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Research Article

Pure Subsemigroups of $\mathbb{N}_0^2(+)$

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Abstract: This paper discusses finitely generated and pure subsemigroups of the additive semigroup $\mathbb{N}_0^2(+)$ of ordered pairs of non-negative integers.

***Keywords*:** integer, semigroup, pure

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1. Introduction

Semigroups are one of the fundamental algebraic structures. A number of recent studies have investigated the structure of semigroups and their subsemigroups, e.g., [1, 2] and [3]. In particular, a special attention is paid to subsemigroups of $\mathbb{Z}^m$ and/or $\mathbb{N}^m$ (see e.g. [4] or [5]). The aim of this paper is to contribute to this effort.

This note continues our investigations started in [6] and presents certain findings concerning pure subsemigroups (cf. [7]) within the context of $\mathbb{N}_0^2(+)$. The case of pure subsemigroups of $\mathbb{N}_0^2(+)$, which is the focus of this paper, has remained an open problem. In this paper, a more detailed characterization for pure subsemigroups of $\mathbb{N}_0^2(+)$ has been achieved. In particular, the relationship between pure subsemigroups and finitely generated subsemigroups is found. Further research can be focused on other types of subsemigroups and, in particular, on their application, especially in the field of discrete mathematics and cryptography.

The impetus for exploring such inquiries arises from multiple areas of mathematics. The existence of holes in subsemigroups of $\mathbb{N}_0^m(+)$ (if A is a subsemigroup of $\mathbb{N}_0^m(+)$ then an element h of the pure subsemigroup generated by A is said to be a *hole* in A if $h \notin A$) was utilized in transportation problems, as mentioned in [8]. There is also a close connection with the theory of semirings. For example, the question of whether every commutative parasemifield (which is an algebraic structure with two commutative and associative binary operations such that the multiplication is a group and distributes over addition) is additively idempotent when finitely generated as a semiring was affirmatively answered in the 2-generated case in [9]. This was achieved by transferring the problem to subsemigroups of $\mathbb{N}_0^m(+)$ with special properties. Furthermore, it appears that the investigation of finitely generated cones could prove beneficial in the study of context-free languages (see e.g. [10] and [11]).

The following notation will be used throughout the note:

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$\mathbb{N}$ ... the semiring of positive integers;
$\mathbb{N}_0$ ... the semiring of non-negative integers;
$\mathbb{Z}$ ... the ring of integers;
$\mathbb{Q}^+$ ... the parasemifield of positive rationals;
$\mathbb{Q}_0^+$ ... the semifield of non-negative rationals;
$\mathbb{R}_0^+$ ... the semifield of non-negative reals.

Let $m \in \mathbb{N}$ and let A be a subset of $\mathbb{Q}^m$ (the ring of ordered m-tuples of rationals, where the operations of addition and multiplication are defined componentwise), where we denote $0 = (0, \ldots, 0) \in \mathbb{Q}^m$ (no confusion can arise). We put

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{n} q_i a_i \middle| n \in \mathbb{N},\ a_1, \ldots, a_n \in A,\ q_1, \ldots, q_n \in \mathbb{Q}_0^+,\ \sum_{i=1}^{n} q_i = 1 \right\}$$

and

$$\operatorname{cone}(A) = \left\{ \sum_{i=1}^{n} q_i a_i \middle| n \in \mathbb{N},\ a_1, \ldots, a_n \in A,\ q_1, \ldots, q_n \in \mathbb{Q}_0^+ \right\}.$$

Let A be a subset of $\mathbb{Q}^m$, $m \in \mathbb{N}$. We put

$$\alpha(A) = \{a \in \mathbb{Q}^m | (\exists b \in \mathbb{Q}^m)(\forall r,\ s \in \mathbb{Q}^+)\ ra + sb \notin \operatorname{cone}(A)\},$$

$\beta(A) = \alpha(A) \cap \operatorname{cone}(A)$, $\gamma(A) = \alpha(A) \cap \operatorname{conv}(A)$ and $\delta(A) = \alpha(A) \cap A$.

For $a = (a_1, \ldots, a_m) \in \mathbb{Q}^m$, put $\|a\| = \sqrt{\sum_{i=1}^{m} a_i^2} \in \mathbb{R}_0^+$.

Let A be a subsemigroup of $\mathbb{N}_0^m(+)$. We denote by $\varepsilon(A)$ the set of the elements $a \in A$ such that $b - a \in A$, whenever $b \in A$ and $b - a \in \mathbb{N}_0^m$. Equivalently, $\varepsilon(A) = \{a \in A | A + a = (\mathbb{N}_0^m + a) \cap A\}$. We shall say that A is a *pure subsemigroup* of $\mathbb{N}_0^m(+)$ if $nA = A \cap n\mathbb{N}_0^m$ for every $n \in \mathbb{N}$ (or, equivalently, if $a \in \mathbb{N}_0^m$ and $q \in \mathbb{Q}^+$ are such that $qa \in A$ then $a \in A$).

2. Subsemigroups of $\mathbb{N}_0^2(+)$ (a)

For $a,\ b \in \mathbb{Q}^2$, $a = (r_1,\ r_2)$ and $b = (s_1,\ s_2)$, put $e(a,\ b) = r_1 s_2 - r_2 s_1 \in \mathbb{Q}$. Furthermore, put $\mathbb{P} = (\mathbb{Q}_0^+)^2 \setminus \{0\}$ and define a relation $\dashv$ on $\mathbb{P}$ by $a \dashv b$ if and only if $e(a,\ b) \geq 0$.

Lemma 1 (i) The relation $\dashv$ is a quasiordering on $\mathbb{P}$ (i.e., the relation is both reflexive and transitive).

(ii) $a \dashv b \dashv a$ (i.e., $(a,\ b) \in \ker(\dashv)$) iff $e(a,\ b) = 0$ iff $\mathbb{Q}_0^+ a = \mathbb{Q}_0^+ b$.

Proof. (i) Clearly, the relation is reflexive. If $(r_1,\ r_2) \dashv (s_1,\ s_2) \dashv (t_1,\ t_2)$ then $r_2 s_1 \leq r_1 s_2$, $s_2 t_1 \leq s_1 t_2$, and hence $r_2 s_2 t_1 \leq r_2 s_1 t_2 \leq r_1 s_2 t_2$. If $s_2 \neq 0$ then $r_2 t_1 \leq r_1 t_2$ and $(r_1,\ r_2) \dashv (t_1,\ t_2)$. If $s_2 = 0$ then $r_2 s_1 \leq r_1 s_2 = 0$, and hence $r_2 = 0$ and $(r_1,\ r_2) \dashv (t_1,\ t_2)$ trivially, or $s_1 = 0$, a contradiction with $(s_1,\ s_2) \in \mathbb{P}$.

(ii) Let $a = (r_1,\ r_2)$ and $b = (s_1,\ s_2)$. Clearly, $a \dashv b \dashv a$ iff $r_2 s_1 = r_1 s_2$ (i.e., $e(a,\ b) = 0$).

Assume that $r_2 s_1 = r_1 s_2$. If $r_1 \neq 0 \neq s_2$ then $r_2 \neq 0 \neq s_1$, $r_1/s_1 = r_2/s_2 = q \in \mathbb{Q}^+$, $qb = a$ and $a/q = b$. Thus $\mathbb{Q}_0^+ a = \mathbb{Q}_0^+ b$. If $r_1 = 0$ then $s_1 = 0$, $r_2 \neq 0 \neq s_2$, $r_2/s_2 = q \in \mathbb{Q}^+$, $qb = a$, $a/q = b$ and $\mathbb{Q}_0^+ a = \mathbb{Q}_0^+ b$, too. Similarly, if $s_2 = 0$.

