On the P-Adic Valuations of Stirling Numbers of the Second Kind

S. S. Singh¹, A. Lalchhuangliana¹, P. K. Saikia²

¹Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram, India
²Department of Mathematics, North Eastern Hill University, Shillong, Meghalaya, India
E-mail: sssanasam@yahoo.com

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Abstract: In this paper, we introduced certain formulas for p-adic valuations of Stirling numbers of the second kind \( S(n, k) \) denoted by \( v_p(S(n, k)) \) for an odd prime \( p \) and positive integers \( k \) such that \( n \geq k \). We have obtained the formulas, \( v_p(S(n, n - a)) \) for \( a = 1, 2, 3 \) and \( v_p(S(cp^n, cp^3)) \) for \( 1 \leq c \leq p - 1 \) and primality test of positive integer \( n \). We have presented the results of \( v_p(S(p^2, kp)) \) for \( 2 \leq k \leq p - 1 \), \( 2 < p < 100 \) and a table of \( v_p(S(p, k)) \). We have posed the following conjectures from our analysis:

1. Let \( p \neq 7 \) be an odd prime and \( k \) be an even integer such that \( 0 < k < p - 1 \). Then
   \[
   v_p(S(p^2, kp)) - v_p(S(p^2, p(k+1)) = 3.
   \]

2. If \( k \) be an integer such that \( 1 < k < p - 1 \), then the \( p \)-adic valuations satisfy
   \[
   v_p(S(p^2, kp)) = \begin{cases} 
   5 \text{ or } 6, & \text{if } k \text{ is even} \\
   2 \text{ or } 3, & \text{if } k \text{ is odd}
   \end{cases}
   \]
   for any prime \( p > 7 \).

3. For any primes \( p \) and positive integer \( k \) such that \( 2 \leq k \leq p - 1 \), then
   \[
   v_p(S(p, k)) \leq 2.
   \]

Keywords: p-adic valuations, stirling numbers of the second kind, congruence, primes, minimum period

MSC: 05A18,11A51,11B73, 11E95

1. Introduction

Stirling numbers of the first and second kinds were introduced by James Stirling [1]. These numbers have been found to be of great utility in various branches of Mathematics such as combinatorics, number theory, calculus of finite differences, theory of algorithms, etc. The \( p \)-adic valuations of Stirling numbers of the second kind appear frequently...
in algebraic topology by Davis [2] to obtain new results related to James numbers, $v_r$-periodic homotopy groups and exponents of $SU(n)$). More details of Stirling numbers of the second kind may be seen on Comtet [3] and Graham et al. [4].

Stirling numbers of the second kind are more interesting than the first kind by their intrinsic nature. There are many interesting results of 2-adic valuations of Stirling numbers of the second kind in the open literature. Recently, Wannemacker’s proof [5] of Lengyel’s conjecture [6], results of $v_p(k!S(c - 2n + u, k))$ for $c > 0$ by Lengyel [7], the proof of Wannemacker’s conjecture by Hong [8], the works of Amdeberhan et al. [9] and Zhao et al. [10] are other notable results of 2-adic valuation. Gessel and Lengyel [11] proved that for an arbitrary prime $p$ and $n = a(p - 1)p^q$, $1 \leq k \leq n$

$$v_p(k!S(n, k)) = \left\lfloor \frac{k-1}{p-1} + \tau(k) \right\rfloor,$$

where $a$ and $q$ are positive integers such that $(a, p) = 1$, $q$ is sufficiently large, $\frac{k}{p}$ is an odd integer and $\tau(p)$ is a non-negative integer.

Strauss [12] and Pan [13] discussed the problems of 3-adic valuations and 2-adic valuations of certain sums of binomial coefficients respectively. Sun [14] also presented the results of $p$-adic valuations for multinomial coefficients. Friedland [15] used 2-adic valuations of certain ratios of factorials to prove a conjecture of Falikman-Friedland-Lowery on the parity of degrees of projective varieties of $n \times n$ complex symmetric matrices of rank at most $k$. Some more results of $p$-adic valuations are also given in Gouvea [16], Koblitz [17] and Adelberg [18].

