Research Article

Structural Identifiability of Systems with Multiple Nonlinearities

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Abstract: The structural identifiability (SI) problem considers for dynamical systems with multiple nonlinearities under uncertainty. It shows the widely used paradigm based on a priori parametric identifiability is not applicable in this case. The geometric framework (GF) is derive from the system phase portrait and reflects the system nonlinear part properties under uncertainty. GF gives a conception of the system nonlinear part. The SI analysis problem interprets as a solution to the structural identification problem. The concept of S-synchronizability, which is the basis for estimation structural identifiability, introduce. Conditions of identifiability and structural identifiability are obtained. The constant excitation impact of input is studied on structural identifiability of the system. It shows that the input, which is constantly excited, can give the insignificant GF. Conditions obtain for the existence of insignificant frameworks. Approaches are proposed to the estimation of structural identifiability systems with two nonlinearities and difficulties are noted. It is shown that a priori information is critical about the relation of variables. The approach is proposed to SI estimation based on the analysis of the influence graph.

Keywords: structural identifiability, structure, geometric framework, nonlinearity, excitation constancy, importance graph, S-synchronizability, degree of non-identifiability

1. Introduction

The problem of identifying dynamic systems is one of the most relevant areas of study. Foundational results on the parametric identification of systems are obtained. At the same time, research is continuing to evaluate the parametric identifiability of dynamic systems. Identifiability is a condition for obtaining adequate models. Many studies have devoted to solving this problem [1-7]. Various approaches and methods are used to estimate identifiability: Taylor expansion [7]; series generation method [8]; similarity transformation [9] and differential algebra. The possibility of further application of parametric identification procedures is the result of applying these methods and approaches. The requirement of identifiability is reduced to estimating the non-degeneracy of the information matrix formed from the input and output data of the system. In identification theory, this requirement is equivalent to the condition of constant excitation. Call this direction parametric identification (IP).

This parametric paradigm is the main direction of research in identification theory and transformed into nonlinear systems. Many authors study on parametric identification of nonlinear systems [1-2, 5-7, 10-11]. In [1], the approach based on the analysis of the system’s output sensitivity is used to study identifiability. The effectiveness of this approach is shown for estimating the identifiability of the system parameter combination. In [11], local IP conditions are obtained.
for various types of experimental data. A critical analysis of the approaches used to estimate the identifiability of biological models given in [7]. Methods for identifiability estimating nonlinear systems are based on the approaches described in the first paragraph of this section. The requirements for the data analysis used in the IP task solution consider in [5]. The study of various types of identifiability (global, local, structural, and practical) is described in many works [1, 5]. Most studies on IP are associated with a priori identifiability.

Note that the identifiability problem of nonlinear systems has its peculiarities and relevance. To show this, consider the second-order system with hysteresis

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-3 & -4
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \phi(y) + \begin{bmatrix}
0 \\
1
\end{bmatrix} u,
\]

where \(y = x_1\),

\[
\phi(y) = \begin{cases}
2.2, & \text{if } (y - d > 2.2) \& (y' > 0), \\
y - d, & \text{if } (y - d \leq 2.2) \& (y' > 0), \\
1.5, & \text{if } (y - d \leq 1.5) \& (y' > 0), \\
2.2, & \text{if } (y > 2.2) \& (y' < 0), \\
y, & \text{if } (y \leq 2.2) \& (y' < 0), \\
1.5, & \text{if } (y \leq 1.5) \& (y' < 0),
\end{cases}
\]

\[
d = 1
\]

Show the impact of the input \(u(t)\) on the nonlinear properties of the system (1). This influence reflects on the system of identifiability. Let \(u_{6-0.5}(t) = 6 - 0.5\sin(0.1\pi t)\). The phase portrait of the system and the hysteresis shows in Figure 1. We see that the system (1) is identifiable.

**Figure 1.** Results of structure estimation for \(u_{6-0.5}(t)\)

Figure 2 presents the system properties for \(u_{6-0.4}(t) = 6 - 0.4\sin(0.1\pi t)\). Such input does not guarantee the identifiability of the system based on experimental data. We cannot conclude about the properties of the system from the phase portrait. Nonlinearity has a form that complicates decision-making. Presented results show that the identifiability problem of nonlinear systems is relevant. The input plays a significant role in solving the identifiability problem. This
problem complicated when multiplicity nonlinearities are in the system. As a rule, parametric identification methods are approximation and level the influence of nonlinearity. The example shows that the structural identifiability estimation is possible only if the system structure is known. The structural identifiability problem is not solved by parametric methods under uncertainty.

![Figure 2. Results of structure estimation for $u_{6,-0.5}(t)$](image)

Works on IP problems do not consider the estimation problem the system structure. Therefore, the concept of structural identifiability does not reflect the essence of the identifiability problem. But this terminology is used in problems of assessing identifiability. Therefore, in this section, we adhere to this terminology to continue analyzing the results obtained.

The concept of identifiability and smooth identifiability are introduced for nonlinear systems in [12]. In [13], a relationship studies between the identifiability and observability of nonlinear biological systems. The structural identifiability of time series is described by nonlinear regression and autoregressive equations studied in [14]. In [15], the observability controllability and reachability joint estimate for nonlinear processes obtained. The identification problem of parameters is considered for systems with several nonlinearities in [16-18]. The structure of nonlinearity specify a priori. is not studied.

So, the system identifiability is understood as the possibility to estimate its parameters. Methods are based on the information matrix nondegeneracy estimation. Similar results are obtained in the parametric estimation theory. They are based on checking the constant excitation (CE) condition of the input and output of the system. As a rule, the model structure specifies a priori, and the essence of the structural identifiability is not always clear. The identifiability of nonlinear system is reduced to the IP problem in the following sense: how the nonlinear system structure (form, dependence) to estimate under uncertainty. The SI problem is not studied in this form. But these are the structural aspects of the system identifiability. The question: input provides the structural identifiability of the system as structural identification, does not consider. This formulation is proposed in [19]. In [20], results presented for systems with a single nonlinearity given the solution to this problem.

In this paper, we consider the structural identifiability problem of the dynamical system with nonlinearities under uncertainty. It is very complex problem since the methods for formalizing the system structure have not been developed. The concept of SI ($h$-identifiability) is introduced in [19]. The proposed approach solves the problem of estimating the nonlinear system structure. It is based on the analysis of geometric frameworks that the state of the nonlinear system reflected. Below we give the summary and generalization of the results obtained in [19-20].

### 2. Problem statement

Consider the system
\[
\dot{X} = AX + B_u \Phi(Y) + B_\phi U,
\]

\[
Y = C^T X,
\]

where \(X \in \mathbb{R}^n\), \(U \in \mathbb{R}^p\), \(Y \in \mathbb{R}^k\) are state vector, input and output; \(A \in \mathbb{R}^{n \times n}\), \(B_u \in \mathbb{R}^{n \times p}\), \(B_\phi \in \mathbb{R}^{n \times q}\), \(C \in \mathbb{R}^{k \times n}\) are matrices of corresponding dimensions; \(\Phi(Y) \in \mathbb{R}^q\) is nonlinear vector function. \(A\) is Hurwitz matrix.