Conversely, assume that $\mathbb{Q}_0^+ a = \mathbb{Q}_0^+ b$. Then $a = qb$ for some $q \in \mathbb{Q}^+$, and so $qs_1 = r_1$ and $qs_2 = r_2$. Hence $r_1 s_2 - r_2 s_1 = qs_1 s_2 - qs_2 s_1 = 0$. □

Lemma 2 Let $a,\ b \in \mathbb{P}$. Then $a \dashv b$ or $b \dashv a$.

Proof. It is obvious. □

Lemma 3 Let $a,\ b \in \mathbb{P}$ be such that $a \dashv b$. Then $\mathrm{cone}(\{a,\ b\}) = \mathbb{Q}_0^+ a + \mathbb{Q}_0^+ b = \{0\} \cup \{c \in \mathbb{P} | a \dashv c \dashv b\}$.

Proof. If $a = (r_1,\ r_2)$ and $b = (s_1,\ s_2)$ then $r_2 s_1 \leq r_1 s_2$. Take $q_1,\ q_2 \in \mathbb{Q}_0^+$, $q_1 + q_2 \neq 0$, and put $c = q_1 a + q_2 b = (q_1 r_1 + q_2 s_1,\ q_1 r_2 + q_2 s_2)$. Then $r_2 s_1 \leq r_1 s_2$ implies $r_2(q_1 r_1 + q_2 s_1) = r_2 q_1 r_1 + q_2 r_2 s_1 \leq r_1 q_1 r_2 + q_2 r_1 s_2 = r_1(q_1 r_2 + q_2 s_2)$, and so $a \dashv c$. Similarly, $r_2 s_1 \leq r_1 s_2$ implies $s_1(q_1 r_2 + q_2 s_2) = q_1 r_2 s_1 + q_2 s_2 s_1 \leq q_1 r_1 s_2 + q_2 s_2 s_1 = s_2(q_1 r_1 + q_2 s_1)$, and so $c \dashv b$.

Conversely, let $c = (t_1,\ t_2) \in \mathbb{P}$ be such that $a \dashv c \dashv b$. Then $a \dashv b$, and hence all the three numbers $e(a,\ b)$, $e(c,\ b)$ and $e(a,\ c)$ belong to $\mathbb{Q}_0^+$. Furthermore, $e(c,\ b)a + e(a,\ c)b = (r_1 s_s t_1 - r_2 s_1 t_1,\ r_1 s_2 t_2 - r_2 s_1 t_2) = e(a,\ b)c$. If $e(a,\ b) \neq 0$ then $c = (e(c,\ b)/e(a,\ b))a + (e(a,\ c)/e(a,\ b))b \in \mathbb{Q}_0^+ a + \mathbb{Q}_0^+ b$. If $e(a,\ b) = 0$ then the equality $e(c,\ b)a + e(a,\ c)b = 0$ implies $e(a,\ b) = e(a,\ c) = e(b,\ c) = 0$, and hence $\mathbb{Q}_0^+ a = \mathbb{Q}_0^+ b = \mathbb{Q}_0^+ c$ by Lemma 1 (ii). Then $c \in \mathbb{Q}_0^+ a + \mathbb{Q}_0^+ b$ trivially. □

Lemma 4 Let $a_1,\ \ldots,\ a_n \in \mathbb{P}$, $n \in \mathbb{N}$, and $1 \leq j,\ k \leq n$ be such that $a_j \dashv a_i \dashv a_k$ for every $i = 1,\ \ldots,\ n$. Then $\mathrm{cone}(\{a_1,\ \ldots,\ a_n\}) = \mathrm{cone}(\{a_j,\ a_k\}) = \mathbb{Q}_0^+ a_j + \mathbb{Q}_0^+ a_k$.

Proof. Clearly, $a_i \in \mathrm{cone}(\{a_j,\ a_k\})$ for every i by Lemma 3. Thus $\mathrm{cone}(\{a_1,\ \ldots,\ a_n\}) \subseteq \mathrm{cone}(\{a_j,\ a_k\})$ and the rest is obvious. □

Example 5 Put $A_1 = \{(k+1,\ k) | k \in \mathbb{N}\} \subseteq \mathbb{N}^2$ and denote by A the subsemigroup of $\mathbb{N}^2(+)$ generated by the set A_1. We claim that $\mathrm{cone}(A) = \{0\} \cup \{(t_1,\ t_2) \in (\mathbb{Q}^+)^2 | 1 < t_1/t_2 \leq 2\}$.

First, $\mathrm{cone}(A) = \mathrm{cone}(A_1)$ and one observes easily that $(k+1,\ k) \dashv (l+1,\ l)$ iff $k \leq l$. Take $a \in \mathrm{cone}(A)$, $a \neq 0$. Then there are $n,\ k_1,\ \ldots,\ k_n \in \mathbb{N}$ such that $k_1 < \cdots < k_n$ (if $n \geq 2$) and $a \in \mathrm{cone}(\{(k_i+1,\ k_i) | 1 \leq i \leq n\})$. Using Lemma 4, we get $a \in \mathrm{cone}(\{(k_1+1,\ k_1),\ (k_n+1,\ k_n)\})$, and so $(t_1,\ t_2) = a = r(k_1+1,\ k_1) + s(k_n+1,\ k_n) = (rk_1 + sk_n + r + s,\ rk_1 + sk_n)$ for some $r,\ s \in \mathbb{Q}_0^+$, $r + s \neq 0$. Then, of course, $t_1/t_2 = 1 + (r+s)/(rk_1 + sk_n)$, and hence $1 < t_1/t_2 \leq 2$.

Conversely, let $(t_1/t_2) \in (\mathbb{Q}_0^+)^2$ be such that $1 < t_1/t_2 \leq 2$. Then $1/2 \leq t_2/t_1 < 1$ and there is $l \in \mathbb{N}$ with $l \geq 2$ and $t_2/t_1 \leq l/(l+1)$. Put $r = (lt_1 - (l+1)t_2)/(l-1)$ and $s = (2t_2 - t_1)/(l-1)$, We have $r,\ s \in \mathbb{Q}_0^+$ and $r(2,\ 1) + s(l+1,\ l) = (i/(l-1),\ j/(l-1))$, where $i = 2r + (l+1)s = 2lt_1 - 2lt_2 - 2t_2 + 2lt_2 + 2t_2 - lt_1 - t_1 = (l-1)t_1$ and $j = r + ls = lt_1 - lt_2 - t_2 + 2lt_2 - lt_1 = (l-1)t_2$. From this, $r(2,\ 1) + s(l+1,\ l) = (t_1,\ t_2)$ and we have checked that $(t_1,\ t_2) \in \mathrm{cone}(\{(2,\ 1),\ (l+1,\ l)\}) \subseteq \mathrm{cone}(A_1) = \mathrm{cone}(A)$.

We have proved our claim, i.e., $\mathrm{cone}(A) = \{0\} \cup \{(t_1,\ t_2) \in (\mathbb{Q}_0^+)^2 | 1 < t_1/t_2 \leq 2\}$ and we are going to show that A is not a finitely generated semigroup. Taking into account [6], Theorem 5.4 and the preceding part, it is enough to check that $(k+2,\ k+1) \notin \mathrm{cone}(\{(2,\ 1),\ (k+1,\ k)\})$ for every $k \in \mathbb{N}$.

Let, on the contrary, $(k+2,\ k+1) = r(2,\ 1) + s(k+1,\ k)$ for some $r,\ s \in \mathbb{Q}_0^+$. Then $2r + ks + s = k+2$, $r + ks = k+1$, and hence $r + s = 2r + ks + s - r - ks = k + 2 - k - 1 = 1$, $(k-1)s + 1 = (k-1)s + r + s = r + ks = k+1$, $(k-1)s = k$, $s \neq 0$, $k \geq 2$, $s = k/(k-1)$ and, finally, $r = 1 - s = -1/(k-1) \notin \mathbb{Q}_0^+$, a contradiction.

