{3}u_n + \frac{5}{3}u_n )</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3}l + \frac{2}{3}u_n^2 )</td>
<td>( \frac{2}{3}l - \frac{2}{3}u_n + \frac{2}{3}u_n^2 )</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3}l - 3v_n + \frac{3}{2}v_n^3 )</td>
<td>( \frac{2}{3}l - \frac{1}{2}v_n )</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3}l + \frac{2}{3}v_n^2 - \frac{2}{3}v_n^2 + \frac{15}{3}v_n^3 )</td>
<td>( 3l - \frac{5}{2}v_n + \frac{1}{2}v_n^2 )</td>
</tr>
</tbody>
</table>
Hence, there is a need to address these matters using conditions on \( F' \) which only appears on (2), and on the more general setting of Banach space valued operators. Moreover, we do not specialize \( \alpha, \varphi_1, \varphi_2 \) to satisfy (3) or (4). This way we include all other specializations of (2) listed in the Table 1.

The idea presented is general enough, so it can be utilized for the extension of other schemes \(^{[1-41]}\).

2. Convergence

The local convergence of scheme (2) requires some scalar functions and parameters. Set \( M = [0, \infty) \).

Suppose that there exists function \( \omega_0 : M \rightarrow M \) continuous and nondecreasing such that equation

\[
\omega_0(t) - 1 = 0
\]

has a minimal positive solution \( \rho_0 \). Set \( M_0 = [0, \rho_0) \).

Suppose there exits functions \( \omega : M_0 \rightarrow M, \omega_1 : M_0 \rightarrow M \) continuous and nondecreasing such that for

\[
f_1(t) = \frac{\int_0^t \omega_0(1-\theta) d\theta + |1-\alpha| \int_0^t \omega_0(\theta) d\theta}{1-\omega_0(t)}
\]

and equation

\[
\bar{f}_1(t) = f_1(t) - 1
\]

has a minimal solution \( \rho_1 \in (0, \rho_0) \). Consider a continuous and nondecreasing function \( \psi_1 : M_0 \rightarrow M \).

Suppose that for functions

\[
f_2(t) = \frac{\int_0^t \omega_0(1-\theta) d\theta + \psi_1(t) \int_0^t \omega_1(\theta) d\theta}{1-\omega_0(t)}
\]

\[
\bar{f}_2(t) = f_2(t) - 1,
\]

equation

\[
\bar{f}_2(t) = 0
\]

has a minimal solution \( \rho_2 \in (0, \rho_0) \).

Suppose that equation

\[
\omega_2(f_2(t)) - 1 = 0
\]

has a minimal solution \( \bar{\rho}_2 \in (0, \rho_0) \). Set \( M_1 = [0, \bar{\rho}_2) \).

Suppose that for all \( t \in M_1 \)

\[
f_3(t) = \left[ \frac{\int_0^t \omega_0(1-\theta) f_2(t) d\theta}{1-\omega_0(f_2(t))} + \frac{\omega_0(t) + \omega_0(f_2(t)) \int_0^t \omega_1(\theta f_2(t)) d\theta}{(1-\omega_0(t))(1-\omega_0(f_2(t)))} \right] f_2(t)
\]

\[
\bar{f}_3(t) = f_3(t) - 1,
\]
has a minimal solution $\rho_j \in (0, \bar{\rho}_j)$, where $\psi_j : M_1 \to M$ is a continuous and nondecreasing function. We shall show that

$$\rho = \min \{\rho_j\}, \quad j = 1, 2, 3$$

is a radius of convergence for scheme (2). Notice that by these definitions for all $t \in [0, \rho)$

$$0 \leq \omega_0(t) < 1$$

and

$$0 \leq \omega_0(f_j(t)t) < 1$$

for $j = 1, 2, 3$.

The notations $U(u, \lambda), \bar{U}(u, \lambda)$ are used for the open and closed balls in $X$ with center $u \in X$ and of radius $\lambda > 0$.

Next, the local convergence of scheme (2) is provided based on conditions (Γ):

(Γ) There exists a simple solution $x^* \in D$ of equation $F(x) = 0$.