The nonlinear function \(\phi_i(\xi) \in \Phi\) is smooth and satisfies the condition

\[
\chi \leq \phi(\xi) \leq \gamma_2 \xi^2, \quad \xi \neq 0, \quad \phi(0) = 0, \quad \gamma_1 \geq 0, \quad \gamma_2 < \infty,
\]

where \(\xi \in \mathbb{R}\) is the input of a nonlinear element. \(\xi\) is a linear combination of state variables. For the system (2), the information set is known

\[
I_o = \{U(t), Y(t), t \in J = [t_0, t_k]\}.
\]

Problem: analyze the set \(I_o\) and estimate the structural identifiability of the system (2).

Apply the approach to structural identification proposed in [17]. It is based on the transition into a structural space and the construction of \(S_e\) framework. \(S_e\) reflects the properties of the nonlinear part (2). The analysis \(S_e\) is related to solving the SI problem of the system. To distinguish the approach described from IP, we use the \(h\)-identifiability term (HI).

Let \(q = 1\), \(C = [1, 0, \ldots, 0]^T\), \(B_\phi = B_u = I = [0, 0, \ldots, 0, 1]^T\), \(u \in \mathbb{R}\), \(Y = y\), \(y \in \mathbb{R}\), \(\Phi(Y) = \phi(Y)\). Denote the system (2), (3) with the specified parameters as \(S_{y\phi}\).

Next statements are given for the system \(S_{y\phi}\). \(S_{y\phi}\) is the particular case of the system (2), (3). We will denote \(S_{y\phi}\) as (2) further.

### 3. Method of constructing \(S_{e\phi}\)-framework

The construction of the \(S_{e\phi}\)-framework requires the formation of a set \(I_{N, e\phi}\) containing information about the function \(\phi(y)\). \(S_{e\phi}\) is described by a function \(f_{e\phi} : y \rightarrow e\), where \(e \in \mathbb{R}\) is a variable that reflects the change in the nonlinearity \(\phi(y)\) under uncertainty. Describe the method of obtaining \(I_{N, e\phi}\) [18]. Apply the differentiation operation to \(y(t)\) and denote the obtained variable as \(x_1\). Obtain the information set \(I_{ent} = \{I_o, x_1\}\).

**Remark 1** If the variables measured with an error apply filtering or smoothing procedures.

Select the subset \(I_g \subset I_{ent}\) corresponding to the partial solution of system (2) (steady state). Apply the mathematical model

\[
\dot{x}_1(t) = H^T [1 \ u(t) \ y(t)]^T,
\]

to the selection of the linear component in \(x_1\), where \(H \in \mathbb{R}^3\) is parameters vector. The variable \(x_1\) is defined on the interval \(J_g = J \setminus J_{u\phi}\).

We determine the vector \(H\) as the solution to the problem

\[
\min_H Q(e) \left|_{e = x_1 - x_1} \right. \rightarrow H_{opt}, \quad Q(e) = 0.5e^2.
\]

Find the forecast for the variable \(x_1\) by applying the model (5) \(\forall t \in I_g\) and form the error \(e(t) = \hat{x}_1(t) - x_1(t)\). \(e(t)\) is
the function of nonlinearity $\varphi(y)$. We have $I_{N,g} = \{ y(t), e(t) \, t \in J_g \}$. Apply the notation $y(t)$ supposing $y(t) \in I_{N,g}$.

**Remark 2** The choice of model (5) structure is one of the stages for structural identification of the system (2). The type model is defined by the input and system information.

The phase portrait $S$ is described by the function $\Gamma_{ey} : \{ y \} \to \{ e \}, \forall t \in J_g$ does not always guarantee the decision-maker on system nonlinear properties under uncertainty. Go to the structure space $P_{ve} = (y, e)$. Consider the function $\Gamma_{ey} : \{ y \} \to \{ e \}, \forall t \in J_g$ which on the plane $(y, e)$ describes the change in the framework $S_{ey}$. $I_{N,g}$ contains the information about $\varphi(y)$. Therefore, $S_{ey}$ describes the change in the nonlinear function in the generalized form. The identification of the form $\varphi(y)$ is based on the use of the input satisfied certain conditions. The input must have the property of constant excitation (see below). Such input gives the closed framework $S_{ey}$.

Apply the model (5) and represent the system $S$ as

$$S_y: \begin{cases} \dot{\tilde{X}} = A\tilde{X} + \zeta, \\ \tilde{y} = C^T\tilde{X}, \end{cases}$$

$$S_{\varphi}: e = f(y, x_1),$$

(6)

where $\tilde{X} \in R^n$ is the variable describing the general solution of the system (2); $\zeta \in R$ is a bounded perturbation appearing as the analysis result of the variable $e$.

Consider the identifiability problem of system $S_y, S_{\varphi}$.

### 4. Structural identifiability of nonlinear system $S_{\varphi}$

Consider the system $S_y$ and properties of the set $I_{N,g}$, which allow us to solve the structural identification problem, and, consequently, the identifiability estimation.

Let conditions hold.

- B1. The input is constantly excited at the interval $J$.
- B2. The analysis of $S_{ey}$ gives the solution to the estimation problem of the nonlinear properties of the system $S_{ey}$.

We will state the basic concepts following [20].

**Definition 1** If $u(t)$ satisfies B1 and B2 conditions, then the input $u(t)$ is representative.

Let the framework $S_{ey}$ is closed, and the area $S_{ey}$ is not zero. Denote height $S_{ey}$ as $h(S_{ey})$, where height is the distance between two points on the opposite sides of the framework $S_{ey}$.

**Theorem 1** [20]. Let (i) the linear part of the system $S_{\varphi}$ is stable; (ii) the nonlinearity $\varphi(\cdot)$ satisfies the condition (3); (iii) the input is bounded, and constantly excited; (iv) $h(S_{ey}) \geq \delta_{ey}$, where $\delta_{ey} > 0$. Then the framework $S_{ey}$ is identified on the set $I_{N,g}$.

Theorem 1 is showed conditions in the framework $S_{ey}$. The framework $S_{ey}$ must be closed, hence, it must have the height or distance between opposite points of the framework. It ensures that $S_{ey}$ has a diameter (see below).

**Definition 2** The framework $S_{ey}$ is called $h$-identifiable if theorem 1 holds for $S_{ey}$.

Let $S_{ey}$ be $h$-identifiable. $h$-identifiability features are considered in [19-20].

But a “bad” input exist that is constantly excited. This input gives a so-called “insignificant” $S_{ey}$-framework ($NS_{ey}$-framework). However, the $NS_{ey}$-structure can be $h$-identifiable. The system (2) identification with the $NS_{ey}$-framework gives results which are not typical for the system.

**Conditions of an existence $NS_{ey}$-structure.** Consider a class of nonlinear functions to which the homothety operation is applicable [22].

Let $S_{ey} = P^l_{S_{ey}} \cup P^r_{S_{ey}}$, where $P^l_{S_{ey}}, P^r_{S_{ey}}$ are left and right fragments $S_{ey}$. Determine for $P^l_{S_{ey}}, P^r_{S_{ey}}$ secant

**Contemporary Mathematics** 144 | Nikolay Karabutov
\[ \gamma^*_S = a^* y, \quad \gamma'^*_S = a'^* y, \] (7)

where \( a^* \), \( a'^* \) are numbers determined using the least-squares method (LSM).