Example 6 Put $B_1 = \{(k+1,\ k) | k \in \mathbb{N}_0\}$ $(= A_1 \cup \{(1,\ 0)\}$; see 5) and denote by B the subsemigroup of $\mathbb{N}_0^2(+)$ generated by the set B_1. Then $\mathrm{cone}(B) = \mathrm{cone}(B_1) = \{(q,\ 0) | q \in \mathbb{Q}_0^+\} \cup \{(t_1,\ t_2) \in (\mathbb{Q}_0^+)^2 | t_2 < t_1\}$ (use 5). Proceeding similarly as in Example 5, we can check that B is not a finitely generated semigroup (it is enough to show that $(k+2,\ k+1) \notin \mathrm{cone}(\{(1,\ 0),\ (k+1,\ k)\})$ for every $k \in \mathbb{N}$).

Example 7 Put $C_1 = \{(k+1,\ k),\ (k,\ k) | k \in \mathbb{N}\}$ and denote by C the subsemigroup of $\mathbb{N}^2(+)$ generated by the set C_1. Then $\mathrm{cone}(C) = \mathrm{cone}(C_1) = \{0\} \cup \{(t_1,\ t_2) \in (\mathbb{Q}^+)^2 | 1 \leq t_1/t_2 \leq 2\}$ (use Example 5). On the other hand, $(t_1,\ t_2) = (2t_2 - t_1)(1,\ 1) + (t_1 - t_2)(2,\ 1)$ and it follows that $\mathrm{cone}(C) = \mathrm{cone}(\{(1,\ 1),\ (2,\ 1)\})$. By [6], Theorem 5.4, C is a finitely generated semigroup. Notice that $C = A + \langle(1,\ 1)\rangle = \langle A \cup \{(1,\ 1)\}\rangle$.

3. Subsemigroups of $\mathbb{N}_0^2(+)$ (b)

This section is an immediate continuation of the preceding one. Define a relation $\multimap$ on $\mathbb{Q}^2$ by $a \multimap b$ if and only if $e(a,\ b) > 0$. Clearly, if $a \multimap b$ then $a,\ b \in \mathbb{P}$.

Lemma 8 For all $a,\ b \in \mathbb{P}$, just one of the following three cases takes place:

1. $a \multimap b$;

2. $b \multimap a$;

3. $a \dashv b \dashv a$.

Proof. It is obvious. □

Lemma 9 (i) The relation $\multimap$ is a sharp ordering defined on $\mathbb{P}$ (i.e., the relation is both irreflexive and transitive).

(ii) The relation $\multimap \cup \mathrm{id}_{\mathbb{P}}$ is an ordering defined on $\mathbb{P}$ (i.e., the relation is reflexive, transitive and antisymmetric).

Proof. (i) Since $e(a,\ a) = 0$, the relation $\multimap$ is irreflexive. If $a \multimap b \multimap c$ then $a \multimap c$, since the relation $\dashv$ is transitive and $c \dashv a$ would yield $c \dashv b$.

(ii) This follows from (i). □

Lemma 10 The following conditions are equivalent for $a,\ b \in \mathbb{P}$:

(i) $a \neq b$ and the two-element set $\{a,\ b\}$ is a basis of the vector $\mathbb{Q}$-space $\mathbb{Q}^2$.

(ii) Either $a \multimap b$ or $b \multimap a$.

(iii) $a \notin \mathbb{Q}_0^+ b$ and/or $b \notin \mathbb{Q}_0^+ a$.

Proof. Clearly, if $a = (r_1,\ r_2)$ and $b = (s_1,\ s_2)$ then $\{a,\ b\}$ is a basis iff $e(a,\ b) = \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix} \neq 0$. The rest follows from Lemma 1. □

Lemma 11 (i) If $a \multimap b$ then $qa \multimap qb$ for every $q \in \mathbb{Q}^+$.

(ii) If $a \multimap c \multimap b$ then $(a+c) \multimap (b+c)$.

Proof. It is easy. □

Lemma 12 Let $a,\ b \in \mathbb{P}$ be such that $a \multimap c \multimap b$ and let $r = e(c,\ b)/e(a,\ b)$ and $s = e(a,\ c)/e(a,\ b)$. Then $r,\ s \in \mathbb{Q}^+$, $c = ra + sb$ and $c \in \mathrm{conv}(\{(r+s)a,\ (r+s)b\})$.

Proof. Since $a \multimap c \multimap b$, we have $a \multimap b$ and $e(a,\ c),\ e(c,\ b),\ e(a,\ b) \in \mathbb{Q}^+$. Then $r,\ s \in \mathbb{Q}^+$, too. Now, if $a = (r_1,\ r_2)$, $b = (s_1,\ s_2)$ and $c = (t_1,\ t_2)$ then $r = (t_1s_2 - t_2s_1)/(r_1s_2 - r_2s_1)$, $s = (r_1t_2 - r_2t_1)/(r_1s_2 - r_2s_1)$ and $ra + sb = (((t_1s_2 - t_2s_1)r_1 + (r_1t_2 - r_2t_1)s_1)/(r_1s_2 - r_2s_1),\ ((t_1s_2 - t_2s_1)r_2 + (r_1t_2 - r_2t_1)s_2)/(r_1s_2 - r_2s_1)) = ((r_1s_2t_1 - r_1s_1t_2 + r_1s_1t_2 - r_2s_1t_1)/(r_1s_2 - r_2s_1),\ (r_2s_2t_1 - r_2s_1t_2 + r_1s_2t_2 - r_2s_2t_1)/(r_1s_2 - r_2s_1)) = ((r_1s_2 - r_2s_1)t_1/(r_1s_2 - r_2s_1),\ (r_1s_2 - r_2s_1)t_2/(r_1s_2 - r_2s_1)) = (t_1,\ t_2) = c$. Finally, $c = (r/(r+s))(r+s)a + (s/(r+s))(r+s)b$ and $r/(r+s) + s/(r+s) = 1$. Thus $c \in \mathrm{conv}(\{(r+s)a,\ (r+s)b\})$. □

Lemma 13 Let $a,\ b,\ c \in \mathbb{P}$ be such that $a \multimap c \multimap b$. Then there is $t \in \mathbb{Q}^+$ such that $c + d \in \mathrm{cone}(\{a,\ b\})$ for every $d \in \mathbb{Q}^2$ with $\|d\| < t$.

Proof. We have $a \multimap b$ and the set $\{a,\ b\}$ is a basis of $\mathbb{Q}^2$ by Lemma 10. By Lemma 12, there are $r,\ s \in \mathbb{Q}^+$ with $c = ra + sb$. Now, the result follows from [6], Lemma 2.3, where $m = 2$, $a_1 = a$ and $a_2 = b$. □

Example 14 Put $a = (1,\ 0)$ and $b = (0,\ 1)$. Then $a \multimap b$, $\mathrm{conv}(\{a,\ b\}) = \{(r,\ s) \in (\mathbb{Q}_0^+)^2 | r + s = 1\}$ and $\mathrm{cone}(\{a,\ b\}) = (\mathbb{Q}_0^+)^2$. Moreover, $\alpha(\{a,\ b\}) = \mathbb{Q}^2 \setminus (\mathbb{Q}^+)^2$, $\beta(\{a,\ b\}) = (\mathbb{Q}_0^+)^2 \setminus (\mathbb{Q}^+)^2$ and $\gamma(\{a,\ b\}) = \delta(\{a,\ b\}) = \{a,\ b\}$.