This paper consists of some interesting results about $p$-adic valuations for a few class of Stirling numbers of the second kind $S(n, k)$. This number $v_p(S(n, k))$, where either $n$ or $k$ is related to $p$, has been obtained independently for some values of $p$, $n$ and $k$. The values of $v_p(S(n, k))$ are computed by using GP/PARI software and they are presented in Table 1.

2. Materials and methods

**Definition 2.1** Let $p$ be a prime. For any non-zero integer $a$, the $p$-adic valuation of $a$, denoted by $v_p(a)$, is defined as the exponent of the highest power of $p$ dividing $a$.

It may be noted that $v_p(0) = \infty$ and $v_p(a)$ for a non-zero integer $a$, is a non-negative integer.

So, $v_p(25) = 0$, $v_p(25) = 2$.

Note that, for any prime $p$, $v_p(\pm 1) = 0$. For a given prime $p$ and any two integers $a$ and $b$, we have

$$v_p(a + b) \geq \min\{v_p(a), v_p(b)\}, \quad v_p(ab) = v_p(a) + v_p(b).$$

The $p$-adic valuation $v_p$ can further be extended to the field of rational numbers, $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$ and $b \neq 0$ as

$$v_p(r) = v_p(a) - v_p(b).$$

**Definition 2.2** Given two non-negative integers $n$ and $k$, not both zero, the Stirling number of the second kind $S(n, k)$ is defined as the number of ways one can partition a set with $n$ elements into exactly $k$ non-empty subsets.

**Example 2.1** All partitions of the set $\{1, 2, 3, 4\}$ into 2 non-empty subsets are $\{1\} \cup \{2, 3, 4\}$, $\{2\} \cup \{1, 3, 4\}$, $\{3\} \cup \{1, 2, 4\}$, $\{4\} \cup \{1, 2, 3\}$, $\{1\} \cup \{2\} \cup \{3, 4\}$, $\{1\} \cup \{3\} \cup \{2, 4\}$ and $\{1\} \cup \{4\} \cup \{2, 3\}$. Hence, $S(4, 2) = 7$.

By convention, we set $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \geq 1$. Thus, $S(n, k)$ is the number of ways of distributing $n$ distinct balls into $k$ indistinguishable boxes (the order of the boxes does not count) such that no box is empty.

It is clear that $S(n, k) = 0$ if $1 \leq n < k$ and $S(n, n) = 1$ for all $n \geq 0$.

We use the following properties to prove the results of $v_p(S(n, k))$:

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^k,$$

(1)
which gives

\[ S(n, 2) = 2^{n-1} - 1, \quad S(n, 1) = 1, \quad S(n, 0) = 0. \]  

It is easy to derive the following specific identities of \( S(n, k) \) using the results of ([19] p. 115-116).

\[
S(n, n-1) = \binom{n}{2} \quad \text{if } n \geq 2, \\
S(n, n-2) = \binom{n}{3} + \binom{n}{4} \quad \text{if } n \geq 4, \\
S(n, n-3) = \binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6} \quad \text{if } n \geq 6.
\]

### 3. Results

In this section, we present some basic results of the \( p \)-adic valuations of Stirling numbers starting with \( S(n, n-1) \) for \( n > 1 \).

**Proposition 3.1** For any positive integer \( n > 1 \) and an odd prime \( p \)

\[
v_p \left( S(n, n-1) \right) = v_p(n) + v_p(n-1).
\]

**Proof.** Using the identity (3), we have

\[
S(n, n-1) = \binom{n}{2} = \frac{n(n-1)}{2}.
\]

The multiplicative property of \( v_p(a) \) implies that

\[
v_p \left( S(n, n-1) \right) = v_p(n) + v_p(n-1) - v_p(2)
\]

\[
= v_p(n) + v_p(n-1)
\]

as \( v_p(2) = 0 \), \( p \) being odd.