**Theorem 2** [16]. Let (i) the framework \( S_{ey} \) is \( h \)-identifiable and has the form \( S_{ey} = F^l_{S_{ey}} \cup F^r_{S_{ey}} \), where \( F^l_{S_{ey}} \), \( F^r_{S_{ey}} \) is the left and right fragment of \( S_{ey} \); (ii) secants for \( F^l_{S_{ey}} \), \( F^r_{S_{ey}} \) are described by equations (7). Then \( S_{ey} \) is \( NS_{ey} \)-framework if

\[ \left\| a' - a^* \right\| > \delta_h, \quad \delta_h > 0. \] (8)

**Remark 3** \( NS_{ey} \)-frameworks are often the result of inadequate application of input action.

Introduce designations: \( D_y = \text{dom}(S_{ey}) \) is definition range of the framework \( S_{ey} \), \( D_y = \max_y \{ y(t) - \min_y y(t) \} \) is diameter \( D_y \). Let \( u(t) \in U \) where \( U \) is an acceptable set of inputs for the system (2). The set \( U \) contains representative inputs.

**Definition 3** If \( D_y \) of the structure \( S_{ey} \) has the maximum diameter \( D_y \), the input \( S \)-synchronizes the system (2).

**Definition 4** The input \( u(t) \in U_S \subseteq U \) is the \( S \)-synchronizer system \( S_{ey} \) if the definition range \( D_y \) of the framework \( S_{ey} \) has the maximum diameter \( D_y \).

Consider a reference framework \( S_{ey}^{ref} \). \( S_{ey}^{ref} \) is the framework \( S_{ey} \) reflecting all properties of the function \( \varphi(y) \). Designate by the diameter \( D_y(S_{ey}^{ref}) \) as \( D_y^{ref} \). \( D_y^{ref} \) exists if the input the system (2) is \( S \)-synchronizing.

Definitions 2, 3 show if \( S_{ey} \cong S_{ey}^{ref} \), then \( |D_y - D_y^{ref}| \leq \varepsilon \), where \( \varepsilon \geq 0, \equiv \) is the proximity sign. Elements of the subset \( U_S \) have property

\[ |D_y(S_{ey}(u(t)|_{u \in U_S}) - D_y^{ref}| \leq \varepsilon. \] (9)

Synchronization \( u(t) \in U \) is the choice of the input \( u_h(t) \in U \) such that reflects all features \( \varphi(y) \) in \( S_{ey} \). It is true if \( u(t) \) ensures \( \max_{u_h} D_y \) and \( S_{ey} \neq NS_{ey} \). We interpret the choice \( u_h(t) \in U \) as ensuring synchronization between structures of the model and the system. \( d_{h,y} = \max_{u_h} D_y \) is the condition of \( h \)-identifiability which can represent as

\[ |D_y(S_{ey}(u(t)|_{u \in U_S}) - d_{h,y}| \leq \varepsilon. \] (10)

The condition for \( NS_{ey} \)

\[ |D_y(S_{ey}(u(t)|_{u \in U_S}) - d_{h,y}| > \varepsilon. \] (11)

(10) can be interpreted as proximity domain

\[ Q = \left| S_{ey}(u(t)|_{u \in U_S}) - S_{ey}^{ref} \right|, \] (12)

which is understood as \( |\dot{y}(t) - \dot{y}^{ref}(t)| \leq \varepsilon \) for almost \( \forall t \geq \tilde{t} \).

We will write \( \delta Q \leq \varepsilon \) if frameworks under consideration are close. If the condition \( \delta Q \leq \varepsilon \) is true for \( Q \) for
almost $\forall t \geq t^*$ then the domain $Q_D$ will be called the S-synchronizability area on the set of inputs $\{u_h(t)\}$ or the structural identifiability domain on the set $\{S_y(u_h(t))\}$, where $S_y$ is the phase portrait of the system $S_{yw}$.

So, two criteria (8) and (11) presented for the existence of the insignificant framework. Structure of systems $S_y$ and $S_{yw}$ are structurally unidentifiable in this case.

Let the input $u_h(t)$ synchronize the system $S_{yw}$. If $u(t)$ is S-synchronizing, then we will write $u_h(t) \in S$. Note that a finite set $\{u_h(t)\} \in S$ exists for the system $S_{yw}$. The choice of optimal $u_h(t)$ depends on $d_{h,y}$ and (10). The hold of the condition (10) is one of the prerequisites for SI of the system $S_{yw}$.

Definition 5 If framework $S_{cy}$ is $h$-identified and conditions $\|a' - a''\| \leq \delta_{h,y}$, (9) satisfied, then the framework $S_{cy}$ or the system (2) (system $S_{yw}$) is structurally identified or $h_{y}$-identifiable.

Definition 5 shows if the system (2) is $h_{y}$-identified then the framework $S_{cy} \in S_{yw}$ has the maximum diameter of area $D_{y}$. If $S_{cy}$ is $h_{y}$-identifiable.

Definition 6 The model (5) is SM-identifying if the framework $S_{cy}$ is $h_{y}$-identifiable.

The framework $S_{cy}$ is defined on $u_h(t) \in S$ and $u_h(t) 1$ satisfies condition B1. Therefore, $S_{cy}$ corresponds to the nonlinearity $\phi(y)$ defined on the class

$$\phi(y) \in \mathcal{F}_\phi = \{\phi(y) \in R \mid \phi(y, A), A \in R^{nA}, \alpha_j \in A, \alpha_j \in [\overline{a}_j, \overline{\alpha}_j]\},$$

where $\overline{a}_j, \overline{\alpha}_j$ are some numbers.

Note that the term SM-identifying does not coincide with the concept proposed in [24].

Theorem 3 [19]. Let (i) the input $u(t) \in S$ is constantly excited; (ii) the system $S_{yw}$ phase portrait have $m$ features; (iii) $S_{cy}$-framework is $h_{y}$-identified and contains fragments corresponding to features of the system $S_{yw}$. Then the model (5) is SM-identifying.

The theorem 3 shows if the model (5) is not SM-identifying, then model (5) structure or the informational set (4) need to change.

Let $c_S$ is the center of the framework $S_{cy}$ on the set $J_y = \{y(t)\}$, $c_{y_t}$ is the center of the area $D_{y}$. The set $U_S$ given for the system $S_{yw}$ and (i) exists $\varepsilon \geq 0$ such that $\|c_S - c_{D_y}\| \leq \varepsilon$; (ii) $\|a' - a''\| \leq \delta_{h,y}$, where $a'$, $a''$ are coefficients of secants (8) for $(\mathcal{F}_{S_{cy}^I}, \mathcal{F}_{S_{cy}^r}) \subset S_{cy}$. Then the system $S_{yw}$ is $h_{y}$-identifiable and the input $u_h(t) \in S$, and the structure $S_{cy}$ defines the class $\mathcal{F}_\phi$.