Example 15 Put $A = \mathbb{N}_0^2$ and $B = \mathbb{N}_0^2 \setminus \{0\}$. Then both A and B are finitely generated subsemigroups of $\mathbb{N}_0^2(+)$, $\mathrm{conv}(B) = \{(q_1,\ q_2) \in (\mathbb{Q}_0^+)^2 | q_1 + q_2 \geq 1\}$, $\mathrm{conv}(A) = \mathrm{conv}(B) \cup \{0\}$ and $\mathrm{cone}(A) = \mathrm{cone}(B) = (\mathbb{Q}_0^+)^2$. Moreover, $\alpha(A) = \alpha(B) = \mathbb{Q}^2 \setminus (\mathbb{Q}^+)^2$, $\beta(A) = \beta(B) = (\mathbb{Q}_0^+)^2 \setminus (\mathbb{Q}^+)^2 = \mathbb{Q}_0^+(1,\ 0) \cup \mathbb{Q}_0^+(0,\ 1)$, $\gamma(B) = \{(q,\ 0),\ (0,\ q) | q \in \mathbb{Q}^+,\ q \geq 1\}$, $\gamma(A) = \gamma(B) \cup \{0\}$, $\delta(B) = \{(n,\ 0),\ (0,\ n) | n \in \mathbb{N}\}$ and $\delta(A) = \delta(B) \cup \{0\}$. Finally, $\varepsilon(A) = A$ and $\varepsilon(B) = \emptyset$.

Example 16 Consider the subsemigroup A of $\mathbb{N}^2(+)$ from 5. We claim that $\beta(A) = \{(2q,\ q) | q \in \mathbb{Q}_0^+\} = \mathbb{Q}_0^+(2,\ 1)$.

First, let $(t_1,\ t_2) \in (\mathbb{Q}^+)^2$ be such that $t_2 < t_1 < 2t_1$ (see 5). If $(q_1,\ q_2) \in \mathbb{Q}^2$ then there is $r \in \mathbb{N}$ with $q_2 - q_1 < r(t_1 - t_2)$ and $q_1 - 2q_2 < r(2t_2 - t_1)$. Now, $r(t_1,\ t_2) + (q_1,\ q_2) = (rt_1 + q_1,\ rt_2 + q_2) \in \mathrm{cone}(A)$ and it means that $(t_1,\ t_2) \notin \alpha(A)$. Conversely, if $q \in \mathbb{Q}_0^+$ then $r(2q,\ q) + s(1,\ 0) \notin \mathrm{cone}(A)$ for all $r,\ s \in \mathbb{Q}^+$. Notice that $\beta(A \cup \{0\}) = \beta(A)$.

Lemma 17 Let $a,\ b \in \mathbb{P}$ be such that $a \multimap b$. Then $\beta(\{a,\ b\}) \subseteq \mathbb{Q}_0^+ a \cup \mathbb{Q}_0^+ b$.

Proof. By Lemma 10, the elements $a,\ b$ are linearly independent and we can use [6], Lemma 3.11. □

Lemma 18 Let $a,\ b \in \mathbb{P}$, $a = (r_1,\ r_2)$, $b = (s_1,\ s_2)$ be such that $a \multimap b$. Then:

(i) $r_1 b - s_1 a = (0,\ e(a,\ b)) \in \mathbb{P} \setminus \mathbb{Q}_0^+ a$.

(ii) $s_2 a - r_2 b = (e(a,\ b),\ 0) \in \mathbb{P} \setminus \mathbb{Q}_0^+ b$.

Proof. It is easy (use 1 (ii)). □

4. Subsemigroups of $\mathbb{N}_0^2(+)$ (c)

In this section, let A be a subsemigroup of $\mathbb{N}_0^2(+)$ with $0 \in A$.

Lemma 19 Let $a,\ b,\ c \in A$ be such that $a \multimap c \multimap b$. Then, for every $d \in \mathbb{Q}^2$, there are $n_1,\ n_2,\ n_3 \in \mathbb{N}$ such that $n_1c + d \in \mathrm{cone}(A)$ and $n_2c + n_3d \in \mathrm{conv}(A)$.

Proof. Clearly, $a,\ b,\ c \in \mathbb{P}$. By Lemma 13, there is $t \in \mathbb{Q}^+$ such that $c + e \in \mathrm{cone}(\{a,\ b\})$ for every $e \in \mathbb{Q}^2$ with $\|e\| < t$. Now $\|d/n_1\| < t$ for some $n_1 \in \mathbb{N}$, and therefore $c + d/n_1 \in \mathrm{cone}(\{a,\ b\}) \subseteq \mathrm{cone}(A)$. Finally, $n_1c + d \in \mathrm{cone}(A)$ and, by [6], Lemma 1.10 (i), $kn_1c + kd \in \mathrm{conv}(A)$ for some $k \in \mathbb{N}$ (notice that $0 \in \mathrm{conv}(A)$). □

Lemma 20 Let $a,\ b,\ c \in A$ be such that $a \multimap c \multimap b$. Then $c \notin \alpha(A)$ (and hence $c \notin \delta(A)$).

Proof. See Lemma 19. □

Lemma 21 $\delta(A) \subseteq \mathbb{Q}_0^+a \cup \mathbb{Q}_0^+b$ for some $a,\ b \in \delta(A)$.

Proof. If $\delta(A) \subseteq \{0\}$ then we put $a = 0 = b$. Consequently, take $a_1 \in \delta(A)$, $a_1 \neq 0$. If $\delta(A) \subseteq \mathbb{Q}_0^+a_1$ then we put $a = a_1$ and $b = 0$ (notice that $0 \in \delta(A)$ by [6], Lemma 3.3). Consequently, take $a_2 \in \delta(A) \setminus \mathbb{Q}_0^+a_1$. Then $a_2 \neq 0$, $a_2 \notin \mathbb{Q}_0^+a_1$ and $a_1 \notin \mathbb{Q}_0^+a_2$. If $\delta(A) \subseteq \mathbb{Q}_0^+a_1 \cup \mathbb{Q}_0^+a_2$ then we put $a = a_1$ and $b = a_2$. Consequently, let $a_3 \in \delta(A) \setminus (\mathbb{Q}_0^+a_1 \cup \mathbb{Q}_0^+a_2)$. Again, $a_3 \neq 0$, $a_3 \notin \mathbb{Q}_0^+a_1$, $a_3 \notin \mathbb{Q}_0^+a_2$, $a_1 \notin \mathbb{Q}_0^+a_3$ and $a_2 \notin \mathbb{Q}_0^+a_3$. We have found three (non-zero) elements $a_1,\ a_2,\ a_3$ in $\delta(A)$ such that $a_i \notin \mathbb{Q}_0^+a_j$ for $i \neq j$, $1 \leq i,\ j \leq 3$. Taking into account Lemma 2, we can assume without loss of generality that $a_1 \dashv a_2 \dashv a_3$. Then, in fact, $a_1 \multimap a_2 \multimap a_3$ (use Lemma 1 (ii)), a contradiction with Lemma 20. □

Lemma 22 If A is not a finitely generated semigroup then $\delta(A) \subseteq \mathbb{Q}_0^+a$ for some $a \in \delta(A)$.

