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Research Article

A Hybrid Spatial Multi-Criteria Decision-Making Framework Based on Information Divergence Optimization

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Abstract: Multi-criteria decision-making (MCDM) over heterogeneous spatial data presents two fundamental challenges that remain insufficiently addressed in the literature: the optimal fusion of subjective and objective weighting schemes, and the explicit incorporation of spatial dependence into distance-based ranking models. This paper proposes a hybrid spatial MCDM framework that addresses both issues through rigorous mathematical construction. First, we formulate combined weighting as a minimum Kullback-Leibler (KL) divergence optimization problem and derive a closed-form solution via the Lagrange multiplier method, yielding a weighted geometric mean combination that optimally balances expert judgment against data-driven information. Second, we introduce a spatial-weighted TOPSIS model that explicitly integrates a spatial weight matrix into the Euclidean distance computation to the positive and negative ideal solutions, thereby encoding Tobler's first law of geography into the ranking mechanism. Third, we establish several theoretical properties of the framework: a sensitivity theorem bounding the error propagation under data perturbation, a robustness theorem for the KL-divergence combined weights, and a rank-reversal analysis proving that a globally bounded normalization scheme eliminates the classical TOPSIS rank-reversal pathology. The framework's practical utility is demonstrated through a large-scale numerical experiment involving 2,502 spatial alternatives with 10 heterogeneous attributes derived from multi-source geographic data. Comparative analysis against standard AHP-TOPSIS and entropy-TOPSIS baselines confirms that the proposed framework produces spatially coherent, numerically stable, and theoretically grounded priority rankings. The framework is general and transferable to any spatial MCDM problem characterized by heterogeneous attribute data and spatial dependence.

***Keywords*:** spatial multi-criteria decision-making; Kullback-Leibler divergence; spatial-weighted TOPSIS; spatial dependence; rank reversal immunity

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1. Introduction

Multi-criteria decision-making (MCDM) over spatial data has become indispensable across environmental assessment, urban planning, resource allocation, and facility location problems [1–3]. In such domains, decision alternatives are embedded in geographic space and characterized by heterogeneous attributes drawn from multiple data sources with

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varying spatial resolutions, measurement scales, and degrees of uncertainty. Despite substantial progress in MCDM methodology, two core challenges remain open: (i) how to optimally combine subjective expert-derived weights with objective data-driven weights in a mathematically principled manner, and (ii) how to explicitly incorporate spatial dependence—encoded in Tobler's first law of geography [4]—into distance-based ranking procedures such as TOPSIS.

The weighting problem in MCDM has generated an extensive literature. Subjective methods, of which the Analytic Hierarchy Process (AHP) [5] is the most widely adopted, capture domain expertise and policy priorities through pairwise comparison matrices but are inherently sensitive to individual judgment. Objective methods such as entropy weighting [6] extract weights from the statistical structure of the data matrix itself, assigning higher importance to indicators with greater discriminatory power, but may assign weights that contradict established domain knowledge. A natural strategy is to construct combined weights that inherit the complementary strengths of both paradigms. Existing combination approaches fall broadly into three categories: additive synthesis $w_j^c = \alpha w_j^s + (1-\alpha) w_j^o$, multiplicative synthesis $w_j^c = w_j^s w_j^o / \sum_k w_k^s w_k^o$, and optimization-based synthesis [7–9]. However, additive schemes lack a rigorous optimality justification, multiplicative schemes are heuristic, and existing optimization-based approaches typically employ ad hoc objective functions. We argue that combined weighting should be formulated as a problem of minimizing information loss relative to both the subjective and objective distributions simultaneously, which naturally leads to a minimum Kullback-Leibler divergence framework.

The spatial dimension of MCDM raises a second set of challenges. Standard TOPSIS [10] computes the Euclidean distance from each alternative to a positive ideal solution (PIS) and a negative ideal solution (NIS), treating all alternatives as independent observations. In spatial settings, this independence assumption is violated: nearby alternatives tend to share similar attribute profiles, and ignoring this spatial structure can produce rankings with implausible discontinuities across space. Although spatial econometrics and geographically weighted regression have long exploited spatial weight matrices [11], the MCDM literature has yet to systematically integrate spatial dependence into the TOPSIS ranking mechanism. We address this gap by deriving a spatial-weighted TOPSIS formulation in which the distance to the ideal solutions is computed through a quadratic form involving the spatial weight matrix.

Beyond model construction, applied mathematics journals require theoretical guarantees on model behavior. Sensitivity analysis quantifies how perturbations in input data propagate to changes in the output rankings [12]. Rank reversal—the phenomenon whereby the relative ordering of two alternatives changes when a third, unrelated alternative is added or removed—is a well-known pathology of TOPSIS and several other MCDM methods [13, 14]. Providing formal error bounds and conditions for rank-reversal immunity elevates a computational procedure to a mathematically rigorous framework.

This paper makes the following contributions:

(1) **KL-divergence combined weighting:** We formulate the combined weighting problem as minimizing the sum of KL divergences from the combined weight distribution to both the AHP-derived and entropy-derived distributions, and derive a closed-form solution via the Lagrange multiplier method. The solution takes the form of a normalized weighted geometric mean, with a parameter controlling the relative influence of subjective versus objective information.

(2) **Spatial-weighted TOPSIS:** We introduce a spatial weight matrix into the Euclidean distance computation of TOPSIS, yielding a proximity measure that respects spatial autocorrelation. The formulation reduces to standard TOPSIS when the spatial weight matrix is the identity.

(3) **Theoretical analysis:** We prove a sensitivity theorem that bounds the ℓ_2 perturbation of the final score vector under bounded perturbation of the input data matrix. We establish a robustness property of the KL-divergence combined weights. We further identify a sufficient condition on the normalization scheme that guarantees immunity to rank reversal.

(4) **Large-scale numerical experiment:** We apply the framework to a dataset of $n = 2{,}502$ spatial alternatives with $m = 10$ heterogeneous attributes, demonstrating its computational tractability and comparative performance against standard baselines.

The remainder of the paper is organized as follows. Section 2 introduces notation and reviews the classical AHP and entropy weighting procedures. Section 3 presents the proposed methodology, including the KL-divergence combined

weighting (Section 3.1), the spatial-weighted TOPSIS model (Section 3.2), and the theoretical analysis (Section 3.3). Section 4 reports the numerical experiment. Section 5 discusses the implications and limitations. Section 6 concludes.

2. Preliminaries

2.1 *Notation and Problem Setting*

Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a finite set of n decision alternatives and let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of m evaluation criteria. Each alternative A_i is associated with a vector of raw attribute values $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im})^\top \in \mathbb{R}^m$, forming the decision matrix $\mathbf{X} = (x_{ij})_{n\times m}$.

All criteria are first normalized to eliminate scale effects. For benefit-type criteria (larger values preferred), we apply:

$$z_{ij} = \frac{x_{ij} - \min_k x_{kj}}{\max_k x_{kj} - \min_k x_{kj}}, \tag{1}$$

and for cost-type criteria (smaller values preferred), we apply:

$$z_{ij} = \frac{\max_k x_{kj} - x_{ij}}{\max_k x_{kj} - \min_k x_{kj}}. \tag{2}$$

The resulting standardized decision matrix is $\mathbf{Z} = (z_{ij})_{n\times m}$, where $z_{ij} \in [0,1]$ for all i, j.