Since $\phi(y) \in \mathcal{F}_\phi$ the center $c_{D_y}$ of area $D_{y}$, $c_{y_t}$ is in $J_{y_{cy}}$, where $J_{y_{cy}}$ is some interval.

Let some subset $\{u_{h_{i}}(t)\} \subset U_S \subseteq U$ (i $\geq 1$) whose elements have the property of S-synchronizability exists. The framework $S_{cy}(u_{h_{i}})$ with the diameter $D_{y_{i}}$, of area $D_{y_{i}}$ corresponds to every $u_{h_{i}}(t)$. As $u_{h_{i}}(t) \in S$ the diameter $D_{y_{i}}$ has the property $d_{h_{i}} \Sigma$-optimality. Let the hypothetical framework $S_{cy}$ (the framework $S_{cy}^{rf}$) of the system $S_{yw}$ have diameter $d_{h_{i}} \Sigma$.

Definition 7 The framework $S_{cy_{i}}$ has $d_{h_{i}} \Sigma$-optimality property on the set $U_h$ if $\varepsilon_{\Sigma} > 0$ such that $\|f_{h_{i}D_{y_{i}}} - D_{y_{i}}\| \leq \varepsilon_{\Sigma}$ \forall i = 1, # $U_h$.

Definition 8 If $\{u_{h_{i}}(t)\} = U_{h_{i}} \subset U$ & $(u_{h_{i}}(t) \in S)$, i $\geq 1$ and frameworks $S_{cy_{i}}(u_{h_{i}})$ have $d_{h_{i}} \Sigma$-optimality property, then frameworks $S_{cy_{i}}(u_{h_{i}})$ are structurally indiscrepant on sets $\{u_{h_{i}}(t)\} J_{y_{i}}(u(t) = u_{h_{i}}(t))$.

So, the $h_{y}$-identifiability estimate can be obtained from any input, following definitions 6, 7.

Definition 9 If frameworks $S_{cy_{i}}(u_{h_{i}})$ have $d_{h_{i}} \Sigma$-optimality property, then $S_{cy_{i}}(u_{h_{i}})$ is locally structurally identifiable on the set $U_{h_{i}}$.

Denote the framework $S_{cy_{i}}(u_{h_{i}})$ had $d_{h_{i}} \Sigma$-optimality property as $S_{cy_{i}}^{\Sigma}$, and the locally structurally identified framework $S_{cy_{i}}(u_{h_{i}})$ as $S_{cy_{i}}^{LSI}$. 

Contemporary Mathematics 146 | Nikolay Karabutov
The framework $S_{cy}$ is locally structurally identifiable on the set $U_h \subseteq U_S$ if

$$\left( \exists u_h \in S \right) \Rightarrow \left( S_{cy} \subseteq S^E \right) \Rightarrow S_{cy} \equiv S_{cy}^{LSI} \tag{13}$$

**Remark 4** We consider nonlinearities satisfying condition (3). Therefore, notes made above are valid.

**Definition 10** The framework $S_{cy}$, $(u_i) \notin U_S$ that does not have the $d_h, \Sigma$-optimality property is locally structurally unidentifiable on the set $U_h$.

The framework $S_{cy} \setminus U_h$ that is structurally unidentifiable on the set $U_h$ defines the class $\mathcal{F}_{pe}^N \subset \mathcal{F}_{pe}$.

**Remark 5** The described approach applies to the nonlinear system with the dynamic law of nonlinearity change. In this case, the multilevel analysis gives the solution to the identifiability problem.

The identifiability of system $S$ is considered in [21].

### 5. On excitation constancy effect on identifiability of system

In [23], the excitation constancy influence is studied on the identifiability estimation of the system with hysteresis. It is showed that not every input with the CE property guarantees the structural identifiability of the system. Below we present results that allow estimating the CE impact.

Consider the input $u \in \mathcal{P}_E\alpha$, where

$$\mathcal{P}_E\alpha(t) \geq \alpha$$

holds for $\exists \alpha > 0$ and $\forall t \geq t_0$ on some interval $T > 0$.

Let input $u(t)$ of the system $S_{cy}$ have the property $u(t) \in \mathcal{P}_E\alpha_{\omega_k}$, where

$$u_k(t) : \left( u_k \in \mathcal{P}_E\alpha \right) \& \left( u_k \in \mathcal{P}_E\omega_k \right) \& \left( u_k \in S \right), \quad \mathcal{P}_E\omega_k : \left( \mathcal{P}_E\omega_k \right) = \mathcal{R}\mathcal{F}_k \left( \Omega_k \right) \tag{14}$$

$\mathcal{R}\mathcal{F}(\Omega_k)$ is the model for $u_k(t)$ based on the Fourier series and given on the set of frequencies $\Omega_k = \left\{ \omega_1, \omega_2, \ldots, \omega_k \right\}$.

Let $u_k \in U_k, U_k = U \setminus U_S$. Consequently $u_k \notin S$. For $u_h \in S$ is hold

$$u_h(t) : \left( u_h \in \mathcal{P}_E\alpha \right) \& \left( u_h \in \mathcal{P}_E\omega_h \right) \& \left( u_h \in S \right), \quad \mathcal{P}_E\omega_h : \left( \mathcal{P}_E\omega_h \right) = \mathcal{R}\mathcal{F}_h \left( \Omega_h \right) \tag{15}$$

where $\Omega_k \neq \Omega_h$.

Compare (14), (15) and obtain

$$\left( \mathcal{R}\mathcal{F}_h \left( \Omega_h \right) \neq \mathcal{R}\mathcal{F}_k \left( \Omega_k \right) \right) \Rightarrow S_{cy}^h \neq S_{cy}^k \Rightarrow S_{cy}^h = \mathcal{N}S_{cy} \tag{16}$$

From (16) have

$$\left( \mathcal{D}_y \left( S_{cy}^h \right) \neq \mathcal{D}_y \left( S_{cy}^k \right) \right) \Rightarrow \left[ \mathcal{D}_y \left( S_{cy}^h \right) \geq \mathcal{D}_y \left( S_{cy}^k \right) \right] \tag{17}$$

The definitional domain of frameworks $S_{cy}^h, S_{cy}^k$ do not coincide, and $S_{cy}^h$ is $d_h, \Sigma$-optimal on the set $U_h$. Therefore, the fulfillment of the condition (11) follows from inequality (17). Consequently, the structure of the system $S_{cy}$ nonlinear
part with \( u_k \) has indicators that do not coincide with the structurally identifiable parameters (2) with \( u_h \).

So, the CE condition of the input affects the \( h \)-identifiability of the \( S_y \)-system, and, consequently, the system \( S_y \).

The statement is true.

Theorem 5. Let (i) the input \( u_k \) satisfies to condition (14); (ii) the \( S_y \)-framework corresponds to the input \( u_k \); (iii) there is the input \( u_h \in S \) such that the condition (15) is satisfied; (iv) conditions (16), (17) holds. Then (a) the \( S_y \)-system is structurally unidentifiable by the input \( u_k \); (b) structural parameters of the \( S_y \)-system do not correspond to the parameters system \( S_y \) with the identifiable structure.