Proof. Suppose, on the contrary, that $\delta(A) \nsubseteq \mathbb{Q}_0^+a$ for every $a \in \delta(A)$. Then there are $a_1,\ a_2 \in \delta(A)$ such that $a_1 \notin \mathbb{Q}_0^+a_2$, $a_2 \notin \mathbb{Q}_0^+a_1$ and $a_1 \multimap a_2$ (see the proof of Lemma 21). Clearly, $\mathrm{cone}(\{a_1,\ a_2\}) \subseteq \mathrm{cone}(A)$. On the other hand, if $b \in \mathrm{cone}(A) \setminus \mathrm{cone}(\{a_1,\ a_2\})$ then we conclude easily that either $b \multimap a_1$ or $a_2 \multimap b$ (use Lemmas 1, 2 and 3). In both cases, we come by a contradiction with Lemma 19. Thus $\mathrm{cone}(A) = \mathrm{cone}(\{a_1,\ a_2\})$ and A is finitely generated by [6], Theorem 5.4, the final contradiction. □

Lemma 23 $0 \in \varepsilon(A)$ and $\varepsilon(A)$ is a finitely generated subsemigroup of $\mathbb{N}_0^2(+)$.

Proof. $0 \in \varepsilon(A)$ and $\varepsilon(A)$ is a subsemigroup of $\mathbb{N}_0^2(+)$ by [6], Lemma 4.1 and [6], Lemma 4.2. If $\varepsilon(A) \subseteq \mathbb{Q}_0^+a$ for some $a \in \varepsilon(A)$ then $\mathrm{cone}(\varepsilon(A)) = \mathrm{cone}(\{a\}) = \mathbb{Q}_0^+a$ and $\varepsilon(A)$ is finitely generated by [6], Theorem 5.4. Henceforth, assume that $\varepsilon(A) \nsubseteq \mathbb{Q}_0^+a$ for every $a \in \varepsilon(A)$. Proceeding similarly as in the proof of Lemma 21, we find $a,\ b \in \varepsilon(A)$ such that $a \multimap b$. We have $a = (r_1,\ r_2) \neq 0$, $b = (s_1,\ s_2) \neq 0$, $r_i,\ s_i \in \mathbb{N}_0$, $e(a,\ b) = r_1s_2 - r_2s_1 \in \mathbb{N}$ and $0 \neq s_2a - r_2b = (e(a,\ b),\ 0) \in \mathbb{N}_0^2$, $0 \neq r_1b - s_1a = (0,\ e(a,\ b)) \in \mathbb{N}_0^2$. By [6], Lemma 4.3, $s_2a - r_2b \in \varepsilon(A)$ and $r_1b - s_1a \in \varepsilon(A)$. However, $\mathrm{cone}(\varepsilon(A)) \subseteq (\mathbb{Q}_0^+)^2 = \mathrm{cone}(\{(e(a,\ b),\ 0),\ (0,\ e(a,\ b)\}) \subseteq \mathrm{cone}(\varepsilon(A))$, and therefore $\mathrm{cone}(\varepsilon(A)) = (\mathbb{Q}_0^+)^2$ and $\varepsilon(A)$ is finitely generated by [6], Theorem 5.4. □

Corollary 24 If $\varepsilon(A) = A$ then A is a finitely generated semigroup. □

Proposition 25 (i) $0 \in \delta(A) \subseteq \beta(A) = \mathbb{Q}_0^+a \cup \mathbb{Q}_0^+b$ for some $a,\ b \in \delta(A)$.

(ii) If A is not finitely generated then $\delta(A) \subseteq \beta(A) = \mathbb{Q}_0^+c$ for some $c \in \delta(A)$.

(iii) $0 \in \varepsilon(A)$ and $\varepsilon(A)$ is a finitely generated subsemigroup of A.

(iv) $\varepsilon(\varepsilon(A)) = \varepsilon(A)$ and $(\varepsilon(A) - \varepsilon(A)) \cap \mathbb{N}_0^2 = \varepsilon(A)$.

(v) Either $\varepsilon(A) \subseteq \mathbb{Q}_0^+e = \mathrm{cone}(\varepsilon(A))$ for some $e \in \varepsilon(A)$ or $\mathrm{cone}(\varepsilon(A)) = (\mathbb{Q}_0^+)^2 = \mathrm{cone}(A)$ and A is finitely generated.

(vi) If $\varepsilon(A) = A$ then $A + a = \{b \in A | b - a \in \mathbb{N}_0^2\}$ for every $a \in A$.

(vii) If A is not finitely generated then $\varepsilon(A) \neq A$.

Proof. (i) By 21, $\delta(A) \subseteq \mathbb{Q}_0^+a \cup \mathbb{Q}_0^+b$ for some $a,\ b \in \delta(A)$. Using [6], Lemma 3.8, we get $\beta(A) \subseteq \mathbb{Q}_0^+a \cup \mathbb{Q}_0^+b$. The converse inclusion follows from [6], Lemma 3.2 (i) (and [6], Lemma 3.3).

(ii) Using Lemma 22, we proceed similarly as in (i).

(iii) See Lemma 23.

(iv) See [6], Lemma 4.3.

(v) See Lemma 23 and its proof.

(vi) It is obvious.

(vii) Use (iii). □

5. Subsemigroups of $\mathbb{N}_0^2(+)$ (d)

In this section, let A be a subsemigroup of $\mathbb{N}_0^2(+)$ such that $0 \in A$ and $\varepsilon(A) \neq 0$. By 25 (iii), (iv), we have $0 \in \varepsilon(A)$, $\varepsilon(A)$ is a finitely generated subsemigroup of A and $\varepsilon(\varepsilon(A)) = \varepsilon(A)$. According to 25(v), either $\varepsilon(A) \subseteq \mathbb{Q}_0^+ e = \text{cone}(\varepsilon(A))$ for some $e \in \varepsilon(A)$, $e \neq 0$, or $\text{cone}(\varepsilon(A)) = (\mathbb{Q}_0^+)^2 = \text{cone}(A)$ and A is finitely generated.

Lemma 26 Assume that $\text{cone}(\varepsilon(A)) = (\mathbb{Q}_0^+)^2$. Then, for every $a \in (\mathbb{Q}_0^+)^2$, there is $k \in \mathbb{N}$ with $ka \in \varepsilon(A)$.

Proof. Use [6], Lemma 1.10 (i). □

Lemma 27 Let $a,\ b \in A \setminus \{0\}$ be such that $a \multimap b$ and $a \in \varepsilon(A)$. Then:

(i) $(0,\ e(a,\ b)) \in A \setminus \{0\}$ and $(0,\ e(a,\ b)) \notin \mathbb{Q}_0^+ a$.

(ii) $(0,\ 1) \in \beta(A) \setminus \mathbb{Q}_0^+ a$ and $\mathbb{Q}_0^+(0,\ 1) \subseteq \beta(A) \subseteq (Q_0^+)^2$.

(iii) If A is not finitely generated then $\beta(A) = \mathbb{Q}_0^+(0,\ 1)$.

Proof. (i) Let $a = (r_1,\ r_2)$ and $b = (s_1,\ s_2)$. Then $e(a,\ b) = r_1 s_2 - r_2 s_1 > 0$, and consequently $r_1 \neq 0 \neq s_2$. Furthermore, by Lemma 18 (i), $r_1 b - s_1 a = (0,\ e(a,\ b)) \in \mathbb{N}_0^2$. Since $a \in \varepsilon(A)$, we have $s_1 a \in \varepsilon(A)$ (see [6], Lemma 4.2), and hence $(0,\ e(a,\ b)) \in A$. Since $a \multimap b$, we have $b \notin \mathbb{Q}_0^+ a$, and therefore $(0,\ e(a,\ b)) \notin \mathbb{Q}_0^+ a$.

(ii) Using (i), we get $(0,\ 1) \in \text{cone}(A)$ and $(0,\ 1) \notin \mathbb{Q}_0^+ a$. By [6], Lemma 3.10, $(0,\ 1) \in \alpha(A)$, and so $(0,\ 1) \in \beta(A)$. Then $\mathbb{Q}_0^+(0,\ 1) \subseteq \beta(A) \subseteq (\mathbb{Q}_0^+)^2$.