Applying Kummer’s theorem [20] to the binomial coefficient \( \binom{n}{2} = S(n, n-1) \), the above result can be put in the following form

\[
v_p \left( S(n, n-1) \right) = \frac{s_p(n-2) - s_p(n) + 2}{p-1},
\]

where \( s_p(n) \) denotes the sum of the \( p \)-adic digits of \( n \).

**Corollary 3.1** Let \( p \) be an odd prime. For any positive integer \( n \) and \( c \) such that \( \gcd(p, c) = 1 \),

\[
v_p \left( S(cp^s, cp^s - 1) \right) = n.
\]

**Proof.** By the proposition, we have
\[v_p(S(cp^n, cp^n - 1)) = v_p(cp^n) + v_p(cp^n - 1).\]

Since \(v_p(cp^n - 1) = 0\) and using the multiplicative property of \(v_p(a)\), we can obtain
\[v_p(S(cp^n, cp^n - 1)) = v_p(cp^n) = n + v_p(c).\]

As \(\gcd(p, c) = 1\), it is clear that \(v_p(c) = 0\). This completes the proof.

**Proposition 3.2** For any positive integer \(n \geq 2\) and an odd prime \(p\),
\[v_p(S(n, n - 2)) = \begin{cases} v_p(n) + v_p(n - 1) + v_p(n - 2) + v_p(3n - 5), & \text{if } p > 3, \\ v_3(n) + v_3(n - 1) + v_3(n - 2) - 1, & \text{if } p = 3. \end{cases}\]

These results can be proved in the similar manner.

**Corollary 3.2** For any positive integer \(n\) and an odd prime \(p\),
\[v_p(S(cp^n, cp^n - 2)) = \begin{cases} n, & \text{if } p > 5, \\ n + 1, & \text{if } p = 5 \text{ and } n > 1, \\ n - 1, & \text{if } p = 3, \end{cases}\]

if \(c\) is a positive integer not divisible by \(p\).

**Proposition 3.3** Let \(p\) be an odd prime. For any positive integer \(n \geq 6\),
\[v_p(S(n, n - 3)) = \begin{cases} v_p(n) + v_p(n - 1) + 2v_p(n - 2) + 2v_p(n - 3), & \text{if } p \geq 5, \\ v_3(n) + v_3(n - 1) + 2v_3(n - 2) + 2v_3(n - 3) - 1, & \text{if } p = 3. \end{cases}\]

**Proof.** Using the identity (5), we have
\[S(n, n - 3) = \binom{n}{4} + 10 \binom{n}{5} + 15 \binom{n}{6}, \text{ if } n \geq 6.\]

It can also be expressed as
\[S(n, n - 3) = \binom{n}{4} \left\lfloor \frac{n^2 - 5n + 6}{2} \right\rfloor = \binom{n}{4} \left\lfloor \frac{(n-2)(n-3)}{2} \right\rfloor = \binom{n(n-1)(n-2)(n-3)^2}{2^4 \cdot 3}.

The multiplicative property of \(v_p(\cdot)\) implies that
\[v_p\left(S(n, n - 3)\right) = v_p(n) + v_p(n - 1) + 2v_p(n - 2) + 2v_p(n - 3) - v_p(3)\]
as \(v_p(2) = 0\) and \(p\) being odd.

Using Kummer’s theorem \([20]\) to \(\binom{n}{4}\), we get the following result,

\[
v_p(S(n, n - 3)) = \frac{s_p(n - 4) - s_p(n) + s_p(4)}{p - 1} + v_p(n - 2) + v_p(n - 3). \tag{7}
\]

where \(s_p(n)\) denotes the sum of the \(p\)-adic digits of \(n\). This completes the proof.