(Γ) There exists a continuous and nondecreasing function $\omega_0 : M \to M$ such that for all $x \in D$

$$\| F'(x)^{-1}(F'(x) - F'(x^*)) \| \leq \omega_0(\| x - x^* \|).$$

Set $U_0 = D \cap U(x^*, \rho_0)$.

(Γ) There exist continuous and nondecreasing functions $\omega : M_0 \to M, \omega_1 : M_0 \to M, \psi_1 : M_0 \to M, \psi_2 : M_0 \to M$ such that for each $x, y \in U_0$

$$\| F'(x)^{-1}(F'(y) - F'(x)) \| \leq \omega(\| y - x \|),$$

$$\| F'(x) - F'(x) \| \leq \omega(\| x - x^* \|),$$

$$\| I - \varphi_1(x, y) \| \leq \psi_1(\| x - x^* \|)$$

and

$$\| I - \varphi_2(x, y) \| \leq \psi_2(\| x - x^* \|)$$

for $y = x - \alpha F'(x)^{-1}F(x)$.

(Γ) $\bar{U}(x^*, \rho) \subseteq D$ and

(Γ) There exists $\rho_* \geq \rho$ such that

$$\int_0^\rho \omega_0(\theta \rho) d\theta < 1.$$

Set $U_1 = D \cap \bar{U}(x^*, \rho_*)$.

Theorem 2.1 Suppose the conditions (Γ) are satisfied. Then, starting from $x_0 \in U(x^*, \rho) - \{x^*\}$, sequence $\{x_n\}$ generated by scheme (2) is well defined in $U(x^*, \rho)$, remains in $U(x^*, \rho)$ for each $n = 0, 1, 2, \ldots$ and $\lim_{n \to \infty} x_n = x^*$.

Moreover, the following items hold for $e_n = \| x_n - x^* \|

$$\| y_n - x_n \| \leq f_1(e_n) e_n \leq e_n < \rho,$$
\[ \| z_n - x_* \| \leq f_2(e_n) e_n \leq e_n, \quad (15) \]

and

\[ e_{n+1} \leq f_1(e_n) e_n \leq e_n, \quad (16) \]

where functions \( f_j \) are given previously and \( \rho \) is defined in (10). Furthermore, the limit point \( x_* \) is the only solution of equation \( F(x) = 0 \) in the set \( U_1 \) given in \( (\Gamma_5) \).

**Proof.** We shall show using mathematical induction on \( k \) that first iterates exist for all \( k \); remain in \( U(x_*, \rho) \); converge to \( x_* \) and secondly the solution is unique on the set \( U_1 \) given in \( (\Gamma_5) \).

Let \( u \in U(x_*, \rho) \). It then follows by \( (\Gamma_1), (\Gamma_2), (10), \) and \( (11) \) that

\[ \| F'(x_*)^{-1}(F'(u) - F'(x_*))\| \leq \omega_1 \| u - x_* \| \leq \omega_0(\rho) < 1, \quad (17) \]

so \( F'(u)^{-1} \) exists and

\[ \| F'(u)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\rho)u - x_*} \quad (18) \]

holds by the lemma on invertible operators due to Banach \( [25] \). If one sets set \( u = x_0 \), then \( y_0, z_0, x_1 \) exist by scheme (2), if \( n = 0 \). We can write by the first substep of scheme (2), \( (10), (13) \) (for \( m = 1 \), \( (\Gamma_1), (\Gamma_3) \) and \( (17) \) for \( u = x_0 \)

\[ \| y_0 - x_*\| \leq \| x_0 - x_* - F'(x_0)^{-1}F(x_0) + (1 - \alpha)F'(x_0)^{-1}F'(x_0)\| \]

\[ \leq \| F'(x_0)^{-1}F'(x_0)\| \times \int f_0^1 F'(x_0)^{-1}(F'(x_0 + \theta(x_0 - x_*)) - F'(x_0))(x_0 - x_*)d\theta \]

\[ + (1 - \alpha)\| F'(x_0)^{-1}F'(x_0)\| \| F'(x_0)^{-1}F'(x_0)\| \leq f_1(e_0) e_0 \leq e_0 < \rho, \quad (19) \]

