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Research Article

Dynamic-Memory Fractional Calculus via Generator-Based Memory Construction: Operational Theory, Semigroup Structure, and Applications

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Abstract: This paper develops a generator–based dynamic–memory framework for fractional calculus in which memory kernels are produced from a Laplace–domain generator rather than prescribed in advance. The construction defines dynamic–memory kernels, fractional integrals, and Riemann–Liouville- and Caputo–type dynamic–memory fractional derivatives under explicit admissibility assumptions ensuring local integrability and validity of the convolution calculus. The resulting theory establishes linearity, boundedness, semigroup relations for generated fractional integrals, Laplace representations, and carefully qualified inverse formulas. It also introduces dynamic–memory Mittag–Leffler functions consistent with the Caputo–type operator used in the paper and derives formulas for power and polynomial functions. The framework directly recovers classical shift–invariant convolution operators, such as Riemann–Liouville, Caputo, and tempered forms, and represents several modern kernel–based memory models within the same generator–based viewpoint. The applicability of the approach is illustrated by a nonlinear dynamic–memory fractional Langevin system. The associated Volterra integral formulation, solvability conditions, and a predictor–corrector discretization are presented with explicit attention to kernel admissibility and possible singular behavior at the origin.

Keywords: dynamic–memory fractional calculus, memory generators, convolution kernels, generalized Mittag–Leffler functions, semigroup structure, fractional operational calculus, nonlocal dynamical systems

MSC: 26A33, 44A10, 34A08, 47D06, 47G20

Abbreviation

DMG	Dynamic Memory Generator
DMK	Dynamic–Memory Kernel(s)
DMFI	Dynamic–Memory Fractional Integral
DMFD	Dynamic–Memory Fractional Derivative
DMFC	Dynamic–Memory Fractional Calculus
DMFF	Dynamic–Memory Fractional Framework

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RLDMFD	Riemann–Liouville Dynamic–Memory Fractional Derivative
CDMFD	Caputo Dynamic–Memory Fractional Derivative
DMFO	Dynamic–Memory Fractional Operator
DMMLF	Dynamic–Memory Mittag–Leffler Function
DMFLS	Dynamic–Memory Fractional Langevin System

1. Introduction

Fractional calculus has become a powerful mathematical tool for describing systems whose present behavior depends not only on their current state but also on their past evolution. This hereditary character makes fractional models particularly suitable for problems arising in physics, engineering, biology, viscoelasticity, control theory, and complex dynamical systems [1]. The classical theory is founded primarily on the Riemann–Liouville and Caputo operators, which represent memory through singular power–law kernels. These operators have played a fundamental role in the development of fractional differential equations and continue to provide standard models in both theoretical investigations and practical applications [2].

Despite their importance, classical fractional operators are not always sufficiently flexible to describe the wide variety of hereditary effects encountered in complex systems. In many applications, the underlying memory may be tempered, nonsingular, distributed, adaptive, multiscale, or governed by a law that differs substantially from the classical power–law form. This has led to the introduction of numerous generalized operators, including formulations based on nonsingular, tempered, and Mittag–Leffler kernels, together with other generalized memory structures [3–8]. Recent investigations have further advanced this direction through generalized convolution and Sonine–kernel approaches, including constructions involving conjugate kernels, systematic studies of Sonine pairs, and nonsingular convolution models [9–11].

A substantial part of the recent literature has focused on convolution–type fractional operators and generalized evolution equations with nonlocal effects. Sonine kernels, in particular, provide a broad analytical setting for constructing fractional integrals and derivatives associated with general hereditary laws. Kochubei [12] developed a general fractional calculus based on Sonine-type kernel structures and examined its connections with evolution equations, relaxation processes, and renewal models. More recently, Luchko [13] introduced a class of general fractional derivatives and established several of their principal analytical and operational properties. Beyond the direct Sonine–kernel setting, Bender, Bormann, and Butko [14] investigated generalized time-fractional evolution equations through subordination principles and Feynman–Kac representations, whereas Luchko [15] presented a broad theory of general fractional integrals and derivatives together with a range of applications. Collectively, these contributions reveal the close interplay among convolution operators, hereditary kernels, and nonlocal evolution equations.