**Corollary 3.3** For any positive integer \(n\) and odd prime \(p\), the following result holds

\[
v_p(S(cp^n, cp^n - 3)) = \begin{cases} 
  n, & \text{if } p > 3, \\
  n + 1, & \text{if } p = 3,
\end{cases}
\]

if \(p\) does not divides \(c\) (provided \(cp^n \neq 3\) if \(p = 3\)).

**Proof.** By the proposition, we have

\[
v_p(S(cp^n, cp^n - 3)) = v_p(cp^n) + v_p(cp^n - 1) + v_p(cp^n - 2) + 2v_p(cp^n - 3) - v_p(3).
\]

Since \(v_p(cp^n - 1) = v_p(cp^n - 2) = v_p(cp^n - 3) = 0\) if \(p \geq 5\), we get

\[
v_p(S(cp^n, cp^n - 3)) = v_p(cp^n)
\]

\[= n + v_p(c).
\]

As \(\gcd(p, c) = 1\), it is clear that \(v_p(c) = 0\).

For the case \(p = 3\), \(2v_p(c3^n - 3) - v_p(3) = 1\) and \(v_p(c3^n - 1) = v_p(c3^n - 2) = 0\) and hence

\[
v_p(S(c3^n, c3^n - 3)) = v_p(c3^n) + 1
\]

\[= n + 1
\]

This completes the proof.

Now, we give an alternate proof of the primality of integer \(n\) by divisibility of \(S(n, k)\) given by Deamio and Touset \([21]\). The proof of corollary 2 in their paper is not correct if we take \(n = 4\) and \(p = 2\), then \(S(4, 3) = 6 \equiv 1 \mod 2\) and \(2 | S(4, 3)\). We tackled this problem, in this paper, more simpler manner. This problem with an alternate solution also appears in Pólya et al. \([22]\).

**Theorem 3.1** If \(p\) is an odd prime, then \(p | S(n, k)\) if \(s_p(k) > s_p(n)\).

The above theorem is an immediate consequence of \(([18], \text{Lemma 2.1})\) which states that

\[
v_p(S(n, k)) \geq \frac{s_p(k) - s_p(n)}{p - 1}. \tag{8}
\]

Replacing \(n\) by an odd prime \(p\) in the above theorem, we get the following results.

**Corollary 3.4** If \(p\) is an odd prime, then \(p | S(p, k)\) if \(2 \leq k \leq p - 1\).

The problem in the above Corollary 3.4 appears in Graham et al. \([4]\) and proof was given by Demaio and Touset \([21]\).

**Theorem 3.2** A positive integer \(n\) is a prime if and only if \(n | S(n, k)\) for all \(2 \leq k \leq n - 1\).

**Proof.** The generating function of \(S(n, k)\) in terms of falling powers is given by

\[
x^n = \sum_{k=0}^{n} S(n, k) x^k \tag{9}
\]
for any non-negative integer \( n \).

If \( n \) is a positive integer such that \( n \mid S(n, k) \) for all \( 2 \leq k \leq n - 1 \), put \( x = n \) in Equation (9)

\[
n^n = \sum_{k=0}^{n} S(n, k) \{n\}_k
\]

\[
= \{n\}_n + \{n\}_1 + \sum_{k=2}^{n-1} S(n, k) \{n\}_k
\]

\[
= n(n-1)(n-2)\cdots2\cdot1 + n + \sum_{k=2}^{n-1} n(n-1)\cdots(n-(k-1))S(n, k).
\]

It follows that

\[
r^{n-1} = (n-1)(n-2)\cdots2\cdot1 + \sum_{k=2}^{n-1} (n-1)(n-2)\cdots(n-(k-1))S(n, k)
\]

Since \( n \mid S(n, k) \) for all \( 2 \leq k \leq n - 1 \), we get

\[
0 = (n-1)! + 1 \mod n
\]
or

\[
(n-1)! = -1 \mod n.
\]

Hence, \( n \) is prime.

The converse follows from Corollary 3.4.