Obtain the non-identifiability degree estimate of the \( S_y \)-system. Let the phase portrait \( S \) constructed for the system \( S_y \). Definitional domains of \( S \) and \( S \)-frameworks are coincident. Therefore, the diameter \( D(S_y) \) of the framework \( S \) is known. Consider the set \( \{u_i(t)\} \) having the property \( PEP \). Determine for each \( u_i(t) \) the structure \( S_y \), and obtain \( D_y(u_i) \).

Suppose \( d_{h,y} = \max \{D_y(u_i)\} \), and denote the corresponding input as \( u_h \). Determine \( d_{h,y} = |D_y(u)\) for all inputs \( V = \{u_i(t)\} \). Since \( u_h \in S \), therefore \( d_{h,y} > D_y, \forall j \geq 1 \). Then evaluate the non-identifiability degree as

\[
SI_j = SI(S_y, j) = \frac{d_{h,y} - d_{y,j}}{d_{h,y}} \tag{18}
\]

(18) shows that system \( S_y \) is structurally identifiable if \( SI_j \to 0 \). The structural identifiability area \( Q_D \) is defined by the condition (10).

Remark 6 If fragments \( F_S, F_S \) select on the phase portrait \( S \), then the estimate for the non-identifiability is defined as

\[
SI = SI(S) = \frac{d_f(S)}{d_f(S)} \tag{19}
\]

where \( d_{f}(F_S), d_{f}(F_S) \) are diameters of fragments \( F_S, F_S \). The system \( S_y \) is structurally identifiable if \( SI(S) \leq O(2) \) where \( O(2) \) is neighborhood 1.

The input amplitude can influence on the SI of nonlinear systems. Modify conditions (14), (15)

\[
u_k(t) : (u \in PE \) \& \( (u_h \in S), \) \( u_h(t) = Rf_k(G_k, \Omega_k), \tag{19}
\]

\[
u_h(t) : (u_h \in PE \) \& \( (u_h \in S), \) \( u_h(t) = Rf_h(G_h, \Omega_h), \tag{20}
\]

where \( G_k, G_h \) are model \( RF_k, RF_h \) parameter vectors.

Present models \( RF_k, RF_h \) as

\[
RF_k(G_k, \Omega_k) = g_h RF_k(G_h, \Omega_h), \quad RF_h(G_h, \Omega_h) = g_h RF_h(G_h, \Omega_h),
\]

where \( RF_k(G_h, \Omega_h), RF_h(G_h, \Omega_h) \) are modifications of models (14), (15); \( g_h = \max_{i \leq 1} g_{h,i} \), \( i = 1, \# \Omega \), \( g_{h,i} \) is an element \( G_h \), \( g_{h,i} = \max_{i \leq 1} g_{h,i} \), \( g(p = k, h) \) denotes the generalized amplitude of the input.

Condition (16) is transformed into the form
\[ g_h \mathcal{RF}_h \left( \tilde{G}_h, \Omega_h \right) \neq g_k \mathcal{RF}_k \left( \tilde{G}_k, \Omega_k \right). \]

Since \( u_h \in S \) then \( g_h \geq g_k \). This conclusion follows from

\[ D_h \left( S(u_h) \right) \geq D_h \left( S(u_k) \right) \Rightarrow \mathcal{RF}_h \left( \tilde{G}_h, \Omega_h \right) \geq \mathcal{RF}_k \left( \tilde{G}_k, \Omega_k \right), \]

and the model \( \mathcal{RF}_h \left( \tilde{G}_h, \Omega_h \right) \) approximates the input ensuring S-synchronization of the system \( S_{yw} \).

Obtain \( d_{h,z}\)-optimality of the diameter \( D_h(S^h) \) from \( S(u_h) \Rightarrow S^h \). The framework \( S^h \) does not have this property (see (20)). Therefore, the input \( u_k \notin S \), which has a smaller generalized amplitude, gives the diameter \( D_k(S^k) \).

**Theorem 6** Let (i) the input \( u_k \) of the system \( S_y \) satisfies the condition (19); (ii) the framework \( S^k \) corresponds to input \( u_k \); (iii) there is an input \( u_h \in S \) such that the condition (20) holds; (iv) conditions (16), (17) are hold. Then (a) the \( S_y \)-system is structurally non-identifiable by the input \( u_k \); (b) structural parameters of the system \( S_y \) do not correspond to the system \( S_{yw} \) with an identifiable framework \( S^h \) if \( g_h \geq g_k \).

**Example 1** Consider the nonlinear system with Bouc-Wen hysteresis (system \( S_{BW} \))

\[
\begin{align*}
mx + cx + F(x, z, t) &= f(t), \\
F(x, z, t) &= a k z(t) + (1 - \alpha) k dz(t), \\
\dot{z} &= d^{-1} \left( ax - \beta |x|^{\alpha} \text{sign}(z) - \gamma |z|^{\beta} \right).
\end{align*}
\]

where \( m > 0 \) is weight, \( c > 0 \) is damping, \( F(x, z, t) \) is the restoring force, \( d > 0, n > 0, k > 0, \alpha \in (0, 1), y(t) = x(t), u(t) = f(t) \) is exciting force, \( a, \beta, \gamma \) are some numbers.

Let \( n = 1.5, c = 1, m = 1, \beta = 0.5, \alpha = 0.7, k = 0.6, d = a = 1 \). The set \( I_o \) has the form \( I_o = \{ u(t), y(t), t \in [0; t_e] \} \).

Consider four variant inputs

\[
\begin{align*}
u_0(t) &= 2 - 2 \sin(0.15\pi t), \\
u_1(t) &= 2 - 2 \sin(0.35\pi t), \\
u_2(t) &= 2 - 2 \sin(0.5\pi t), \\
u_3(t) &= 2 - 2 \sin(0.15\pi t) + 0.2 \sin(0.35\pi t).
\end{align*}
\]

Denote phase portraits of the system \( S_{BW} \) with inputs (24) as \( S_i(t = 0, 3) \). Definitional domains the phase portrait and hysteresis coincide (see Figure 3 for the case \( S_0 \)).

Calculate diameters for the phase portrait definitional domain

\[
\begin{align*}
D_{S,0}(S_0) &= 3.75, \\
D_{S,1}(S_1) &= 1.728,
\end{align*}
\]
\[ D_{y,2}(S_2) = 1.08, \]
\[ D_{y,3}(S_3) = 3.967. \]  \hspace{1cm} (25)

**Figure 3.** Phase portrait and output of the hysteresis for the system with \( u_0 \)

Diameter calculation is based on the analysis of frameworks. Definitional domains of frameworks \( S_{cy}, S \) and the hysteresis coincide (see Figures 5, 6). Hence, the diameter is the definitional domain \( S_{cy} \) and \( S \). Results are obtained for the system \( S_{BW} \) steady state. The analysis is showed \( u_0(t) \in S \). We assume that the system \( S_{BW} \) with the phase portrait \( S_0 \) is the standard and \( D_{h,y} = D_{y,0}(S_0) \). The degree of non-identifiability of the system \( S_{BW} \) for various \( u_i \):

\[ SI_1 = 0.549, \]
\[ SI_2 = 0.718, \]
\[ SI_3 = -0.035. \]  \hspace{1cm} (26)

**Figure 4.** To structural identifiability assessment of system \( S_{BW} \)

(a) diameter estimate framework \( S_i \) in ranked space

(b) adequacy estimate of framework \( S_i \)
We see that the $S_{BW}$-system with $u_1$, $u_2$ is structurally non-identifiable, and the $S_{BW}$-system with input $u_3$ is structurally indistinguishable from input $u_0$. So, frameworks $S_{ey}, 1(u_1)$, $S_{ey}, 2(u_2)$ are frameworks of class $\mathcal{NS}_{ey}$, and the framework $S_{ey}, 3(u_3)$ belongs to class $\mathcal{LSI}_{ey}$.