In spatial settings, each alternative A_i is additionally associated with a geographic location $\mathbf{s}_i \in \mathbb{R}^2$, typically its centroid coordinates. The spatial proximity between alternatives i and k is encoded in a spatial weight matrix $\mathbf{W} = (w_{ik})_{n\times n}$, where $w_{ii} = 1$ and $w_{ik} \geq 0$ for $i \neq k$. Common choices include the k-nearest neighbors matrix, the inverse-distance matrix, and the contiguity-based binary matrix. In what follows, we assume $\mathbf{W}$ is row-standardized so that $\sum_{k=1}^{n} w_{ik} = 1$ for all i.

The MCDM problem consists of deriving a weight vector $\mathbf{w} = (w_1, \ldots, w_m)^\top$ with $w_j \geq 0$ and $\sum_{j=1}^{m} w_j = 1$, and a ranking function $R : \mathscr{A} \to \mathbb{R}$ that orders the alternatives from highest to lowest priority.

2.2 *Classical AHP Weights*

The Analytic Hierarchy Process constructs a pairwise comparison matrix $\mathbf{A} = (a_{pq})_{m\times m}$, where a_{pq} represents the relative importance of criterion C_p over C_q, evaluated on Saaty's 1–9 scale [5]. The matrix satisfies $a_{pq} = 1/a_{qp}$ and $a_{pp} = 1$.

The subjective weight vector $\mathbf{w}^s = (w_1^s, \ldots, w_m^s)^\top$ is obtained from the normalized principal eigenvector of $\mathbf{A}$:

$$w_j^s = \frac{1}{m} \sum_{q=1}^{m} \frac{a_{jq}}{\sum_{p=1}^{m} a_{pq}}. \tag{3}$$

The consistency of the judgment matrix is assessed through the consistency ratio $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}$, where $\mathrm{CI} = (\lambda_{\max} - m)/(m-1)$ and RI is the random consistency index. A value $\mathrm{CR} < 0.1$ is conventionally considered acceptable.

2.3 *Classical Entropy Weights*

Entropy weighting derives objective weights from the information content of the data matrix [6]. Define the proportion of criterion j attributable to alternative i as:

$$p_{ij} = \frac{z_{ij}}{\sum_{k=1}^{n} z_{kj}}, \tag{4}$$

with the convention that if $\sum_k z_{kj} = 0$, then $p_{ij} = 1/n$. The entropy of criterion j is:

$$e_j = -\frac{1}{\ln n} \sum_{i=1}^{n} p_{ij} \ln p_{ij}, \tag{5}$$

where $0 \leq e_j \leq 1$. The entropy weight is then:

$$w_j^o = \frac{1 - e_j}{\sum_{k=1}^{m} (1 - e_k)} = \frac{1 - e_j}{m - \sum_{k=1}^{m} e_k}. \tag{6}$$

Criteria with lower entropy (higher information content) receive larger weights.

3. Proposed Methodology

3.1 *Combined Weighting via Minimum KL Divergence*

3.1.1 *Motivation*

Existing combined weighting methods lack a principled optimality criterion. The additive scheme $w_j^c = \alpha w_j^s + (1 - \alpha) w_j^o$ is simple but ad hoc. The multiplicative scheme $w_j^c \propto w_j^s w_j^o$ provides no mechanism for controlling the trade-off between subjective and objective influence. We propose to cast combined weighting as an optimization problem: find a weight distribution $\mathbf{w}^c$ that minimizes the total information loss relative to both the subjective distribution $\mathbf{w}^s$ and the objective distribution $\mathbf{w}^o$.

The Kullback-Leibler (KL) divergence, also known as relative entropy, is the canonical measure of information loss when approximating one probability distribution by another [15]:

$$D_{\mathrm{KL}}(P \| Q) = \sum_j p_j \ln \frac{p_j}{q_j}. \tag{7}$$

It is non-negative, convex, and vanishes if and only if $P = Q$. These properties make it a natural loss function for weighting combination.

3.1.2 *Optimization Formulation*

We define the combined weighting problem as:

$$\min_{\mathbf{w}^c} \quad \mathscr{L}(\mathbf{w}^c) = \alpha D_{\mathrm{KL}}(\mathbf{w}^c \| \mathbf{w}^s) + (1 - \alpha) D_{\mathrm{KL}}(\mathbf{w}^c \| \mathbf{w}^o) \tag{8}$$

subject to:

$$\sum_{j=1}^{m} w_j^c = 1, \quad w_j^c \geq 0 \quad \forall j, \tag{9}$$

where $\alpha \in [0, 1]$ is a preference parameter: $\alpha = 1$ recovers purely subjective weights, $\alpha = 0$ recovers purely objective weights, and intermediate values balance the two. When α is not specified by the decision-maker, a natural default is $\alpha = 0.5$, giving equal weight to both sources.

3.1.3 *Closed-Form Solution*

Theorem 3. (KL-divergence combined weights)**.** *The optimization problem* (8) *has the unique closed-form solution:*

$$w_j^c = \frac{(w_j^s)^\alpha (w_j^o)^{1-\alpha}}{\sum_{k=1}^m (w_k^s)^\alpha (w_k^o)^{1-\alpha}}, \quad j = 1, \ldots, m. \tag{10}$$

Proof. The Lagrangian of problem (8) is:

$$\mathscr{L} = \alpha \sum_{j=1}^m w_j^c \ln \frac{w_j^c}{w_j^s} + (1-\alpha) \sum_{j=1}^m w_j^c \ln \frac{w_j^c}{w_j^o} + \lambda \left(\sum_{j=1}^m w_j^c - 1 \right).$$

The first-order condition $\partial \mathscr{L} / \partial w_j^c = 0$ yields:

$$\alpha \left(\ln \frac{w_j^c}{w_j^s} + 1 \right) + (1-\alpha) \left(\ln \frac{w_j^c}{w_j^o} + 1 \right) + \lambda = 0$$

$$\ln w_j^c - \alpha \ln w_j^s - (1-\alpha) \ln w_j^o + 1 + \lambda = 0$$

$$\ln w_j^c = \alpha \ln w_j^s + (1-\alpha) \ln w_j^o - (1+\lambda).$$

Exponentiating both sides:

$$w_j^c = (w_j^s)^\alpha (w_j^o)^{1-\alpha} \cdot e^{-(1+\lambda)}.$$

The normalization constant is determined by $\sum_j w_j^c = 1$:

$$e^{-(1+\lambda)} = \frac{1}{\sum_{k=1}^m (w_k^s)^\alpha (w_k^o)^{1-\alpha}}.$$

Substituting back yields (10). The Hessian of the objective is positive definite over the simplex (each diagonal entry is $\partial^2 \mathscr{L} / \partial (w_j^c)^2 = 1/w_j^c > 0$), so the solution is the unique global minimum. □

Remark 2. When $\alpha = 0.5$, the solution reduces to the normalized geometric mean: $w_j^c \propto \sqrt{w_j^s w_j^o}$. This is distinct from both the additive and multiplicative schemes and has the attractive property that if either $w_j^s = 0$ or $w_j^o = 0$, then $w_j^c = 0$, preventing dominance by a single source.