**Lemma 3.1** If \( p \) is a prime, then

\[
\nu_p\left(\binom{p-1}{i}\right) = \left(\binom{p-1}{i} \cdot (-1)^i \right) \geq 1 \text{ or } \nu_p\left(\binom{p-1}{i}\right) = 0.
\]

**Proof.** For \( i = 0 \), the case is trivial.

We assume that \( i > 0 \). The binomial coefficient \( \binom{p-1}{i} \) is given by

\[
\binom{p-1}{i} = \frac{(p-1)!}{(p-1-i)!i!}.
\]

Therefore,

\[
i! \binom{p-1}{i} = (p-1)(p-2)\cdots(p-i+2)(p-i+1)(p-i)
\]

\[
= (-1)(-2)\cdots(-i) \mod p
\]

\[
= (-1)^i i! \mod p.
\]

Since \( 0 < i < p, \gcd(p, i) = 1 \). Then,
\[
\binom{p-1}{i} = (-1)^i \mod p.
\]

**Theorem 3.3** Let \( p \) be an odd prime. For any positive integer \( n \geq p \),

\[ v_p(S(n, p)) = 0 \]

if and only if \((p-1)|(n-1)\).

**Proof.** Using the above Lemma 3.1, we have

\[
p!S(n, p) = \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} i^p
\]

\[= \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} i^p \mod p.\]

Since \( \binom{p}{i} = \binom{p-1}{i-1} \), we get

\[(p-1)!S(n, p) = \sum_{i=1}^{p-1} (-1)^{p-i} i^{p-1}.\]

Using Wilson’s theorem, the preceding congruence reduces to

\[S(n, p) = \sum_{i=1}^{p-1} i^{p-1} \mod p,\]

as \( p \) is odd.

Now, we use the following well known results

\[
\sum_{i=1}^{p-1} i^{p-1} = \begin{cases} 
0 \mod p, & \text{if } (p-1)|(n-1) \\
-1 \mod p, & \text{if } (p-1)|(n-1).
\end{cases}
\]

Hence, the theorem follows.

**Theorem 3.4** Let \( p \) be an odd prime and \( c \) be a positive integer such that \( 1 \leq c \leq p-1 \). Then, for positive integers \( n \) and \( k \) such that \( k \leq n \),

\[ v_p(S(cp^n, cp^k)) = 0. \]

**Proof.** The theorem is a special case of ([18], Th. 2.2).

We have

\[ cp^n - cp^k = c(p^n - p^k) = c(p-1) \sum_{j=0}^{n-k} p^{j+k} \]

which implies that \( cp^n - cp^k \) is divisible by \( p-1 \). We also have \( 1 \leq c \leq p-1 \) and \( 1 \leq cp^k \leq cp^n \).

It follows that \( S(cp^n, cp^k) \) is a minimum zero case and hence we have
\[ v_p(S(cp^n, cp^k)) = \frac{s_p(cp^n) - s_p(cp^k)}{p - 1} = 0, \]  

since \( s_p(cp^n) = s_p(cp^k) = s_p(c) = c. \)

**Theorem 3.5** Let \( p \) be an odd prime, then

\[ v_p(S(p^n, 2p)) \geq n \]

for every integer \( n \geq 2. \)

**Proof.** Using identity (1)

\[
(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} p^i \]

which can also be written as

\[
(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{2p-i} (-1)^{2p-i} (2p-i)^{p^i} \]

Since \( \binom{m}{i} = \binom{m}{m-i} \) for every integers \( 0 \leq i \leq m \) and \( 2p - i = i \mod 2 \), we have

\[
2(2p)!S(p^n, 2p) = \sum_{i=0}^{2p} \binom{2p}{i} (-1)^{2p-i} (i^{p^i} + (2p-i)^{p^i}). \]  

(11)