Obtained results are confirmed by Figure 4. It shows system outputs in an integrated form. Rectangular areas represent estimates of diameters within the specified limits. They confirm estimates (26).

---

Figure 5. Comparison of $S_{BW}$-system hysteresis with different inputs

So, the frequency properties of the input influence the identifiability of the system significantly. It is relevant for nonlinear systems, where the minor change in input properties affects the estimation of structural parameters. This conclusion is confirmed by Figure 5, where the Bouc-Wen model (23) output shown at different inputs. We see that $u(t)$ changes the definition domain and the actual range of the hysteresis.

The area $Q_D$, which confirms conclusions, shown in Figure 6. Notation in Figure 6: 1 is $S_0$, 2 is $S_2$, 3 is $S_1$, 4 is $S_3$.

---

Figure 6. Structural identifiability domain

6. On structural identifiability of system with two nonlinearities

Consider the system $S_{y\phi}$ with two nonlinearities
\[ B_{\varphi} \varphi(y) = B_{\varphi,1} \varphi_1(y) + B_{\varphi,2} \varphi_2(y), \] 

(27)

where \( \varphi(y) \) satisfies the condition (3).

This case is more complex and has features. In this case, the decision-making on the structural identifiability of the \( S_{\varphi} \)-system is based on the approach described in section 4. But the analysis of \( S_{\varphi} \)-framework properties may be incomplete. There may be a situation where \( S_{\varphi} \) is partially the \( \mathcal{NS}_{\varphi} \)-structure. Consider this case.

Consider the \( S_{\varphi} \)-framework (Figure 7 reflects the steady-state) for the second-order system \( S_{y\varphi} \), (27). Apply Theorem 2 and obtain that condition (8) does not hold. Hence, \( S_{\varphi} = \mathcal{NS}_{\varphi} \).

Figure 7. Example \( S_{\varphi} \)-framework for the system \( S_{y\varphi} \), (27)

Figure 7 shows that one nonlinearity is identifiable, and the other non-linearity is don’t identifiable. Consider mapping \( \Gamma_{y\varphi} : \{ y \} \rightarrow \{ k_{y\varphi} \} \), \( k_{y\varphi} = \frac{e}{y} \) to the decision making. \( \Gamma_{y\varphi} \) corresponds to the \( S_{\varphi} \)-framework (see Figure 8).

Figure 8. \( S_{\varphi} \)-framework

The analysis \( S_{\varphi} \) shows that the nonlinearity (denote it as \( \varphi_1 \)) dominates and is identifiable, and the nonlinearity \( \varphi_2 \) is non-identifiable. Present the framework \( S_{\varphi} \) as \( S_{\varphi} = S_{\varphi}^{id} \cup \mathcal{NS}_{\varphi} \) where \( S_{\varphi}^{id} = S_{\varphi}(\varphi_1) \), \( \mathcal{NS}_{\varphi} = S_{\varphi}(\varphi_2) \).

Definition 11 The system \( S_{y\varphi} \), (27) is called partially structurally identifiable or identifiable on the level \( \varphi_1 \) under \( u \).
if the fragment \( S_{ey}^{id} \) of framework \( S_{ey} \) is \( h \)-identifiable, and unidentifiable at the level \( \varphi_2 \) if \( S_{ey}(\varphi_2) = \mathcal{NS}_{ey} \).

**Definition 12** The subsystem \( S_{ey}^{ee} \) of the system \( S_{ey}^{ee} \), (27) is called identifiable under \( u \in S \) if the fragments \( S_{ey}(\varphi_1) \), \( S_{ey}(\varphi_2) \) of the framework \( S_{ey} = S_{ey}(\varphi_1) \cup S_{ey}(\varphi_2) \) are \( h \)-identifiable.

**Definition 13** The subsystem \( S_{ey}^{ee} \) of the system \( S_{ey}^{ee} \), (27) called structurally identifiable under \( u \in S \) if the fragments \( S_{ey}(\varphi_1) \), \( S_{ey}(\varphi_2) \) of the framework \( S_{ey} = S_{ey}(\varphi_1) \cup S_{ey}(\varphi_2) \) are \( h \)-identifiable, and the conditions of theorem 4 are satisfied for each fragment \( S_{ey}(\varphi_1), S_{ey}(\varphi_2) \).

In this case, the estimation of the \( S_{ey}^{ee} \)-system identifiability is based on the fragmentation of the framework \( S_{ey} \). Apply the smoothing operation on the fragment \( S_{ey}(\varphi_1) \). Obtain the estimate \( \hat{S}_{ey}(\varphi_2) \) of the framework \( S_{ey}(\varphi_2) \). The estimate of framework \( S_{ey}(\varphi_1) \) formed as \( \hat{S}_{ey}(\varphi_1) = S_{ey} \setminus \hat{S}_{ey}(\varphi_2) \). If the condition (27) does not satisfy for nonlinearities, then the system (2) structural identifiability analysis is the complicate problem. In this case, the system \( h_{\delta}^{\text{id}} \)-identifiability estimate requires an extension of the approach proposed in previous sections. Demonstrate this with the example.

**Example 2** Consider the system consisting of the nonlinear actuator and a controlled object. The object has linear and quadratic friction. The actuator has saturation. The system of equations has the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
c\varphi_2(u)
\end{bmatrix} +
\begin{bmatrix}
0 \\
c\varphi_1(x_2)
\end{bmatrix},
\]

where \( \varphi_1(x_2) = x_2^2 \text{sign}(x_2) \) is the quadratic friction, \( \varphi_2(u) = \text{sat}(u) \) is dry friction, \( x = x_1 \) is the rotation angle of the object shaft, \( u \) is current excitation winding of the actuator, \( y \) is the output, \( c_1 = 2, c = 1 \), \( u(t) = 3\sin(0.1\pi t) \).

Experimental information has the form \( I_o = \{u(t), y(t), t = [0, t_f] \}, t_f < \infty \). Construct frameworks \( S, S_{ey} \) (see Figure 9).