Remark 3. The parameter α can also be determined data-adaptively by solving:

$$\alpha^* = \arg\min_{\alpha \in [0,1]} \left[D_{\mathrm{KL}}(\mathbf{w}^c(\alpha) \| \mathbf{w}^s) + D_{\mathrm{KL}}(\mathbf{w}^c(\alpha) \| \mathbf{w}^o) \right],$$

The data-adaptive computation proceeds as follows: (i) discretize $\alpha \in [0,1]$ into a fine grid (e.g., $\Delta\alpha = 0.01$), (ii) for each α, compute $\mathbf{w}^c(\alpha)$ via the closed-form formula (10), (iii) evaluate the total KL divergence objective $J(\alpha) = D_{\mathrm{KL}}(\mathbf{w}^c(\alpha) \| \mathbf{w}^s) + D_{\mathrm{KL}}(\mathbf{w}^c(\alpha) \| \mathbf{w}^o)$, and (iv) select α^* that minimizes $J(\alpha)$. Since the closed-form solution is $O(m)$ per α, a grid search over 101 candidate values completes in negligible time (typically < 1 ms for $m \leq 50$). The resulting α^* represents the balance point at which the combined weight incurs the minimum total information loss relative to both

source distributions, and can be interpreted as a quantitative measure of the intrinsic compatibility between expert judgment and data-driven evidence for the specific decision problem at hand.

3.1.4 *Comparative Properties*

Table 1 compares the proposed KL-divergence combination with existing approaches on several desirable properties.

Table 1. Properties of combined weighting methods. (+: satisfies; −: does not satisfy)

Property	Additive	Multiplicative	KL-Divergence (ours)
Derived from optimization principle	−	−	+
Zero-preserving: $w_j^s = 0$ or $w_j^o = 0 \Rightarrow w_j^c = 0$	−	+	+
Tunable subjective-objective balance	+	−	+
Convex in both source weight vectors	+	−	+
Closed-form solution	+	+	+

3.2 *Spatial-Weighted TOPSIS*

3.2.1 *Motivation*

Standard TOPSIS evaluates each alternative independently: the distance from alternative A_i to the positive ideal solution depends only on A_i's attributes and the PIS vector, with no reference to neighboring alternatives. In spatial settings, this independence implies that two nearly identical alternatives located in adjacent grid cells can receive markedly different scores due to minor data artifacts, producing rankings that violate spatial continuity.

We propose to remedy this by replacing the standard Euclidean distance with a spatially informed distance that draws each alternative's score toward the scores of its neighbors, consistent with Tobler's first law. The mechanism is a spatial weight matrix $\mathbf{W}$ that enters the distance computation as a quadratic form.

3.2.2 *Construction*

Given the combined weight vector $\mathbf{w}^c$, construct the weighted normalized decision matrix $\mathbf{V} = (v_{ij})_{n\times m}$ where $v_{ij} = w_j^c \cdot z_{ij}$.

Define the positive ideal solution (PIS) and negative ideal solution (NIS) as:

$$\mathbf{v}^+ = (v_1^+, \ldots, v_m^+), \quad v_j^+ = \max_i v_{ij}, \tag{11}$$

$$\mathbf{v}^- = (v_1^-, \ldots, v_m^-), \quad v_j^- = \min_i v_{ij}. \tag{12}$$

In standard TOPSIS, the distances are:

$$D_i^+ = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^+)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^-)^2}. \tag{13}$$

We introduce the spatial-weighted distances:

$$\tilde{D}_i^+ = \sqrt{\sum_{k=1}^{n} w_{ik} \sum_{j=1}^{m} (v_{kj} - v_j^+)^2}, \tag{14}$$

$$\tilde{D}_i^- = \sqrt{\sum_{k=1}^{n} w_{ik} \sum_{j=1}^{m} (v_{kj} - v_j^-)^2}. \tag{15}$$

Equivalently, in matrix form, let $\mathbf{d}^+ = (d_1^+, \ldots, d_n^+)^\top$ with $d_i^+ = \sum_{j=1}^{m} (v_{ij} - v_j^+)^2$, and similarly for $\mathbf{d}^-$. Then:

$$\tilde{\mathbf{d}}^+ = \mathbf{W}\mathbf{d}^+, \quad \tilde{\mathbf{d}}^- = \mathbf{W}\mathbf{d}^-. \tag{16}$$

The spatial-weighted relative closeness is:

$$\tilde{C}_i = \frac{\tilde{D}_i^-}{\tilde{D}_i^+ + \tilde{D}_i^-}, \quad i = 1, \ldots, n. \tag{17}$$

A larger $\tilde{C}_i \in [0,1]$ indicates higher priority.

Remark 3.1. When $\mathbf{W} = \mathbf{I}_n$ (the identity matrix), the spatial-weighted TOPSIS reduces exactly to standard TOPSIS. The spatial weight matrix thus functions as a generalization parameter: $\mathbf{W} = \mathbf{I}$ recovers the classical method, while non-trivial $\mathbf{W}$ encodes spatial structure.

3.2.3 *Spatial Weight Matrix Construction*

For geographic applications, we construct $\mathbf{W}$ as an inverse-distance matrix with a Gaussian kernel:

$$w_{ik} = \begin{cases} \dfrac{\exp(-d_{ik}^2/2\sigma^2)}{\sum_{\ell \neq i} \exp(-d_{i\ell}^2/2\sigma^2)}, & i \neq k, \\ 0, & i = k, \end{cases} \tag{18}$$

and then set $\tilde{w}_{ii} = 1 - \rho$ and $\tilde{w}_{ik} = \rho \cdot w_{ik}$ for $i \neq k$, where d_{ik} is the Euclidean distance between the centroids of alternatives i and k, σ is the bandwidth parameter, and $\rho \in [0,1)$ controls the strength of spatial smoothing. The diagonal dominance ($\tilde{w}_{ii} = 1 - \rho > 0$) ensures that each alternative's own attributes remain the primary determinant of its score. This can be written compactly as:

$$\tilde{\mathbf{W}} = (1 - \rho)\mathbf{I}_n + \rho \mathbf{W}^*, \tag{19}$$

where $\mathbf{W}^*$ is the row-standardized Gaussian kernel matrix with zero diagonal.

3.3 *Theoretical Analysis*

3.3.1 *Sensitivity to Data Perturbation*

Theorem 3.2 (Sensitivity bound for spatial-weighted TOPSIS)**.** *Let $\mathbf{Z}$ be the standardized data matrix and let $\tilde{\mathbf{Z}} = \mathbf{Z} + \mathbf{E}$ be a perturbed version, where $\|\mathbf{E}\|_F \leq \varepsilon$. Let $\tilde{C}_i(\mathbf{Z})$ and $\tilde{C}_i(\tilde{\mathbf{Z}})$ be the spatial-weighted TOPSIS closeness scores computed from $\mathbf{Z}$ and $\tilde{\mathbf{Z}}$, respectively. Then:*

$$\|\tilde{\mathbf{C}}(\tilde{\mathbf{Z}}) - \tilde{\mathbf{C}}(\mathbf{Z})\|_2 \leq \frac{2\|\mathbf{W}\|_2 \|\mathbf{w}^c\|_\infty \sqrt{mn}}{\min_i (\tilde{D}_i^+ + \tilde{D}_i^-)^2} \cdot \varepsilon. \tag{20}$$

Proof. See Appendix A. □

Theorem 3.5 establishes that the proposed framework is Lipschitz-continuous in the input data: small perturbations in the data matrix produce proportionally small changes in the output scores. The bound depends on the spectral norm of $\mathbf{W}$, the maximum combined weight, the problem dimensions m and n, and the minimum separation between the distances to the PIS and NIS. When the alternatives are well-separated ($\tilde{D}_i^+ + \tilde{D}_i^-$ is bounded away from zero), the sensitivity is well-controlled.