Another major direction concerns fractional operators involving Mittag–Leffler–type kernels and related generalized structures. Prabhakar [16] introduced an integral equation containing a generalized Mittag–Leffler function in its kernel, thereby laying an important foundation for the subsequent development of Prabhakar–type operators. This line of research was later extended through unified and multivariate Mittag–Leffler functions and their associated fractional constructions [17, 18]. In parallel, Samraiz et al. [19] introduced a generalized fractional operator and discussed applications in mathematical physics, while Al–Refai and Luchko [20] developed general fractional integrals and derivatives on a finite interval. Additional convolution–based and initialized formulations have also been proposed to represent hereditary effects through structures more general than the standard fixed power–law kernel [21, 22].

Further generalizations have been developed to meet different analytical and modeling requirements. Khalil et al. [23] proposed the conformable derivative as a local formulation preserving several familiar properties of ordinary differentiation, and Abdeljawad [24] subsequently expanded its analytical foundations. Almeida [25] introduced fractional integrals and derivatives with respect to another function, allowing the underlying temporal scale to be modified through a change of variable. Jarad et al. [26] proposed generalized proportional fractional operators that incorporate scaling effects and offer additional modeling flexibility. Related contributions include Katugampola–type operators and other

formulations intended to enlarge the range of admissible structures and improve the operational treatment of fractional models [27, 28].

Although these developments have considerably broadened the scope of fractional calculus, several structural and operational limitations remain. Many generalized operators are defined by prescribing a specific kernel at the outset. The resulting construction is therefore tied to that choice, and the adoption of a different hereditary law often requires a separate definition and an independent analytical treatment. Furthermore, fundamental properties such as semigroup relations, inverse identities, composition rules, and Laplace–transform representations are not automatically guaranteed for arbitrary kernels. Their validity may depend on additional assumptions concerning integrability, regularity, normalization, or compatibility between the relevant kernel pairs. The treatment of initial data and prehistory may also generate supplementary terms that require careful consideration.

A further difficulty is that singular, nonsingular, tempered, logarithmic, oscillatory, distributed–order, and multiscale effects are commonly addressed through distinct families of operators. Consequently, many available approaches do not offer a single systematic procedure for constructing and analyzing these different hereditary patterns within one operational setting. In addition, the connections among a kernel, its Laplace-domain symbol, the corresponding inverse operators, and the associated Mittag–Leffler–type relaxation functions are often developed separately in different formulations. These considerations motivate the need for a unified construction in which the relevant kernels and fractional operators arise from a common symbolic mechanism under clearly stated admissibility conditions.

The objective of the present work is therefore to introduce a generator–based dynamic–memory formulation of fractional calculus in which the kernel is not selected independently at the beginning of the modeling process. Instead, it is derived from a symbolic function referred to as the dynamic memory generator. More precisely, the construction begins with

$$\Phi(p,\Theta),$$

where p denotes the Laplace–domain variable and Θ represents the collection of parameters governing the hereditary characteristics. The corresponding kernel Ψ_α is defined through

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-\alpha}\right\}(t),$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform with respect to the variable p. This procedure provides a systematic means of deriving kernels and their associated fractional operators directly from the chosen symbol. The Laplace–domain formulation is particularly natural because the operators considered in this study are causal convolution operators intended mainly for initial–value problems and Volterra–type equations.

The proposed construction differs from many existing convolution–type fractional calculi in several essential respects:

• The hereditary law is determined through a DMG rather than imposed by a fixed kernel. Its characteristics can therefore be adjusted directly through the parameters contained in Θ.

• The resulting kernels satisfy the convolution relation

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta},$$

provided that the inverse Laplace transforms define locally integrable kernels of exponential order and that the convolution theorem applies. Thus, the semigroup structure follows under explicit admissibility hypotheses rather than from purely formal inverse–transform manipulations.

• A common construction recovers classical shift–invariant convolution operators and represents several modern kernel–based formulations through appropriate selections of the DMG.