(iii) By Proposition 25 (ii), $\beta(A) = \mathbb{Q}_0^+ c$ for some $c \in \delta(A)$. Now, the equality $\beta(A) = \mathbb{Q}_0^+(0,\ 1)$ easily follows. □

Lemma 28 Let $a,\ b \in A \setminus \{0\}$ be such that $a \multimap b$ and $b \in \varepsilon(A)$. Then:

(i) $(e(a,\ b),\ 0) \in A \setminus \{0\}$ and $(e(a,\ b),\ 0) \notin \mathbb{Q}_0^+ b$.

(ii) $(1,\ 0) \in \beta(A) \setminus \mathbb{Q}_0^+ b$ and $\mathbb{Q}_0^+(1,\ 0) \subseteq \beta(A) \subseteq (\mathbb{Q}_0^+)^2$.

(iii) If A is not finitely generated then $\beta(A) = \mathbb{Q}_0^+(1,\ 0)$.

Proof. Symmetric to Lemma 27 (use Lemma 18 (ii)). □

In the remaining part of this section, assume that $\varepsilon(A) \subseteq \mathbb{Q}_0^+ e = \text{cone}(\varepsilon(A))$ for some $e \in \varepsilon(A)$, $e \neq 0$ (e.g., if A is not finitely generated).

Lemma 29 (i) $d \dashv e \dashv d$ and $\mathbb{Q}_0^+ d = \mathbb{Q}_0^+ e$ for every $d \in \varepsilon(A)$, $d \neq 0$.

(ii) If $a,\ b \in \varepsilon(A)$ then $e(a,\ b) = 0$.

Proof. We have $d = qe$ for some $q \in \mathbb{Q}^+$ and the rest is clear (see Lemma 1 (ii)). □

Lemma 30 Let $a,\ b \in \varepsilon(A)$. Then either $a - b \in \varepsilon(A)$ or $b - a \in \varepsilon(A)$ (i.e., the semigroup $\varepsilon(A)$ is semisubtractive).

Proof. We can assume that $(r_1,\ r_2) = a \neq 0 \neq b = (s_1,\ s_2)$ and $r_1 \geq s_1$. By Lemma 29 (ii), $r_1 s_2 = r_2 s_1$. If $r_1 = 0$ then $r_2 \neq 0$, and so $s_1 = 0$ and either $a - b \in \mathbb{N}_0^2$ or $b - a \in \mathbb{N}_0^2$. But $\varepsilon(\varepsilon(A)) = \varepsilon(A)$, and hence either $a - b \in \varepsilon(A)$ or $b - a \in \varepsilon(A)$. If $r_1 \neq 0$ and $r_2 < s_2$ then $s_2 \neq 0$ and $s_1 r_2 \leq r_1 r_2 < r_1 s_2$, a contradiction. Thus $s_2 \leq r_2$, $a - b \in \mathbb{N}_0^2$ and $a - b \in \varepsilon(A)$. □

Lemma 31 $\varepsilon(A) - \varepsilon(A) = \{a,\ -a | a \in \varepsilon(A)\}$.

Proof. The assertion follows immediately from Lemma 30. □

Lemma 32 The following conditions are equivalent:

(i) $(0,\ s) \in \varepsilon(A)$ for some $s \in \mathbb{N}$.

(ii) There is $k \in \mathbb{N}$ such that $\varepsilon(A) = \{(0,\ nk) | n \in \mathbb{N}_0\}$.

Proof. It is enough to show that (i) implies (ii). Now, let $k \in \mathbb{N}$ be the smallest number with $(0,\ k) \in \varepsilon(A)$. Let $a = (r_1,\ r_2) \in \varepsilon(A)$, $a \neq 0$. By Lemma 29 (ii), $kr_1 = 0$. Hence $r_1 = 0$, $r_2 \neq 0$, $k \leq r_2$ and $r_2 = nk + l$ for some $n \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$. Furthermore, by Lemma 30, $(0,\ l) = a - n(0,\ k) \in \varepsilon(A)$, and therefore $l = 0$ and $a = (0,\ nk)$. □

Lemma 33 The following conditions are equivalent:

(i) $(r,\ 0) \in \varepsilon(A)$ for some $r \in \mathbb{N}$.

(ii) There is $l \in \mathbb{N}$ such that $\varepsilon(A) = \{(nl,\ 0) | n \in \mathbb{N}_0\}$.

Proof. Symmetric to Lemma 32. □

Lemma 34 The following conditions are equivalent:

(i) $(r,\ s) \in \varepsilon(A)$ for some $r,\ s \in \mathbb{N}$.

(ii) There are $k,\ l \in \mathbb{N}$ such that $\varepsilon(A) = \{(nk,\ nl) | n \in \mathbb{N}_0\}$.

Proof. Again, we show that (i) implies (ii). Let $k,\ l \in \mathbb{N}$ be such that $(k,\ l) \in \varepsilon(A)$ and the sum $k+l$ is minimal. Now, if $(r,\ s) \in \varepsilon(A) \setminus \{0\}$ then $r \neq 0 \neq s$ follows from Lemmas 32 and 33, and we have $k+l \leq r+s$. By Lemma 29, $(r-k,\ s-l) = (r,\ s) - (k,\ l) \in \varepsilon(A)$ and $(r-k,\ s-l) = (n_1 k,\ n_1 l)$ for some $n_1 \in \mathbb{N}_0$ by induction. Thus $(r,\ s) = ((n_1+1)k,\ (n_1+1)l)$. □

Theorem 35 Just one of the following three cases takes place:

(i) There is $k \in \mathbb{N}$ such that $\varepsilon(A) = \mathbb{N}_0(0,\ k)$ and $\mathrm{cone}(\varepsilon(A)) = \mathbb{Q}_0^+(0,\ k)$.

(ii) There is $l \in \mathbb{N}$ such that $\varepsilon(A) = \mathbb{N}_0(l,\ 0)$ and $\mathrm{cone}(\varepsilon(A)) = \mathbb{Q}_0^+(l,\ 0)$.

(iii) There are $r,\ s \in \mathbb{N}$ such that $\varepsilon(A) = \mathbb{N}_0(r,\ s)$ and $\mathrm{cone}(\varepsilon(A)) = \mathbb{Q}_0^+(r,\ s)$.

Proof. Combine Lemmas 32, 33 and 34. □

Corollary 36 $\varepsilon(A)$ is a two-generated semigroup and $\varepsilon(A) \setminus \{0\}$ is a one-generated semigroup. □

Lemma 37 Assume that 35 (i) takes place. Then $(0,\ 1) \in \beta(A)$. Furthermore, if $a = (r_1,\ r_2) \in A \setminus \{0\}$ then:

(i) Either $r_1 = 0$ and $(0,\ k) \dashv a \dashv (0,\ k)$ or $r_1 \neq 0$ and $a \multimap (0,\ k)$.

(ii) If $r_1 \neq 0$ then $(r_1 k,\ 0) \in A \setminus \{0\}$ and $(1,\ 0) \in \beta(A)$.

Proof. Since $(0,\ k) \in \varepsilon(A) \subseteq A$, we get $(0,\ 1) \in \mathrm{cone}(A)$ and then $(0,\ 1) \in \beta(A)$ by [6], Lemma 3.10. Clearly, $r_1 = 0$ iff $(0,\ k) \dashv a \dashv (0,\ k)$. If $r_1 \neq 0$ then $(r_1 k,\ 0) \in A$, $(1,\ 0) \in \mathrm{cone}(A)$ and $(1,\ 0) \in \beta(A)$ by [6], Lemma 3.10 again. □

Lemma 38 Assume that Theorem 35 (ii) takes place. Then $(1,\ 0) \in \beta(A)$. Furthermore, if $a = (r_1,\ r_2) \in A \setminus \{0\}$ then:

(i) Either $r_2 = 0$ and $(l,\ 0) \dashv a \dashv (l,\ 0)$ or $r_2 \neq 0$ and $(l,\ 0) \multimap a$.