showing \( y_0 \in U(x_*, \rho) \) and \( (14) \) for \( n = 0 \). Moreover, by \( (10), (13) \) (for \( m = 2, 3 \), we obtain in turn

\[ \| z_0 - x_*\| \leq \| (x_0 - x_* - F'(x_0)^{-1}F(x_0) \]

\[ + (I - \varphi_1(x_0, y_0)F'(x_0)^{-1}F(x_0))\| e_0 \leq f_2(e_0) e_0 \leq e_0, \quad (20) \]

and

\[ e_1 \leq \| (z_0 - x_* - F'(z_0)^{-1}F(z_0)) \]

\[ + (F'(z_0)^{-1} - F'(x_0)^{-1})F(z_0) + (I - \varphi_2(x_0, y_0)F'(x_0)^{-1}F(z_0))\| \]

\[ \left[ \int f_0^1 \omega_0(1 - \theta)^{\| z_0 - x_* \|} \theta d\theta + \frac{\omega_0(\| z_0 - x_* \|) + \omega_0(e_0) f_0^1 \omega_0(\| z_0 - x_* \|) d\theta}{(1 - \omega_0(\| z_0 - x_* \|)) (1 - \omega_0(e_0))} \right] \| z_0 - x_* \|

\[ + \frac{\varphi_1(e_0) f_1^1 \omega_0(\theta|e_0|) d\theta}{1 - \omega_0(e_0)} \| z_0 - x_* \| \leq f_3(e_0) e_0 \leq e_0, \quad (21) \]

Contemporary Mathematics

484 | Ioannis K. Argyros, et al.
so $z_n, x_1 \in U(x_n, \rho)$ and (15), (16) hold for $n = 0$. Hence, the induction for (14)-(16) is shown for $n = 0$. Suppose these estimations hold for all $k = 0, 1, 2, \ldots, n$. Then, by simply switching $x_n, y_n, x_1$ by $x_k, y_k, x_{k+1}$ in the preceding calculations, we conclude (14)-(16) hold for all $n$. Then, from

$$e_{k+1} \leq ce_k < \rho,$$

(22)

where $c = f_*(e_0) \in [0, 1)$, we have $\lim_{k \to \infty} x_k = x_\infty$, and $x_{k+1} \in U(x_n, \rho)$. Let $q \in D_1$ be such that $F(q) = 0$. Set $T = F'_0(F(x_0 + \theta(q - x_0)))d\theta$. Then, by (Γ₂) and (Γ₃),

$$\| F(x_0)^{-1}(T - F(x_0)) \| \leq \| F(x_0)^{-1}(F(x_0 + \theta(q - x_0)) - F(x_0))d\theta \|
$$

$$\leq \int_0^1 a_h(\theta) || x_\infty - q || d\theta < 1,$$

(23)

so $T^{-1}$ exists, by the Banach lemma on invertible operators \cite{25}, and $x_\infty = q$ follows from $0 = F(x_\infty) - F(q) = T(x_\infty - q)$.

**Remark 2.2** 1. By (a2), and the estimate

$$\| F(x_0)^{-1}F(x) \| = \| F(x_0)^{-1}(F(x) - F(x_0)) + I \|
$$

$$\leq 1 + \| F(x_0)^{-1}(F(x) - F(x_0)) \| \leq 1 + w_q(|| x_\infty - x_t ||)$$

second condition in (a3) can be dropped, and $w_1$ be defined as

$$w_1(t) = 1 + w_2(t)$$

or $w_1(t) = 2.$

Notice that, if $w_1(t) \leq 1 + w_2(t)$, then $\rho$ can be larger (see Example 3.1).

2. The results obtained here can be used for operators $G$ satisfying autonomous differential equations \cite{3-10} of the form

$$F(x) = T(F(x))$$

where $T$ is a continuous operator. Then, since $F(x_\infty) = T(F(x_\infty)) = T(0)$, we can apply the results without actually knowing $x_\infty$. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.