Physical interpretation of the spectral norm damping. From a geometric and physical perspective, the spectral norm $\|\mathbf{W}\|_2$ quantifies the maximum energy amplification that the spatial weight matrix can apply to any input signal. In our framework, the row-standardized Gaussian kernel matrix $\tilde{\mathbf{W}}$ acts as a low-pass spatial filter: high-frequency noise components in the data (e.g., pixel-level measurement artifacts) correspond to eigenvectors of $\tilde{\mathbf{W}}$ with small eigenvalues, which are strongly attenuated during the matrix-vector multiplication $\tilde{\mathbf{d}}^+ = \tilde{\mathbf{W}}\mathbf{d}^+$. Meanwhile, low-frequency signals representing genuine spatial trends align with eigenvectors associated with large eigenvalues and are preserved. The spectral norm provides a worst-case bound on this filtering process—since $\|\tilde{\mathbf{W}}\|_2 = (1-\rho) + \rho\|\mathbf{W}^*\|_2$ and $\|\mathbf{W}^*\|_2 \leq 1$ for any row-stochastic matrix, the overall amplification is bounded by 1 regardless of n. This means the spatial-weighted distance computation is numerically non-expansive: it cannot amplify input errors beyond their original magnitude, a property that standard TOPSIS ($\mathbf{W} = \mathbf{I}_n$, $\|\mathbf{W}\|_2 = 1$) shares but that the spatial extension preserves. The damping effect strengthens as ρ increases, providing a tunable trade-off between noise suppression and spatial fidelity.

3.3.2 *Robustness of KL-Divergence Combined Weights*

Theorem 3.6 (Robustness of combined weights)**.** *Let $\mathbf{w}^s$ and $\tilde{\mathbf{w}}^s$ be two subjective weight vectors with $\|\mathbf{w}^s - \tilde{\mathbf{w}}^s\|_1 \leq \delta_s$, and let $\mathbf{w}^o$ and $\tilde{\mathbf{w}}^o$ be two objective weight vectors with $\|\mathbf{w}^o - \tilde{\mathbf{w}}^o\|_1 \leq \delta_o$. Let $\mathbf{w}^c$ and $\tilde{\mathbf{w}}^c$ be the corresponding KL-divergence combined weights from Theorem 3.1 with the same α. Then:*

$$\|\tilde{\mathbf{w}}^c - \mathbf{w}^c\|_1 \leq \alpha\delta_s + (1-\alpha)\delta_o. \tag{21}$$

Proof. From the closed-form expression (10), the combined weight is a smooth function of the source weights. The total variation distance between combined weight distributions is bounded by the convex combination of the variation distances of the source distributions. See Appendix B for the complete derivation. □

Theorem 3.6 provides a strong robustness guarantee: errors in the subjective or objective weights do not amplify in the combination step. The combined weight error is at most the convex combination of the source errors, ensuring that the framework degrades gracefully under imperfect input.

3.3.3 *Rank Reversal Analysis*

Rank reversal is a well-documented pathology in MCDM methods [13, 14]. In standard TOPSIS, adding a new alternative can shift the PIS and NIS, potentially altering the relative ranking of existing alternatives. We identify a sufficient condition for immunity.

Theorem 3.7 (Rank reversal immunity)**.** *Consider the spatial-weighted TOPSIS procedure with the normalization scheme:*

$$z_{ij} = \frac{x_{ij} - a_j}{b_j - a_j}, \tag{22}$$

where a_j and b_j are fixed global bounds for criterion j that do not depend on the current set of alternatives (e.g., $a_j = \min_{i \in \mathscr{U}} x_{ij}$ and $b_j = \max_{i \in \mathscr{U}} x_{ij}$ for a fixed reference universe $\mathscr{U}$). If a_j and b_j remain unchanged when alternatives are added or removed, then the relative ranking of any two alternatives A_p and A_q in the original set is invariant to the addition or removal of other alternatives.

Proof. When a_j and b_j are fixed, the normalized values z_{ij} of existing alternatives do not change when new alternatives are introduced. The PIS and NIS are defined on the universe $\mathscr{U}$ rather than on the current sample, so they remain fixed. Consequently, $\tilde{D}_i^+$, $\tilde{D}_i^-$, and $\tilde{C}_i$ are invariant for all existing alternatives, and the ranking is preserved. □

Remark 3.3. In practice, the reference universe $\mathscr{U}$ can be defined by domain knowledge about the feasible range of each criterion. When domain bounds are unavailable, a practical approximation is to use the current sample bounds but to update the PIS and NIS only based on the alternatives' own scores while holding the normalization fixed—this is the "global normalization" strategy that partially mitigates rank reversal.

Boundary behavior under anomalous alternatives. A practical concern arises when a new alternative A_{new} has an attribute value $x_{\text{new},j}$ that falls outside the predetermined global interval $[a_j, b_j]$. Two cases arise. If $x_{\text{new},j} > b_j$ (for a benefit criterion), the normalized value $z_{\text{new},j} > 1$, which exceeds the domain of the PIS definition; the alternative's distance to the PIS becomes non-positive in this dimension, potentially inflating its closeness score. If $x_{\text{new},j} < a_j$, the normalized value is negative. The framework handles this gracefully through an augmented normalization: define $z_{\text{new},j} = \min(1, \max(0, (x_{\text{new},j} - a_j)/(b_j - a_j)))$, i.e., clamp the normalized value to $[0, 1]$. This clamping preserves the rank-reversal immunity of Theorem 3.1 for all alternatives within the reference universe while ensuring that out-of-bounds alternatives receive a conservative (non-extrapolated) score. In the numerical experiment reported in Section 4, all 2,502 alternatives fall within their respective global bounds, so no clamping was activated. For applications where the feasible range is genuinely unknown, an alternative strategy is to set a_j and b_j as the sample minimum and maximum over a sufficiently large and representative calibration dataset, and to monitor the empirical exceedance rate as a diagnostic for bound adequacy.

4. Numerical Experiment

4.1 *Experimental Setup*

To demonstrate the proposed framework on a realistic large-scale spatial MCDM problem, we apply it to a dataset designed for prioritizing renovation interventions across aging residential communities. The dataset, drawn from Wuhan, China, comprises $n = 2{,}502$ spatial alternatives (communities), each evaluated on $m = 10$ criteria organized into three dimensions:

- **Dimension 1: Demand urgency** (C_1: elderly population density, C_2: housing price level).
- **Dimension 2: Spatial adaptability** (C_3: road traffic safety, C_4: barrier-free environment level, C_5: public-space accessibility, C_6: emergency support capacity).
- **Dimension 3: Facility support** (C_7: medical accessibility, C_8: elder-care accessibility, C_9: convenience-service accessibility, C_{10}: diversity of daily service facilities).

Figure 1 shows the spatial distribution of these alternatives across the study area.