![Figure 9. Frameworks S, S_{ey}](image)

Apply the results of section 4 and obtain the system structural identifiability. The decision-making about the nonlinearity structure does not give the analysis of frameworks. It is caused by the nonlinearity of the input. The input \( \varphi_2(u) \) on the interval \( J_o = [4; 8.5] \) is constant, and the condition CE is not satisfied. We can assume (see Figure 9) that \( \varphi_2(u) = \text{sat}(u) \). The working interval for \( y \) is equal to \( J_y = [2; 4] \cup [8.5; 10] \). The application of model (5) in this case is inefficient. Therefore, perform the analysis of the dependence \( \dot{x}_2 \) on available variables. The coefficient of determination between \( \dot{x}_2 \) and \( x_2 \) is \( r_{\dot{x}_2 x_2}^2 = 0.995 \), \( r_{\dot{x}_2 y}^2 = 0.916 \). Therefore, there is the dependency between \( \dot{x}_2 \) and \( x_2 \). Apply the method of hierarchical immersion (MHI) to correction structural relationships [23]. Execute the following steps. Let
1. The mathematical model has the form

\[
\hat{\phi}_2(u) = \begin{cases} 
1.8, & \text{if } u > 1.8, \\
u, & \text{if } u \leq 1.8, \\
-1.8, & \text{if } u < -1.8. 
\end{cases}
\]  

(29)

2. Exclusion of influence \( y \). Enter the misalignment \( \varepsilon = \hat{x}_2 - \hat{x}_2 \) and obtain the model

\[
\hat{\varepsilon} = -0.2038y + 0.933, \quad r_{\varepsilon y}^2 = 0.95. 
\]  

(30)

3. Determine the misalignment \( \pi = \varepsilon - \hat{\varepsilon} \) and approximate it with the linear model

\[
\hat{\pi} = 0.424x_2 - 0.559, \quad r_{x2\pi}^2 = 0.94. 
\]  

(31)

So, the model (31) is adequate. Apply the model

\[
\hat{\varepsilon}_2 = -0.37|x_2| x_2 - 0.45, \quad r_{x2\varepsilon}^2 = 0.97. 
\]  

(32)

to increase the accuracy of the approximation \( \pi(t) \).

**Remark 7** The implementation of MHI is based on checking the SI of the framework at each stage. Next, the mathematical model design. This stage is the prerequisite for obtaining the adequate model.

Thus, the analysis confirms the structural identification possibility of the nonlinear system (28) and its identifiability on interval \( J_y \). It is shown that the model (5) application depends on the structure of the system. Therefore, it is not possible to propose a general method to the choice of the model structure for the system with multiple nonlinearities. The approach depends on the specifics of the system under study. This conclusion illustrates considered examples, and confirms the versatility and complexity of the SI problem under consideration.

The system (2) identifiability depends on structural relationships. As a result, the indirect influence of one variable on another can occur. This case is typical for sequentially connected parts of the system. Such systems are typical objects for the identification [16, 18]. In this case, the influence graph between the system variables is used. It is a polyhedron whose cross-section over some variable reflects the relationships between all variables of the system. Next, apply MHI, construct geometric frameworks to estimate the identifiability of each subsystem.

**Figure 10.** Confirmation of system (28) structural identifiability with cubic nonlinearity
**Remark 8** The friction introduction with cubic nonlinearity does not break the system (28) work. This conclusion is valid for the system (28) with a higher degree of friction the system (28) structural identifiability with \( \varphi_1(x_2) = \varphi_{1(4)}(x_2) = x_2^3 \) follows from Figure 10. It reflects the functional dependency for the system (28), where \( x_{2(3)}, x_{2(4)} \) are the change rotation rate of the object shaft angle with quadratic and cubic nonlinearity. The SI check is based on the analysis of the model properties \( \dot{x}_{2(4)} = 1.0472x_{2(3)} + 0.0026, r_{2(4)}^2 = 0.998 \) (see Figure 10). The system (28) with \( \varphi_1(x_2) = \varphi_{1(3)}(x_2) = x_2^2 \text{sign}(x_2) \) is SI. Therefore, the system (28) with \( \varphi_{1(4)}(x_2) \) based the analysis \( \dot{x}_{2(4)} = 1.0472x_{2(3)} + 0.0026 \) and Figure 10 is also structurally identifiable.

**Example 3** Consider the system for generating self-oscillations containing the object (variables \( y_1, y_2 \)), nonlinear (variable \( y_3 \)) and linear (variable \( y_4 \)) meters, and the linear amplifier-converter with the nonlinear actuator (variable \( y_5 \))

\[
\dot{Y} = AY + DF(Y)
\]

\[
Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & k_0 \\ 0 & 0 & \frac{-1}{T_1} & 0 & 0 \\ 0 & k_2 & \frac{1}{T_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{T_3} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F(Y) = \begin{bmatrix} f(y_1) \\ f_2(y_4) \end{bmatrix}
\]

The input of the system is the variable \( u = y_5, f_i(x) (i = 1, 3) \) is the saturation function with the dead zone

\[
f_i(x) = \begin{cases} c, & \text{if } x \geq d_{2,i}, \\ 2(x - d_{1,i}), & \text{if } -d_{1,i} < x < d_{2,i}, \\ 0, & \text{if } -d_{1,i} \leq x \leq d_{1,i}, \\ 2(x + d_{1,i}), & \text{if } -d_{1,i} < x, \\ -c, & \text{if } x < -d_{2,i}, \end{cases}
\]

Figure 11. Object phase portrait
The object phase portrait is shown in Figure 11. It confirms the availability of self-oscillations in the system.

Apply the approach described in section 4 and obtain the object linearity. Build the influence graph (IG) to perform further analysis. The graph is shown for the nonlinear meter (NM) with the output \( y_3 \) in Figure 12. The derivative of the variable \( y_i \) denoted as \( dy_i \) in Figure 12. We consider connections that exceed 75%. We have relations \( \dot{y}_4, y_1, y_5 \) for \( y_3 \), for \( \dot{y}_3 \) we have \( \dot{y}_5, y_2, y_4 \). Apply MHI and exclude variables that influence \( \dot{y}_3 \). We choose \( y_5 \) since \( y_2 \) has the indirect effect influence. Construct the framework \( S_{\dot{y}_3, y_5} \) described by the function \( y_{\dot{y}_3, y_5} : \dot{y}_5 \rightarrow \dot{y}_3 \), and determine the secant

\[
\dot{y}_3 = -1.241 \dot{y}_5 + 0.0094. \tag{35}
\]

Figure 12. Influence graph for the nonlinear meter

Introduce the misalignment \( \varepsilon_{\dot{y}_3} = \dot{y}_3 - \dot{y}_3 \) and construct the framework \( S_{\varepsilon_{\dot{y}_3}, y_1} \) described by the function \( y_{\varepsilon_{\dot{y}_3}, y_1} : y_1 \rightarrow \varepsilon_{\dot{y}_3} \) (see Figure 13).

Figure 13. Estimation of nonlinearity structure for NM

Figure 13 shows that NM contains the nonlinearity described by the saturation function with the dead zone. The dead zone width depends on properties \( y_2 \). The model (35) structure is based on the analysis of the influence graph for \( y_3 \). Variables \( y_i \in \varphi E_{y_i}, i = 1,3 \). Analysis of the framework \( S_{\varepsilon_{\dot{y}_3}, y_1} \) shows that it satisfies conditions of theorem 4. Therefore, the object and the nonlinear meter are structurally identifiable.
Next, we perform the analysis of the linear meter (LM) structure. $y_1, y_3$ influence $y_4$, and $y_2, y_3, y_5$ influence $y_4$. The most relationship is between $y_4$ and $y_3$. The secant structure $S_{y_4 y_3}$ has the form

$$\hat{y}_4 = -1.1737 y_3 + 0.0689, \ r^2_{\hat{y}_4 y_3} = 0.88.$$  \hspace{1cm} (36)

![Figure 14. Linearity estimation of LM](image)

The linear meter is $h$-identifiable. Perform the next MHI stage. Enter the residual $\epsilon_{y_3, \hat{y}_3} = y_3 - \hat{y}_3$ and estimate the influence of variables $y_2, y_5$. Obtain $r^2_{\epsilon_{y_3, \hat{y}_3} y_5} = 0.23, \ r^2_{\epsilon_{y_3, \hat{y}_3} y_2} = 0.73$. So, $\hat{y}_4$ depends on $y_2$ (see Figure 14) linearly.