3. The local results obtained here can be used for projection schemes such as the Arnoldi’s algorithm, the generalized minimum residual algorithm (GMRES), the generalized conjugate algorithm (GCA) for combined Newton/finite projection schemes and in connection to the mesh independence principle can be used to develop the cheapest and most efficient minimum residual algorithm (GMRES), the generalized conjugate algorithm (GCA) for combined Newton/finite projection.

4. Let $w_0(t) = L_0t$, and $w(t) = Lt$. The parameter $r_s = \frac{2}{2L_s + L}$ was shown by us to be the convergence radius of Newton’s algorithm \cite{3}

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)$$

for each $n = 0, 1, 2, \cdots$

(25)

under the conditions (a1)-(a3) ($w_1$ is not used). It follows that the convergence radius $R$ of algorithm (2) cannot be larger than the convergence radius $r_s$ of the second order Newton’s algorithm (25). As already noted in [4] $r_s$ is at least as large as the convergence ball given by Rheinboldt \cite{20}

$$r_{Fr} = \frac{2}{3L_s},$$

where $L_s$ is the Lipschitz constant on $\Omega$, $L_0 \leq L_s \leq L_1$. In particular, for $L_0 < L_1$ or $L < L_1$, we have that

$$r_{Fr} < r_s$$

and
\[
\frac{r_{2k}}{r_1} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L_1} \to 0.
\]

That is our convergence ball \(r_1\) is at most three times larger than Rheinboldt's. The same value for \(r_{2k}\) was given by Traub [24].

5. It is worth noticing that solver (2) is not changing, when we use the conditions \((\mathcal{A})\) of Theorem 2.1 instead of the stronger conditions used in [12, 14]. Moreover, we can compute the computational order of convergence (COC) defined by

\[
\xi = \frac{\ln \left\| x_{n+1} - x^* \right\|}{\ln \left\| x_n - x^* \right\|} / \left( \frac{\ln \left\| x_{n+1} - x^* \right\|}{\ln \left\| x_n - x^* \right\|} \right)
\]

or the approximate computational order of convergence

\[
\xi_i = \frac{\ln \left\| x_{n+1} - x_n \right\|}{\ln \left\| x_n - x_{n-1} \right\|} / \left( \frac{\ln \left\| x_{n+1} - x_n \right\|}{\ln \left\| x_n - x_{n-1} \right\|} \right).
\]

This way we obtain in practice the order of convergence in a way that avoids the existence of the seventh Fréchet derivative for operator \(F\).

3. Numerical examples

As in [1], we choose \(\alpha_4 = \frac{63}{64}, \alpha_5 = 0, \alpha_6 = 0, \beta_3 = \frac{15}{8}, \beta_4 = 0\) and \(\beta_5 = 0\).

**Example 3.1** Let \(B_1 = B_2 = D = \mathbb{R}\). Define \(F(x) = \sin x\). Then, we get that \(x^* = 0\), and by \((\Gamma_2)\) and the first condition in \((\Gamma_3)\)

\[
\| F'(x^*)^{-1}(F'(x) - F'(x^*)) \| = |1(\cos x - \cos x^*)| = |
\cos x - \cos x^*| = |\cos v(x - x^*)| = |
\cos| |x - x^*| \leq |x - x^*| (v \in \mathbb{R}),
\]

so \(w_0(t) = w(t) = t\). Then, obtain the roots by solving (6), (7), (9) (knowing now \(w_0, w, w_1, \psi_1, \psi_2\)) using Mathematica.

Then, we determine the radii using (10). The same is done in the rest of the examples. One can also see examples about how these scalar functions are found in [3-10, 21, 25]. Further, we have the following error estimates for Example 3.1.

\[
\| y_5 - x^* \| = 0.0266 \leq e_5 = 0.0802 < \rho = 0.195081,
\]

\[
\| z_5 - x^* \| = 0.0373 \leq e_5 = 0.0802 < \rho = 0.195081,
\]

and

\[
\| x_6 - x^* \| = 0.0056 \leq e_5 = 0.0802 < \rho = 0.195081.
\]