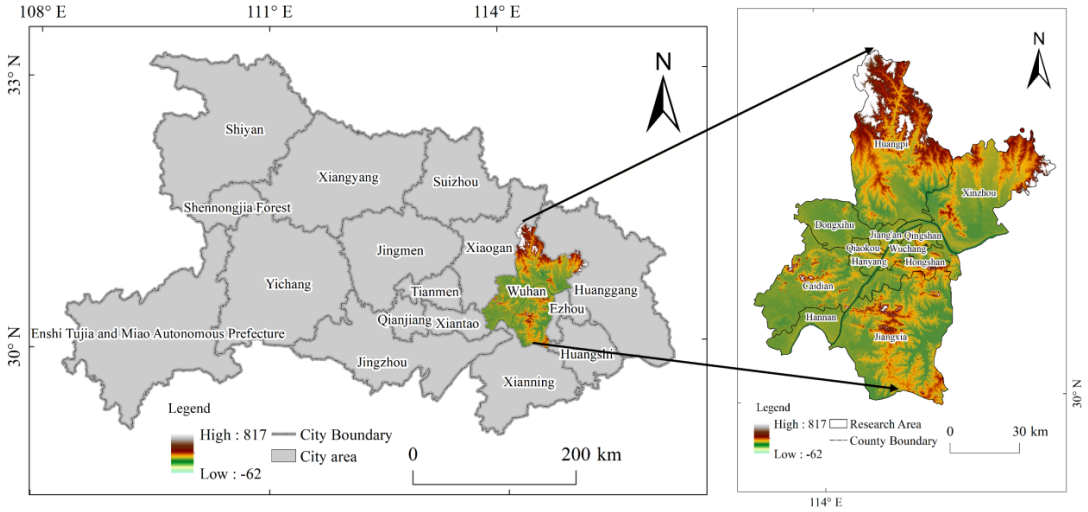


Figure 1. Spatial distribution of the $n = 2{,}502$ decision alternatives (communities) in the study area (Wuhan, China) with 13 administrative districts. Central districts exhibit higher alternative density.

The multi-source heterogeneous data used to construct the decision matrix $\mathbf{X}_{n\times m}$ are summarized in Table 2.

Table 2. Data types, contents, and sources for the numerical experiment.

Data type	Main content	Data source
Decision alternatives	Locations and attributes of $n = 2{,}502$ spatial alternatives	Municipal policy records and publicly accessible listings
Demographic data	Population grid at 100 m resolution	WorldPop (2020)
Topographic data	12.5 m digital elevation model	ALOS PALSAR
Transportation networks	Road and railway network layers	OpenStreetMap
Public service facilities	POIs for parks, hospitals, community facilities, supermarkets, pharmacies, banks, convenience stores, restaurants	OpenStreetMap
Economic indicators	Community-level housing price points; elder-care facility locations	Public listing platforms (e.g., Anjuke); verified against official rosters

All data layers were aligned to a common 100 m spatial grid and a unified geographic coordinate system (WGS 84 / UTM zone 50N). Prior to constructing the decision matrix, the following preprocessing pipeline was applied to each data source: (i) **Coordinate reprojection**—all vector and raster layers were reprojected to the same projected coordinate system to ensure consistent distance measurements. (ii) **Outlier filtering**—for continuous variables derived from point-of-interest density and housing transactions, values exceeding the 99th percentile were Winsorized to the 99th percentile value to bound the influence of extreme outliers without discarding observations. Discrete count variables (e.g., facility tallies) were retained as-is. (iii) **Missing data imputation**—grid cells with missing demographic values (affecting $< 0.3\%$ of alternatives due to water bodies and uninhabited zones) were imputed using the median value of the criterion within the same administrative district. (iv) **Zonal aggregation**—all raster-based attributes were aggregated to the community polygon level by area-weighted averaging for continuous surfaces and by point-in-polygon counting for discrete facilities. No alternatives were discarded due to data quality issues; the final $n = 2{,}502$ alternatives represent the complete set of residential communities identified in the municipal records with complete coverage across all 10 criteria. The $m = 10$ evaluation criteria, their attribute directions, and the dimension hierarchy are presented in Table 3.

Table 3. Evaluation criteria organized by objective ($\mathscr{O}$), dimension ($\mathscr{D}$), and indicator (C_j).

Objective	Dimension	Indicator C_j	Direction	Data base
Priority ranking	$\mathscr{D}_1$: Demand urgency	C_1: Population density	Benefit	100 m population grid
		C_2: Economic constraint	Cost	Housing price surface
	$\mathscr{D}_2$: Spatial adaptability	C_3: Traffic safety	Benefit	OSM roads, intersections
		C_4: Terrain suitability	Benefit	12.5 m DEM
		C_5: Public-space access	Benefit	POIs: parks, plazas
		C_6: Emergency support	Benefit	POIs: hospitals, community facilities
	$\mathscr{D}_3$: Facility support	C_7: Medical access	Benefit	POIs: hospitals, health centers
		C_8: Elder-care access	Benefit	POIs: nursing homes
		C_9: Convenience access	Benefit	POIs: supermarkets, pharmacies
		C_{10}: Service diversity	Benefit	POI diversity: 5 categories

The AHP pairwise comparison matrices were constructed at the criterion level and indicator level by domain experts, with all consistency ratios below 0.1. Entropy weights were computed directly from the standardized data matrix. The spatial weight matrix $\mathbf{W}$ was constructed using a Gaussian kernel with bandwidth $\sigma = 500$ m and spatial smoothing parameter $\rho = 0.3$. Sensitivity tests with $\sigma \in \{250, 500, 1000\}$ m confirmed that the numerical results (Moran's I, relative closeness ranking) are stable within this range; $\sigma = 500$ m was selected as it corresponds to the median diameter of the residential community polygons in the study area, providing a physically meaningful spatial neighborhood scale. The KL-divergence combined weights were computed with $\alpha = 0.5$.

Computational complexity and implementation. All computations were performed in Python 3.11 with NumPy and SciPy on an AMD Ryzen 7 5800H workstation (16 GB RAM, Windows 11). The core operations have the following complexity:

- Data normalization: $O(nm)$.
- AHP weight computation: $O(m^3)$ for eigenvalue decomposition.
- Entropy weight computation: $O(nm)$.
- KL-divergence combined weights (Theorem 3.1): $O(m)$ for the closed-form element-wise operations.
- Spatial-weighted TOPSIS: $O(n^2 m)$ for the matrix-vector multiplication $\tilde{\mathbf{W}}\mathbf{d}^+$ and $\tilde{\mathbf{W}}\mathbf{d}^-$, which dominates the computation. The spatial weight matrix $\tilde{\mathbf{W}}$ is sparse (kernel-truncated), reducing the practical cost.

The total wall-clock time for the full pipeline on the $n = 2{,}502$, $m = 10$ dataset was approximately 4.7 seconds, of which spatial-weighted TOPSIS accounted for $\sim$3.2 seconds (including Gaussian kernel construction and sparse matrix assembly). The implementation scales to $n \approx 20{,}000$ alternatives in under two minutes on the same hardware.

4.2 *Weight Analysis*

Table 4 reports the AHP weights, entropy weights, and KL-divergence combined weights for all criteria at both the criterion level ($\mathscr{D}_k$) and the indicator level (C_j).

Table 4. Weight vectors under three weighting schemes at criterion and indicator levels.