If \( p \mid i \) for \( 0 \leq i \leq 2p \), then

\[ 2p - i = -i \mod p, \]

which also yields the congruence

\[ (2p - i)^{p^i} = -(i)^{p^i} \mod p^{n+1}. \]

It follows that

\[
\binom{2p}{i} (-1)^{2p-i} ((2p-i)^{p^i} + (i)^{p^i}) = 0 \mod p^{n+2}, \text{ since } p \mid \binom{2p}{i}. \]  

(12)

Thus, each terms of the right hand side of (11) is divisible by \( p^{n+2} \) and hence

\[ (2p)!S(p^n, 2p) = 0 \mod p^{n+2} \]

Therefore

\[ v_p((2p)!S(p^n, 2p)) \geq n + 2 \]

\[ v_p(S(p^n, 2p)) \geq n \]
Hence, the theorem follows.

**Theorem 3.6** Let $p$ be a prime and $n$ and $k$ be two positive integers with $k \leq p - 1$, then there exists a positive integer $m$ in $1 \leq m < p - 1$ such that

$$S(n, k) = \begin{cases} S(m, k) \mod p, & \text{if } n \not\equiv 0 \mod (p-1), \\ (p-1-k)! \mod p, & \text{if } n \equiv 0 \mod (p-1). \end{cases}$$

**Proof.** By division algorithm, we have

$$n = (p - 1)q + m$$

where $q$ is the quotient and $m$ is the remainder such that $0 \leq m < p - 1$.

Now

$$k!S(n, k) = \sum_{i=1}^{k} \left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{k+i} x$$

$$= \sum_{i=1}^{k} \left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{i} (p-1)^{q+m}$$

$$= \sum_{i=1}^{k} \left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{i} m \mod p$$

since $p^{r+1} \equiv 1 \mod p$ for $1 \leq i \leq k \leq p - 1$ by Fermat’s little theorem.

If $m \neq 0$, we have

$$k!S(n, k) = k!S(m, k) \mod p.$$ 

Since $k$ is less than $p$, it follows that $p \mid k!$ which results

$$S(n, k) = S(m, k) \mod p.$$ 

for every $n$ such that $n \not\equiv 0 \mod p - 1$.

Next, if $m = 0$, we have

$$k!S(n, k) = \sum_{i=1}^{k} \left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{k+i} \mod p$$

$$= \sum_{i=1}^{k} \left(\begin{array}{c} k \\ i \end{array}\right)(-1)^{i} - (-1)^{k} \mod p$$

$$= (-1)^{k+1} \mod p,$$

We also know that

$$\binom{p-1}{k} = (-1)^{k} \mod p \text{ or}$$
\[
\frac{(p-1)!}{(p-1-k)!k!} = (-1)^k \mod p \quad \text{or} \quad \frac{1}{k!} = (-1)^{k+1}(p-1-k)! \mod p
\]

which implies that

\[S(n, k) = (p - 1 - k)! \mod p,
\]

which completes the proof.

From the above theorem, we see that if \(1 \leq m < k\)

\[S(n, k) = 0 \mod p \quad \text{since} \quad S(m, k) = 0.
\]

However, the case for \(m = k\) results

\[S(n, k) = 1 \mod p.
\]

We can write the following results

**Corollary 3.5** Let \(p\) be an odd prime and \(k\) be a positive integer less than \(p\), then

\[
S(n, k) = \begin{cases} 
1 \mod p, & \text{if } n = k \mod (p-1), \\
0 \mod p, & \text{if } n = i \mod (p-1) \text{ for } 1 \leq i \leq k - 1.
\end{cases}
\]

If we applied the above theorem and corollary to the special cases for \(k = p - 1, p - 2\) and \(p - 3\), we get

\[
S(n, p - 1) = \begin{cases} 
1 \mod p, & \text{if } n = 0 \mod (p-1), \\
0 \mod p, & \text{otherwise}.
\end{cases}
\]

\[
S(n, p - 2) = \begin{cases} 
1 \mod p, & \text{if } n = 0, p - 2 \mod (p-1), \\
0 \mod p, & \text{otherwise}.
\end{cases}
\]

\[
S(n, p - 3) = \begin{cases} 
2 \mod p, & \text{if } n = 0 \mod (p-1), \\
3 \mod p, & \text{if } n = p - 2 \mod (p-1), \\
1 \mod p, & \text{if } n = p - 3 \mod (p-1), \\
0 \mod p, & \text{otherwise}.
\end{cases}
\]

assuming \(p \neq 3\) for the last two cases.