**Remark 9** We consider indirect relationships reflected the influence of system previous elements.

Consider the linear amplifier-converter with the nonlinear actuator (LACNA). The influence graph gives for $y_5$ variables $y_2, y_3, y_4, y_5$. The LACNA phase portrait is shown in Figure 15. We see that the processes are nonlinear in LACUNA.

![Figure 15. LACUNA phase portrait](image)

**Remark 10** The SI estimation of the $S_{y_5 y_3}$ framework is based on the analysis of the values range. Consider frameworks $S_{y_5, y_3}, S_{y_5, y_2}$ and construct secants
\[ \dot{y}_{5,2} = -0.2593y_2 + 0.0281, \quad r_{\dot{y}_{5,2}}^2 = 0.90, \]  
(37)

\[ \dot{y}_{5,3} = -0.5019y_3 - 0.0239, \quad r_{\dot{y}_{5,3}}^2 = 0.77, \]  
(38)

to exclude the influence \( y_2 \) and \( \dot{y}_3 \). In Figure 16, frameworks \( S_{\dot{y}_{3,3}}, S_{\dot{y}_{3,2}} \) shown.

Frameworks analysis shows that the processes are nonlinear in LACUNA. The final decision cannot be made on the LACUNA nonlinearity.

Introduce residuals \( \epsilon_{5,3} = \dot{y}_5 - \dot{y}_{5,3}, \quad \epsilon_{5,2} = \dot{y}_5 - \dot{y}_{5,2} \) and apply MHI. The analysis shows that obtained results do not allow deciding on the LACUNA structure. Consider the \( S_{\dot{y}_{3,4}} \)-framework (see Figure 17). Figure 16 confirms that LACUNA contains the saturation element with the dead zone. But this representation contains some indistinctness. So, consider the framework \( S_{\dot{y}_{3,4}} \) and determine the secant \( \dot{y}_{5,\dot{y}_4} = -0.411\dot{y}_4 + 0.024 \) for it, introduce the misalignment \( \epsilon_{5,\dot{y}_4} = \dot{y}_5 - \dot{y}_{5,\dot{y}_4} \). Next, consider the framework \( S_{\dot{y}_{5,\dot{y}_4}} \) (see Figure 17). The \( S_{\dot{y}_{5,\dot{y}_4}} \)-framework gives the complete representation of the nonlinearity \( f_3(y_4) \) type (form). The framework \( S_{\dot{y}_{5,\dot{y}_4}} \) satisfies the conditions of theorem 4. Therefore, LACUNA is structurally identifiable.
Let the measurement information be known for the system (33)

\[ I_0 = \{ y_3(t), y_4(t), y_5(t), t \in [0, t_k] \}, \quad t_k < \infty. \] (39)

Construct the influence graph to estimate SI of the system (33). Graph components presented above.

We have \( \dot{y}_3 = \varphi(\dot{y}_3, \dot{y}_5) \), \( \dot{y}_4 = \varphi(\dot{y}_3, y_5) \), \( \dot{y}_5 = \varphi(\dot{y}_3, y_4) \), where \( \varphi \) is the connection symbol of graph elements. The SI estimate presented for \( y_5 \) in Figure 16 confirms the identifiability of the system at this level. The relationship \( y_5 = v(\dot{y}_3) \) allows estimating the dead zone of the function \( f_3(y_4) \). The analysis of \( y_5 \) relationship \( \dot{y}_4 = \varphi(\dot{y}_3, y_5) \) confirms the linearity of this subsystem. The framework \( S_{y_5, y_4} \) is shown in Figure 18, and confirms the existence of the saturation class with the variable dead zone.

\( S_{y_5, y_4} \) is structurally identifiable and confirms the above conclusion. The variable \( y_4 \) is a function of \( y_2 \) (see (33)). So, obtain the identifiability of the function \( f_1(y_2) \).

Denote the graph defined on \( I_o \) and contains connections in the system (33) as \( G \).

**Definition 14** The relationship between the variables \( (y_i, y_j) \in G \) of the system (33) is significant if the framework \( S_{y_i, y_j} \) ensures \( h \)-identifiability of the system (33) and corresponds to this relationship.

The subset of graph \( G \) significant for obtaining connections at the \( k \)-element analysis level is denoted as \( G_{k, k} \subset G \).

**Theorem 7** If the system (33) input \( y_5 \) is \( S \)-synchronizing, and frameworks \( S_{y_i, y_j} \) are defined on graph \( G_{k, k} \subset G \), and the conditions of theorem 4 are satisfied, then system (33) is \( h_\delta \)-identifiable.

The proof of theorem 7 follows from the condition \( y_5 \in S \) fulfillment and the subgraph \( G_{k, k} \) existence which the frameworks \( S_{y_i, y_j} \) had the \( d_\delta, \Sigma \)-optimality property.

Continue the LACNA analysis. Consider the framework \( S_{y_5, y_2, y_1} \), select fragments \( \bar{f}_{\delta}^{\ell}, f_{\delta}^{r} \), and approximate \( f_{\delta}^{\ell}, f_{\delta}^{r} \) with secants (7) on \( y_1 \). Have \( a^\ell = -0.485, a^r = -0.347 \). Let \( \delta_\delta = 0.04 \). Calculate \( ||a^\ell - |a^r| || = 0.081 \), apply theorem 1 and obtain \( S_{y_5, y_2, y_1} = \infty \). Next, apply theorem 7 and stop the hierarchical immersion method.

So, it shows the structural identifiability problem complexity for the system with multiple nonlinearities. Internal connections can influence on the SI problem solution. These connections can influence the subsystem for studying their identification indirectly. In this case, considering the relationships influence level has great importance. It is a new problem that appears in the SI study of systems with multiple nonlinearities. A priori information is critical in this case. The SM-model synthesis of the analogous model (5) depends on the influence graph.
7. Conclusion

Structural identifiability conditions for the nonlinear system are obtained. They are founded on the property analysis of geometric frameworks. The role of S-synchronizability and the constant excitation is shown in the structural identifiability analysis. Conditions of structural non-identifiability and structural indistinguishability are obtained. Systems with two nonlinearities are considered. Difficulties appearing in the analysis of structural identifiability are noted. The internal organization influence of system elements is noted on the structural identifiability possibility. The SI estimating problem on the set of available dimensions is considered. The influence graph is essential in the structural identifiability analysis. It allows selecting significant variables for SI analysis and synthesize the model for geometric framework design.

References