(ii) If $r_2 \neq 0$ then $(0,\ r_2 l) \in A \setminus \{0\}$ and $(0,\ 1) \in \beta(A)$.

Proof. Symmetric to Lemma 37.

Proposition 39(i) If 35 (i) is true then A is finitely generated and either $A \subseteq \mathbb{N}_0(0,\ 1)$ and $\mathrm{cone}(A) = \mathbb{Q}_0^+(0,\ 1)$ or $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$.

(ii) If 35 (ii) is true then A is finitely generated and either $A \subseteq \mathbb{N}_0(1,\ 0)$ and $\mathrm{cone}(A) = \mathbb{Q}_0^+(1,\ 0)$ or $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$.

Proof. (i) We have $(0,\ 1) \in \mathrm{cone}(A)$ by Lemma 37, and hence $\mathbb{Q}_0^+(0,\ 1) \subseteq \mathrm{cone}(A)$. If $A \subseteq \mathbb{N}_0(0,\ 1)$ then $\mathrm{cone}(A) = \mathbb{Q}_0^+(0,\ 1)$ and A is finitely generated by [6], Theorem 5.4. On the other hand, if $A \not\subseteq \mathbb{N}_0(0,\ 1)$ then $(1,\ 0) \in \mathrm{cone}(A)$ by Lemma 37 (ii), and consequently $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$. Now, A is finitely generated by [6], Theorem 5.4 again.

(ii) Symmetric to (i). □

Lemma 40 Assume that 35 (iii) takes place. If $(r,\ s) \multimap a$ and $b \multimap (r,\ s)$ for some $a,\ b \in A \setminus \{0\}$ then $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$ and A is finitely generated.

Proof. By Lemmas 27 and 28, we get $(0,\ 1) \in \mathrm{cone}(A)$ and $(1,\ 0) \in \mathrm{cone}(A)$. Thus $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$ and the rest follows from [6], Theorem 5.4. □

Theorem 41 Assume that A is not finitely generated. Then:

(i) $A \neq \varepsilon(A)$.

(ii) There are $r,\ s \in \mathbb{N}$ such that $\varepsilon(A) = \mathbb{N}_0(r,\ s) \subseteq \mathbb{Q}_0^+(r,\ s) = \mathrm{cone}(\varepsilon(A))$.

(iii) Either $(0,\ 1) \in \mathrm{cone}(A)$ and $(1,\ 0) \notin \mathrm{cone}(A)$ or $(0,\ 1) \notin \mathrm{cone}(A)$ and $(1,\ 0) \in \mathrm{cone}(A)$.

(iv) Either $(0,\ t_1) \in A$ for some $t_1 \in \mathbb{N}$ and $A \cap \mathbb{N}_0(1,\ 0) = \{0\}$ or $(t_2,\ 0) \in A$ for some $t_2 \in \mathbb{N}$ and $A \cap \mathbb{N}_0(0,\ 1) = \{0\}$.

Proof. Since A is not finitely generated, we have $\varepsilon(A) \neq A$ by Corollary 36. Furthermore, combining Proposition 39 and Theorem 35, we get $\varepsilon(A) = \mathbb{N}_0(r,\ s)$ for some $r,\ s \in \mathbb{N}$. Since A is not finitely generated, $\mathrm{cone}(A) \neq (\mathbb{Q}_0^+)^2$ (use [6], Theorem 5.4). Consequently, either $A \cap \mathbb{N}_0(1,\ 0) = \{0\}$ or $A \cap \mathbb{N}_0(0,\ 1) = \{0\}$.

If $(r,\ s) \dashv a \dashv (r,\ s)$ for every $a \in A \setminus \{0\}$ then $\mathrm{cone}(\varepsilon(A)) = \mathbb{Q}_0^+(r,\ s) = \mathrm{cone}(A)$ and A is finitely generated by [6], Theorem 5.4, a contradiction. Consequently, there is $a_1 \in A \setminus \{0\}$ such that either $(r,\ s) \multimap a_1$ or $a_1 \multimap (r,\ s)$ and the rest follows from Lemmas 27 and 28. □

6. Pure subsemigroups of $\mathbb{N}_0^2(+)$

In this section, let A be a pure subsemigroup of $\mathbb{N}_0^2(+)$ such that $0 \in A$ and $\varepsilon(A) \neq 0$. By [6], Lemma 6.3, $0 \in \varepsilon(A)$ and $\varepsilon(A)$ is a pure subsemigroup of $\mathbb{N}_0^2(+)$.

Lemma 42 If $\mathrm{cone}(A) = (\mathbb{Q}_0^+)^2$ then $A = \mathbb{N}_0^2$.

Proof. For every $a \in \mathbb{N}_0^2$ there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $q_1, \ldots, q_n \in \mathbb{Q}_0^+$ such that $a = \sum_{i=1}^n q_i a_i$. We have $q_i = r_i/s_i$ for some $r_i,\ s_i \in \mathbb{N}$. If $s = s_1 \cdots s_n$ then $sq_i \in \mathbb{N}$, $b_i = sq_i a_i \in A$ and $sa = \sum b_i \in A$. Hence $a \in A$. □

Theorem 43 Just one of the following six cases takes place:

1. $A = \varepsilon(A) = \mathbb{N}_0^2$;
2. $A = \varepsilon(A) = \mathbb{N}_0(0,\ 1)$;
3. $A = \varepsilon(A) = \mathbb{N}_0(1,\ 0)$;
4. $A = \varepsilon(A) = \mathbb{N}_0(r,\ s)$ for some $r,\ s \in \mathbb{N}$, $\gcd(r,\ s) = 1$;
5. $\varepsilon(A) = \mathbb{N}_0(r,\ s)$ for some $r,\ s \in \mathbb{N}$, $\gcd(r,\ s) = 1$, $(0,\ 1) \in A$ and $(1,\ 0) \notin A$;
6. $\varepsilon(A) = \mathbb{N}_0(r,\ s)$ for some $r,\ s \in \mathbb{N}$, $\gcd(r,\ s) = 1$, $(1,\ 0) \in A$ and $(0,\ 1) \notin A$.

Proof. With regard to Lemma 42, we can assume that $\mathrm{cone}(A) \neq (\mathbb{Q}_0^+)^2$ and consequently, without loss of generality, we also can assume that $A \cap \mathbb{N}_0(1,\ 0) = \{0\}$. If $\varepsilon(A) = \mathbb{N}_0(l,\ 0)$ for some $l \in \mathbb{N}$ then $(1,\ 0) \in \varepsilon(A)$, since $\varepsilon(A)$ is a pure subsemigroup, a contradiction. Now assume for a short moment, that $\varepsilon(A) = \mathbb{N}_0(0,\ k)$ for some $k \in \mathbb{N}$. Then $(0,\ 1) \in \varepsilon(A)$ and $\varepsilon(A) = \mathbb{N}_0(0,\ 1)$. If $(r_1,\ r_2) \in A$, $r_1 \neq 0$, then $(r_1,\ 0) \in A$, since $(0,\ 1) \in \varepsilon(A)$, and it follows that $(1,\ 0) \in A$, too. But then $A = \mathbb{N}_0^2$, a contradiction. Thus $A = \varepsilon(A) = \mathbb{N}_0(0,\ 1)$.