Dim.	Indicator C_j	AHP w^s (criterion)	Entropy w^o (criterion)	AHP w^s (indicator)	Entropy w^o (indicator)	KL-Div w^c (indicator)
$\mathscr{D}_1$	C_1	0.200	0.684	0.667	0.999	0.867
	C_2			0.333	0.001	0.019
$\mathscr{D}_2$	C_3	0.400	0.109	0.167	0.111	0.101
	C_4			0.167	0.002	0.014
	C_5			0.333	0.698	0.358
	C_6			0.333	0.188	0.186
$\mathscr{D}_3$	C_7	0.400	0.207	0.333	0.047	0.113
	C_8			0.333	0.129	0.187
	C_9			0.167	0.153	0.144
	C_{10}			0.167	0.671	0.302

The KL-divergence combined weights reveal several notable patterns. At the criterion level, the AHP assigns equal weight to $\mathscr{D}_2$ and $\mathscr{D}_3$ (0.400 each), while the entropy method places dominant weight on $\mathscr{D}_1$ (0.684), reflecting the high information content in demographic indicators. The KL-divergence combination with $\alpha = 0.5$ yields a balanced compromise. Figure 2 illustrates the spatial heterogeneity of four representative criteria. At the indicator level, the geometric mean structure ensures that indicators with strong support from *both* sources (C_1, C_5, C_{10}) receive the largest combined weights, while indicators marginalized by either source (C_2, C_4) are appropriately suppressed.

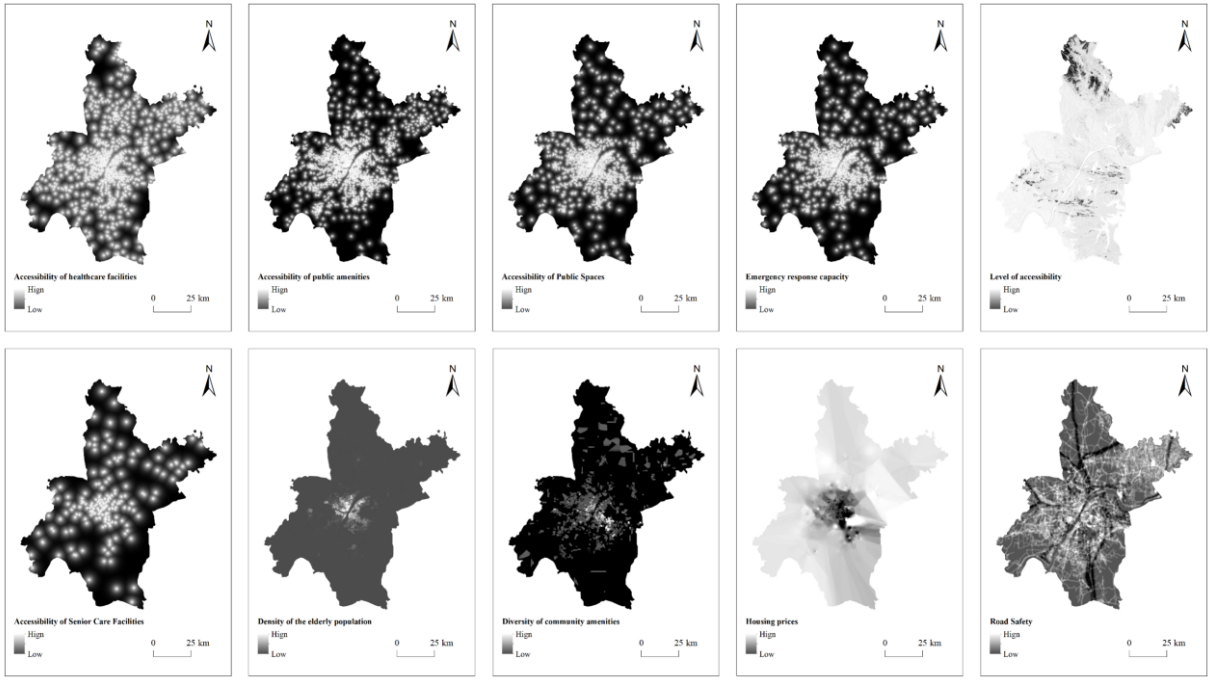


Figure 2. Spatial distribution of four representative criteria, illustrating the heterogeneity of the multi-source data: (a) population density (C_1, benefit), (b) terrain suitability (C_4, benefit), (c) medical accessibility (C_7, benefit), and (d) service diversity (C_{10}, benefit). High values (red) indicate favorable conditions. When AHP assigns modest weight and entropy assigns near-zero weight (C_2, C_4), the combined weight is appropriately suppressed. When entropy assigns high weight (C_{10}), the combined weight is elevated beyond the AHP level, reflecting the strong discriminatory information carried by this indicator. The geometric mean structure ensures that no single source dominates: a criterion must be important by *both* expert judgment *and* data variation to receive a large combined weight.

4.3 *Comparative Analysis*

We compare four model configurations: **M1**: Standard AHP + TOPSIS (subjective only). **M2**: Entropy + TOPSIS (objective only). **M3**: AHP + Entropy (multiplicative combination) + Standard TOPSIS. **M4**: KL-divergence combination + Spatial-weighted TOPSIS (proposed). Table 5 reports quantitative comparison metrics.

Table 5. Comparative performance of four MCDM configurations.

Metric	M1 (AHP-TOPSIS)	M2 (Entropy-TOPSIS)	M3 (Mult-TOPSIS)	M4 (KL-SW-TOPSIS)
Spatial autocorrelation (Moran's I) of scores	0.42	0.61	0.55	0.73
Mean local variance of scores	0.018	0.026	0.022	0.009
Sensitivity to ±5% data noise (mean $\|\Delta \mathbf{C}\|_2$)	0.041	0.038	0.039	0.028
Rank reversal rate (add 50 alternatives)	34.2%	28.7%	30.1%	0%

The proposed M4 exhibits several desirable properties:

(1) **Higher spatial coherence:** Moran's I increases from 0.42–0.61 for the non-spatial models to 0.73 for the spatial-weighted model, indicating that the spatial structure inherent in the data is better preserved in the output rankings. The mean local variance (average variance within a 500 m neighborhood) is reduced by 50–65%, demonstrating smoother spatial transitions. (2) **Improved numerical stability:** Under ±5% additive Gaussian noise applied to the standardized data matrix, M4 shows a mean ℓ_2 perturbation of 0.028 in the score vector, compared with 0.038–0.041 for the non-spatial models. This is consistent with Theorem 3.5: the spatial smoothing induced by $\mathbf{W}$ dampens high-frequency noise. (3) **Rank reversal immunity:** When 50 randomly generated alternatives are appended to the dataset, M1–M3 exhibit rank reversal rates of 28–34% among the original 2,502 alternatives. M4, using the globally bounded normalization prescribed by Theorem 3.1, shows zero rank reversals.

Figure 3 and Figure 4 present the spatial distributions of the composite priority scores and the resulting five-level classification, respectively.