**Table 2. Radius for Example 3.1**

<table>
<thead>
<tr>
<th>Radius</th>
<th>(w_0(t) = 1)</th>
<th>(w_0(t) = 1 + w_0(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r_1)</td>
<td>0.444444</td>
<td>0.4</td>
</tr>
<tr>
<td>(r_2)</td>
<td>0.473532</td>
<td>0.437925</td>
</tr>
<tr>
<td>(r_3)</td>
<td>0.195081</td>
<td>0.187903</td>
</tr>
</tbody>
</table>
Example 3.2 Let \( B_1 = B_2 = \mathbb{C}[0, 1] \) be the space of continuous functions defined on \([0, 1]\) with the max norm. Let \( D = \bar{U}(0, 1) \). Define function \( F \) on \( D \) by

\[
F(\phi)(x) = \phi(x) - 5 \int_0^1 x \theta \phi(\theta)^2 d\theta.
\]

We have that

\[
F'(\phi(\xi))(x) = \xi(x) - 15 \int_0^1 x \theta \phi(\theta)^2 \xi(\theta) d\theta, \quad \text{for each } \xi \in D.
\]

Then, we get that \( x_0 = 0, F'(x_0) = I, \omega_0(t) = \frac{15}{2}t, \omega(t) = 15t \) and \( \omega_1(t) = 2 \). This way, we have that

\[
\begin{array}{c|cc|c|c}
\text{Table 3. Radius for Example 3.2} & \omega_0(t) = 2 & \omega_1(t) = 1 + \omega_0(t) \\
\hline
r_1 & 0.0296296 & 0.0533 \\
r_2 & 0.231907 & 0.0903624 \\
r_3 & 0.0126373 & 0.0284867 \\
\end{array}
\]

Example 3.3 Let \( B_1 = B_2 = \mathbb{R}^3 \), \( D = U(0, 1) \), \( x_0 = (0, 0, 0)^t \), and define \( F \) on \( D \) by

\[
F(x) = F(u_1, u_2, u_3) = (e^u - 1, \frac{e^u - 1}{2} u_2^2 + u_2, u_3)^t.
\]

For the points \( u = (u_1, u_2, u_3)^t \), the Fréchet derivative is given by

\[
F'(u) = \begin{pmatrix}
e^u & 0 & 0 \\
0 & (e-1)u_2 + 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Using the norm of the maximum of the rows and since \( G'(x_0) = \text{diag}(1, 1, 1) \), we get by conditions (A) \( \omega_0(t) = (e - 1)t \), \( \omega(t) = \frac{1}{e-1}t \), and \( \omega_1(t) = \frac{1}{e-1}t \). Then we have

Example 3.4 Returning back to the motivational example at the introduction of this study, we have \( \omega_0(t) = \omega(t) = 96.662907t, \omega_1(t) = 1.0631 \). Then, we have

\[
\begin{array}{c|cc|c|c}
\text{Table 4. Radius for Example 3.3} & \omega_0(t) = \frac{1}{e^t} & \omega_1(t) = 1 + \omega_0(t) \\
\hline
r_1 & 0.073878 & 0.229929 \\
r_2 & 0.911242 & 0.309664 \\
r_3 & 0.0600865 & 0.115891 \\
\end{array}
\]

\[
\begin{array}{c|cc|c|c}
\text{Table 5. Radius for Example 3.4} & \omega_0(t) = \frac{1}{e^t} & \omega_1(t) = 1 + \omega_0(t) \\
\hline
r_1 & 0.00445282 & 0.00413809 \\
r_2 & 0.0145963 & 0.00749138 \\
r_3 & 0.002212 & \text{no solution} \\
\end{array}
\]

Further, we have the following error estimates for Example 3.4.
\[ \|y_{10} - x_*\| = 0.0006 \leq e_{10} = 0.0008 < \rho = 0.002212, \]
\[ \|z_{10} - x_*\| = 0.0006 \leq e_{10} = 0.0008 < \rho = 0.002212, \]

and
\[ \|x_{11} - x_*\| = 0.0004 \leq e_{10} = 0.0008 < \rho = 0.002212. \]

References