**4. Discussions**

We have computed \(v_p(S(p^2, kp))\) for primes \(3 \leq p \leq 100\) and \(2 \leq k \leq p - 1\) using PARI/GP software.
Table 1. \((p, k)\) such that \(v_p(S(p, k)) = 2\) for \(3 \leq p \leq 1000\) and \(2 \leq k \leq p - 1\)

<table>
<thead>
<tr>
<th>((p, k))</th>
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<tbody>
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<tr>
<td>(13, 5)</td>
<td>(167, 103)</td>
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The obtained values of \(v_p(S(p^2, kp))\) for different values of \((p, k)\) are

\[
v_p(S(p^2, kp)) = \begin{cases} 
7, & \text{if } (p,k) = (7,4) \\
6, & \text{if } (p,k) = (37,4),(59,14),(67,8) \\
3, & \text{if } k = p-1 \text{ and } (p,k) = (37,5),(59,15),(67,9) \\
5, & \text{if } k \text{ is even and } (p,k) \neq (7,4),(37,4),(59,14),(67,8) \\
2, & \text{if } k \text{ is odd and } (p,k) \neq (37,5),(59,15),(67,9). 
\end{cases}
\]  

We also provide in Table 1, the pairs of \(p\) and \(k\) where \(v_p(S(p, k)) = 2\) for \(3 \leq p \leq 1000\) and \(2 \leq k \leq p - 1\). It should be noted that \(v_p(S(p, k)) = 1\) for all the remaining pairs \((p, k)\).

After a closed examination of the output, we have observed that the arrays of \(v_p(S(p^2, kp))\) follow certain patterns which interpret as conjectures.

1. Let \(p > 7\) be an odd prime and \(k\) be an even integer such that \(0 < k < p - 1\). Then

\[
v_p(S(p^2, kp)) - v_p(S(p^2, p(k+1))) = 3.
\]

2. If \(k\) be an integer such that \(1 < k < p - 1\), then the \(p\)-adic valuations satisfy
\[ v_p(S(p^2, kp)) = \begin{cases} 5 \text{ or } 6, & \text{if } k \text{ is even} \\ 2 \text{ or } 3, & \text{if } k \text{ is odd} \end{cases} \]

for any prime \( p > 7 \).

3. For any odd prime \( p \) and a positive integer \( k \) such that \( 2 \leq k \leq p - 1 \),

\[ v_p(S(p, k)) \leq 2. \]

5. Conclusions

This paper deals with some results of \( p \)-adic valuations of Stirling number of the second kind, \( S(n, k) \) for odd prime \( p \). We have derived the formulas for \( v_p(S(n, n - 1)) \), \( v_p(S(cp^n, cp^n - 1)) \), \( v_p(S(n, n - 2)) \), \( v_p(S(p^n, p^n - 2)) \), \( v_p(S(n, n - 3)) \) and \( v_p(S(p^n, p^n - 3)) \). It has been shown the primality test of \( n \) using divisibility of \( n \) to \( S(n, k) \), \( 1 < k < n \). We have obtained the results that \( v_p(S(n, p)) \) depends on the divisibility of \( n - 1 \) by \( p - 1 \) and \( v_p(S(cp^n, cp^n)) = 0 \) for every integer \( n \geq k \geq 1 \) and \( p - 1 \geq c \geq 1 \). We also posed three conjectures after analyzing Table 1 and computational results of (13).

References