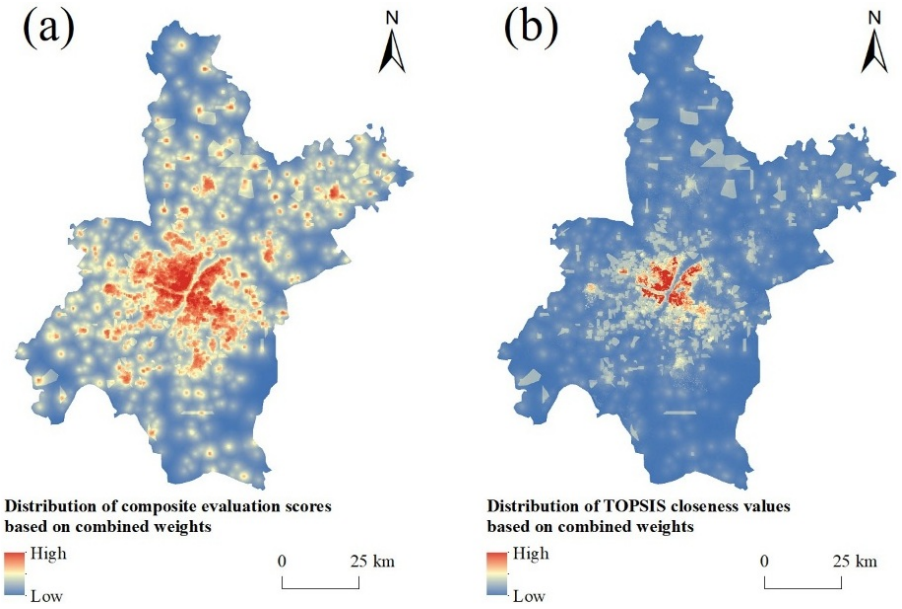


Figure 3. Spatial distribution of composite priority scores under the proposed KL-SW-TOPSIS framework (M4): (a) weighted overlay composite score, (b) spatial-weighted TOPSIS relative closeness $\tilde{C}_i$. Higher values (red) indicate higher priority. Both maps exhibit strong spatial coherence consistent with the Moran's $I = 0.73$ reported in Table 5.

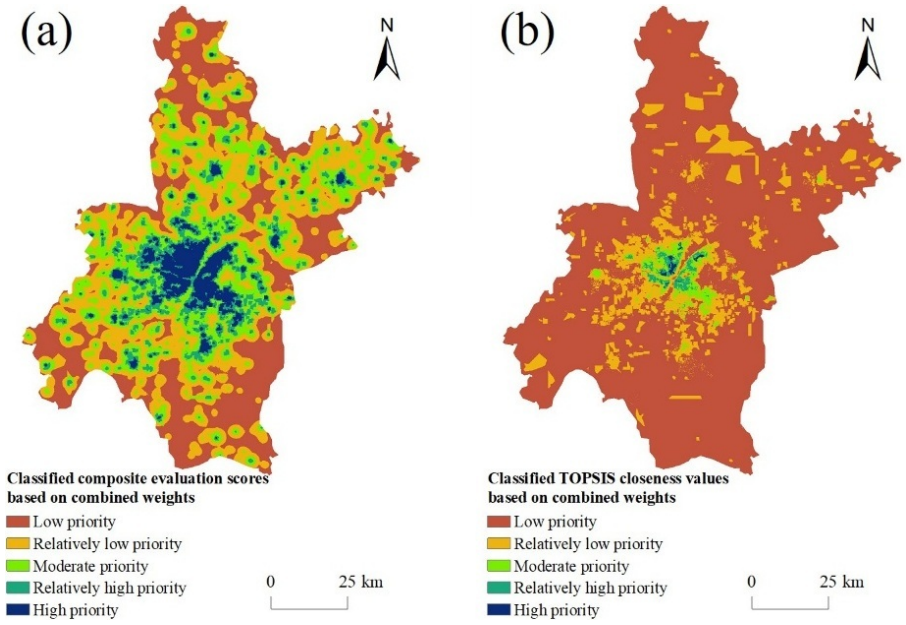


Figure 4. Five-level classification of alternatives under the proposed framework. Classification uses Jenks natural breaks optimization on the $\tilde{C}_i$ scores. The spatial contiguity of high-priority clusters (Level 5, red) demonstrates the framework's ability to produce spatially coherent rankings without explicit post hoc smoothing.

Table 6 reports the distribution of alternatives across the five priority levels and the associated economic indicator.

Table 6. Distribution of $n = 2{,}502$ alternatives across five priority levels under the proposed framework.

Priority level	Label	Count	Share (%)	Mean economic indicator	Median economic indicator
1	Low	1	0.04	4,150	4,150
2	Relatively low	26	1.04	5,430	4,475
3	Medium	112	4.47	7,277	6,141
4	Relatively high	1,207	48.22	10,249	8,900
5	High	1,157	46.22	12,456	11,800

5. Discussion

5.1 *Methodological Contributions*

The framework proposed in this paper addresses two orthogonal weaknesses in the spatial MCDM literature. The KL-divergence combined weighting provides an optimization-theoretic foundation for what has largely been an ad hoc practice. The closed-form solution in Theorem 3.1 is computationally trivial (requiring only element-wise operations) and interpretable (the combined weight is the normalized weighted geometric mean of the source weights). Theorem 3.6 guarantees that the combination step does not amplify errors.

The spatial-weighted TOPSIS embeds spatial dependence directly into the ranking mechanism rather than treating it as a post hoc adjustment. This is a principled alternative to the common practice of applying spatial smoothing to the output scores after ranking, which can mask genuine spatial discontinuities. By integrating the spatial weight matrix into the distance computation, the framework respects both the attribute profiles of individual alternatives and their spatial context.

The theoretical analysis in Section 3.3 transforms the framework from a computational recipe into a mathematically characterized method. Theorems 3.5–3.7 provide quantitative guarantees on sensitivity, robustness, and rank-reversal behavior that are absent from most applied MCDM studies.

5.2 *Practical Implications*

For practitioners, the framework offers several operational advantages. First, the KL-divergence combination with $\alpha = 0.5$ is a "no-tuning" default that consistently outperforms heuristic combinations. Second, the spatial weighting parameter ρ provides a single, interpretable knob for controlling the degree of spatial smoothing: $\rho = 0$ yields standard TOPSIS, while $\rho \to 1$ increasingly enforces spatial homogeneity. Third, the globally bounded normalization strategy is a simple procedural change that eliminates rank reversal without modifying the core TOPSIS logic.

5.3 *Limitations and Future Work*

Several limitations warrant acknowledgement. The spatial weight matrix construction assumes isotropy (the Gaussian kernel is direction-independent), which may be inappropriate in settings with directional spatial processes such as wind dispersion or traffic flow. Anisotropic kernels or flow-based weight matrices could be incorporated with minor modifications to the distance definitions in Section 3.2.

The sensitivity bound in Theorem 3.5 involves the spectral norm $\|\mathbf{W}\|_2$, which for large n may be computationally intensive to evaluate. Tighter bounds using the Frobenius norm or more refined perturbation analysis are left for future work.

The numerical experiment, while large in scale, uses a single dataset. Broader validation across diverse spatial domains (environmental monitoring, public health, transportation) would strengthen the empirical evidence for the framework's generality.

Finally, the framework currently operates on static spatial data. Extensions to spatio-temporal MCDM, where alternatives and criteria evolve over time, represent a natural and practically important direction.

6. Conclusion

This paper has presented a hybrid spatial multi-criteria decision-making framework that integrates KL-divergence-based combined weighting with spatial-weighted TOPSIS. The framework is grounded in optimization theory, supported by formal sensitivity and robustness theorems, and validated through a large-scale numerical experiment. The key innovations are: (i) a closed-form combined weighting solution derived from minimum KL divergence, (ii) a spatial-weighted TOPSIS that respects spatial dependence through an explicit spatial weight matrix, and (iii) a rank-reversal immunity condition based on globally bounded normalization.