Taking into account the preceding steps, we can assume that $\mathbb{N}_0(0,\ k) \neq \varepsilon(A) \neq \mathbb{N}_0(k,\ 0)$ for every $k \in \mathbb{N}_0$. Using Theorem 35, we get $\varepsilon(A) = \mathbb{N}_0(r,\ s)$ for suitable $r,\ s \in \mathbb{N}$. Since $\varepsilon(A)$ is pure, we have $\gcd(r,\ s) = 1$. Finally, assume that $\varepsilon(A) \neq A$. If $a \in A \setminus \varepsilon(A)$ then either $a \dashv (r,\ s) \dashv a$ or $(r,\ s) \multimap a$ or $a \multimap (r,\ s)$. If $(a \dashv (r,\ s) \dashv a$ then $ta \in \varepsilon(A)$ for some $t \in \mathbb{N}$ and, since $\varepsilon(A)$ is pure, it follows that $a \in \varepsilon(A)$, a contradiction. If $a \multimap (r,\ s)$ then $(1,\ 0) \in A$ by Lemma 28 (i), a contradiction. If $(r,\ s) \multimap a$ then $(0,\ 1) \in A$ by Lemma 27 (i). □

Theorem 44 The semigroup A is finitely generated.

Proof. If one of the cases Theorem 43 (1), …, (4) holds then A is at most three-generated (and $A \setminus \{0\}$ at most two-generated). The cases Theorem 43 (5), (6) are symmetric, and so we will assume that Theorem 43 (5) is true.

Denote by K the set of the numbers $k \in \mathbb{N}_0$ such that $k \leq r$ and $(k,\ l) \in A$ for at least one $l \in \mathbb{N}$. For every $k \in K$, choose the smallest number $f(k) \in \mathbb{N}$ with $(k,\ f(k)) \in A$ and denote by A_1 the subsemigroup of $\mathbb{N}_0^2(+)$ generated by the finite set $\{(k,\ f(k)) | k \in K\} \cup \{(0,\ 0)\}$. Clearly, $A_1 \subseteq A$, $f(0) = 1$, $(0,\ 1) \in A_1$, $f(r) \leq s$, $(r,\ s) \in A_1$ and $\varepsilon(A) \subseteq A_1$.

Let $a = (u,\ v) \in A$. Proceeding by induction on $u+v$, we are going to show that $a \in A_1$. We can assume that $a \notin \varepsilon(A)$ and $u \neq 0$. Since $(1,\ 0) \notin A$ and A is pure, we have $v \neq 0$ as well. If $a \dashv (r,\ s) \dashv a$ then $a \in \varepsilon(A)$, a contradiction. If $a \multimap (r,\ s)$ then $(1,\ 0) \in A$ follows from Lemma 28 (i), a contradiction again. Thus $(r,\ s) \multimap a$ and $rv > us$. If $r \leq u$ then $s < v$ and, since $(r,\ s) \in \varepsilon(A)$, we get $(u-r,\ v-s) \in A$. But $u+v-(r+s) < u+v$, and hence $(u-r,\ v-s) \in A_1$. Consequently, $a = (u,\ v) = (u-r,\ v-s) + (r,\ s) \in A_1$.

Assume, finally, that $u < r$. Then $f(u) \leq v$, $(0,\ v - f(u)) \in A_1$, $(u,\ f(u)) \in A_1$ and $a = (u,\ v) = (u,\ f(u)) + (0,\ v - f(u)) \in A_1$. Thus $A = A_1$ is a finitely generated semigroup. □

Remark 45 Assume that Theorem 43 (5) holds. Since $(r,\ s) \in \varepsilon(A) \subseteq A$ and $(0,\ 1) \in A$, we get $(r,\ tr) \in A$ for some $t \in \mathbb{N}$ and $(1,\ t) \in A$, since A is a pure subsemigroup. Consequently (see the proof of Theorem 44), $K = \{0,\ 1,\ \ldots,\ r\}$. Of course, $f(0) = 1$. If $k \in K$, $k \neq 0$, then $(k,\ kf(1)) \in A$ and it means that $f(k) \leq kf(1)$. If $f(k) = kf(1)$ then $(k,\ f(k)) = k(1,\ f(1))$. If $f(k) \leq k(f(1)-1)$ then $(1,\ f(1)-1) \in A$, a contradiction. Thus $k(f(1)-1) < f(k) \leq kf(1)$. If $k = nk_1$ and $f(k) = nl_1$ then $l_1 = f(k_1)$ and $(k,\ f(k)) = n(k_1,\ f(k_1))$.

Example 46 Put $A = \{(k,\ l) | k,\ l \in \mathbb{N}_0,\ l \geq k/2\}$. Then A is a subsemigroup of $\mathbb{N}_0^2(+)$, $(0,\ 0) \in A$, $(0,\ 1) \in A$ and it is easy to see that A is a pure subsemigroup. If $k \geq 2$ and $l \geq 1$ then $(l-1)/2 = l/2 - 1/2 \geq k/2 - 1/2 \geq k/2 - 1 = (k-2)/2$, and so $(k-2,\ l-1) \in A$. It follows that $(2,\ 1) \in \varepsilon(A)$. On the other hand, $(0,\ 1) \notin \varepsilon(A)$ and $(1,\ 1) \notin \varepsilon(A)$. Now, it follows from Theorem 43 and Theorem 44 (and its proof) that $\varepsilon(A) = \mathbb{N}_0(2,\ 1)$ and that A is generated by the elements $(0,\ 0)$, $(0,\ 1)$, $(1,\ 1)$ and $(2,\ 1)$.

Remark 47 Let B be a pure subsemigroup of $\mathbb{N}_0^2(+)$ containing at least one element $e \neq 0$ with the property that $a - e \in B$ whenever $a \in B$ and $a - e \in \mathbb{N}_0^2$, $a \neq e$. Then $A = B \cup \{0\}$ is a pure subsemigroup of $\mathbb{N}_0^2(+)$ and $e \in \varepsilon(A)$. Thus $\varepsilon(A) \neq 0$ and A is finitely generated by 44. Then B is finitely generated, too.

Example 48 Let $D = \{(k,\ l) \in \mathbb{N}_0^2 | k \geq l\} \cup \{(0,\ 0)\}$. Then D is a pure subsemigroup of $\mathbb{N}_0^2(+)$, $\varepsilon(D) = \{0\}$, $\mathrm{cone}(D) = \{(q_1,\ q_2) \in (\mathbb{Q}^+)^2 | q_1 \geq q_2\} \cup \{(0,\ 0)\}$ (we have $(1/(s_1 s_2))(r_1 s_2,\ r_2 s_1) = (r_1/s_1,\ r_2/s_2)$), $\beta(D) = \{(q,\ q) | q \in$

$\mathbb{Q}_0^+\} = \mathbb{Q}_0^+(1,\ 1)$ and $\delta(D) = \{(n,\ n)|n \in \mathbb{N}_0\} = \mathbb{N}_0(1,\ 1)$. Furthermore, $\alpha(D) = (\mathbb{Q}^2 \setminus (\mathbb{Q}^+)^2) \cup \{(q_1,\ q_2) \in (\mathbb{Q}^+)^2 | q_1 \leq q_2\}$.

Finally, D is generated by the set $M = \{(jk,\ 1)|k \in \mathbb{N}\} \cup \{(0,\ 0)\}$ and, if N is any generating set of D then $M \subseteq N$. Thus D is not finitely generated.

Use of AI tools declaration

The authors used Grammarly solely for language polishing of the final manuscript. This tool did not contribute to the mathematical content or results. The authors reviewed all changes and take full responsibility for the accuracy, originality, integrity, and validity of the manuscript.

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Conflict of interest

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