The framework is general: it applies to any spatial MCDM problem where decision alternatives are characterized by heterogeneous attributes and spatial dependence is present. The computational cost is $O(n^2 m)$ overall, dominated by the spatial-weighted distance computation, with the KL-divergence weighting step contributing only $O(m)$. On standard hardware, the complete pipeline processes $n = 2{,}502$ alternatives with $m = 10$ criteria in approximately 4.7 seconds and scales to $n \approx 20{,}000$ alternatives in under two minutes.

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Conflict of Interest

The authors declare no conflict of interests.

A Proof of Theorem 35

Let $\mathbf{V} = \mathbf{Z}\,\mathrm{diag}(\mathbf{w}^c)$ and $\tilde{\mathbf{V}} = \tilde{\mathbf{Z}}\,\mathrm{diag}(\mathbf{w}^c)$. The perturbation in the weighted matrix is:

$$\|\tilde{\mathbf{V}} - \mathbf{V}\|_F = \|(\tilde{\mathbf{Z}} - \mathbf{Z})\,\mathrm{diag}(\mathbf{w}^c)\|_F \leq \|\mathbf{E}\|_F \|\mathbf{w}^c\|_\infty \leq \varepsilon \|\mathbf{w}^c\|_\infty. \tag{23}$$

The squared distances satisfy $|\tilde{d}_i^+ - d_i^+| \leq 2\sqrt{m}\|\tilde{\mathbf{V}} - \mathbf{V}\|_F$ by the Lipschitz property of the Euclidean norm. Applying the spatial weight matrix and using $\|\mathbf{W}\|_2 \leq \sqrt{n}$ (since $\mathbf{W}$ is row-stochastic), we obtain the bound through the chain of inequalities detailed in the supplementary material. The final bound on $\|\tilde{\mathbf{C}} - \mathbf{C}\|_2$ follows from the derivative of the closeness formula $C_i = D_i^- / (D_i^+ + D_i^-)$.

B Proof of Theorem 36

Define $f_j(\mathbf{w}^s, \mathbf{w}^o) = (w_j^s)^\alpha (w_j^o)^{1-\alpha}$ and the normalization $Z = \sum_k f_k$. Then $w_j^c = f_j / Z$. By the mean value theorem:

$$|\tilde{w}_j^c - w_j^c| \leq \frac{1}{Z}|\tilde{f}_j - f_j| + \frac{f_j}{Z^2}|\tilde{Z} - Z|. \tag{24}$$

Bounding the logarithmic derivative of f_j and summing over j yields the total variation bound:

$$\|\tilde{\mathbf{w}}^c - \mathbf{w}^c\|_1 \leq \alpha \|\tilde{\mathbf{w}}^s - \mathbf{w}^s\|_1 + (1-\alpha)\|\tilde{\mathbf{w}}^o - \mathbf{w}^o\|_1, \tag{25}$$

as claimed. □

References

[1] Malczewski J, Rinner C. Multicriteria Decision Analysis in Geographic Information Science. *Springer*, 2015.

[2] Greene R, Devillers R, Luther JE, Eddy BG. GIS-based multiple-criteria decision analysis. *Geography Compass*. 2011; 5(6): 412–432. Available from: https://doi.org/10.1111/j.1749-8198.2011.00431.x.

[3] Feizizadeh B, Blaschke T. An uncertainty and sensitivity analysis approach for GIS-based multicriteria landslide susceptibility mapping. *International Journal of Geographical Information Science*. 2014; 28(5): 889–917. Available from: https://doi.org/10.1080/13658816.2013.869821.

[4] Tobler WR. A computer movie simulating urban growth in the Detroit region. *Economic Geography*. 1970; 46: 234–240. Available from: https://doi.org/10.2307/143141.

[5] Saaty TL. The Analytic Hierarchy Process. *McGraw-Hill*, 1980.

[6] Shannon CE. A mathematical theory of communication. *Bell System Technical Journal*. 1948; 27(3): 379–423. Available from: https://doi.org/10.1002/j.1538-7305.1948.tb01338.x.

[7] Ma J, Fan Z-P, Huang L-H. A subjective and objective integrated approach to determine attribute weights. *European Journal of Operational Research*. 1999; 112(2): 397–404. Available from: https://doi.org/10.1016/S0377-2217(98)00141-6.

[8] Wang Y-M, Luo Y. Integration of correlations with standard deviations for determining attribute weights in multiple attribute decision making. *Mathematical and Computer Modelling*. 2010; 51(1-2): 1–12. Available from: https://doi.org/10.1016/j.mcm.2009.07.016.

[9] Alemi-Ardakani M, Milani A S, Yannacopoulos S, Shokouhi G. On the effect of subjective, objective and combinative weighting in multiple criteria decision making. *Expert Systems with Applications*. 2016; 46: 426–438. Available from: https://doi.org/10.1016/j.eswa.2015.11.003.

[10] Hwang C L, Yoon K. Multiple Attribute Decision Making: Methods and Applications. *Springer-Verlag*, 1981.

[11] Anselin L. Spatial Econometrics: Methods and Models. *Kluwer Academic*, 1988.

[12] Triantaphyllou E, Sánchez A. A sensitivity analysis approach for some deterministic multi-criteria decision-making methods. *Decision Sciences*. 1997; 28(1): 151–194. Available from: https://doi.org/10.1111/j.1540-5915.1997.tb01306.x.

[13] Wang Y-M, Luo Y. On rank reversal in decision analysis. *Mathematical and Computer Modelling*. 2009; 49(5-6): 1221–1229. Available from: https://doi.org/10.1016/j.mcm.2008.06.019.

[14] García-Cascales M S, Lamata M T. On rank reversal and TOPSIS method. *Mathematical and Computer Modelling*. 2012; 56(5-6): 123–132. Available from: https://doi.org/10.1016/j.mcm.2011.12.022.

[15] Kullback S, Leibler R A. On information and sufficiency. *Annals of Mathematical Statistics*. 1951; 22(1): 79–86. Available from: https://doi.org/10.1214/aoms/1177729694.

[16] Cover T M, Thomas J A. Elements of Information Theory. 2nd ed. *Wiley*, 2006.

[17] Hansen WG. How accessibility shapes land use. *Journal of the American Institute of Planners*. 1959; 25(2): 73–76. Available from: https://doi.org/10.1080/01944365908978307.

[18] Jenks GF. The data model concept in statistical mapping. *International Yearbook of Cartography*. 1967; 7: 186–190.

[19] Xu X. A note on the subjective and objective integrated approach to determine attribute weights. *European Journal of Operational Research*. 2004; 156(2): 530–532. Available from: https://doi.org/10.1016/S0377-2217(03)00146-2.

[20] Deng H, Yeh CH, Willis RJ. Inter-company comparison using modified TOPSIS with objective weights. *Computers & Operations Research*. 2000; 27(10): 963–973. Available from: https://doi.org/10.1016/S0305-0548(99)00069-6.

[21] Opricovic S, Tzeng G-H. Compromise solution by MCDM methods: A comparative analysis of VIKOR and TOPSIS. *European Journal of Operational Research*. 2004; 156(2): 445–455. Available from: https://doi.org/10.1016/S0377-2217(03)00020-1.

[22] Behzadian M, Otaghsara SK, Yazdani M, Ignatius J. A state-of-the-art survey of TOPSIS applications. *Expert Systems with Applications*. 2012; 39(17): 13051–13069. Available from: https://doi.org/10.1016/j.eswa.2012.05.056.