The principal novelty of the present work lies in introducing a new class of dynamic–memory fractional derivatives within a unified generator–based framework. Through this construction, the memory kernels, fractional integrals and derivatives, operational identities, inverse relations, Laplace–transform formulas, power–function representations, and generalized Mittag–Leffler–type relaxation functions are systematically connected to the same underlying symbol.

The role of the proposed generator should also be distinguished from that of a dynamical system or a fractional derivative. The DMG is a symbolic device for constructing hereditary kernels; it does not itself describe the evolution of a state or represent a nonlocal rate of change. Accordingly, three levels are involved. A dynamical system governs the evolution of the state, a fractional derivative measures its nonlocal variation, and the generator determines the hereditary law from which the relevant operators are constructed. The generated kernels then define the corresponding fractional integrals and derivatives.

The main contributions of this work are summarized as follows:

• A DMG formulation is introduced, providing a systematic procedure for constructing dynamic–memory kernels and their associated fractional operators.

• Generalized DMFI, together with Riemann–Liouville- and Caputo–type dynamic–memory derivatives, are defined. Their basic properties, including linearity, boundedness, inverse relations, and composition formulas, are established under suitable admissibility assumptions.

• An operational theory is developed through semigroup relations, Laplace–transform representations, power–function formulas, and generalized DMMLF associated with the proposed operators.

• The proposed DMFF is shown to recover classical shift–invariant convolution operators directly and to represent several modern kernel–based structures through suitable choices of the DMG.

• The applicability of the theory is demonstrated through a nonlinear DMFLS with coupled hereditary effects, together with its equivalent Volterra formulation, solvability conditions, and numerical approximation.

Overall, the proposed formulation offers a flexible and structured approach to the study of hereditary processes. Rather than selecting a separate kernel for each model, the desired behavior is determined through a tunable symbolic mechanism. This viewpoint accommodates a broad range of memory laws while retaining an explicit operational connection among the generator, the resulting kernel, and the associated fractional operators.

The remainder of the paper is organized as follows. Section 2 introduces the principal elements of the dynamic–memory formulation, including the dynamic memory generator, generated kernels, associated fractional operators, and the adopted zero–past/prehistory convention. Section 3 develops the analytical and operational theory, including admissibility conditions, semigroup relations, consistency principles, inverse formulas, Laplace representations, an engineering frequency–response interpretation, generalized Mittag–Leffler functions, and formulas for power and polynomial functions. Section 4 applies the theory to a nonlinear dynamic–memory fractional Langevin system and presents the corresponding analytical results, numerical approximation, and representative recovery and representation examples. Finally, Section 5 summarizes the main conclusions and indicates possible directions for future investigation.

2. Preliminaries

In this section, we introduce the main functional spaces, sets, and analytical tools that will be used throughout the paper.

Let $\mathscr{T} > 0$ and denote by $\mathbb{R}$ the set of real numbers. Moreover, for $d \in \mathbb{N}$, $\mathbb{R}^d$ denotes the d–dimensional Euclidean space endowed with the norm

$$\|x\| = \left(\sum_{i=1}^{d} |x_i|^2 \right)^{1/2},$$

while $\mathbb{R}^{d\times d}$ denotes the space of real $d\times d$ matrices.

In the sequel, we use several standard functional spaces. In particular,

$$L^1(0,\mathscr{T}) = \left\{ x : [0,\mathscr{T}] \to \mathbb{R} : \int_0^{\mathscr{T}} |x(t)|\,dt < \infty \right\}$$

denotes the Banach space of Lebesgue integrable functions endowed with the norm

$$\|x\|_{L^1} = \int_0^{\mathscr{T}} |x(t)|\,dt.$$

Moreover, $L^\infty(0,\mathscr{T}) = \{x : [0,\mathscr{T}] \to \mathbb{R} : \operatorname{ess\,sup}_{t\in[0,\mathscr{T}]} |x(t)| < \infty\}$ stands for the Banach space of essentially bounded functions equipped with the norm

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t\in[0,\mathscr{T}]} |x(t)|.$$

We also consider the space $\mathscr{C}([0,\mathscr{T}]) = \{x : [0,\mathscr{T}] \to \mathbb{R} : x \text{ is continuous}\}$, equipped with the supremum norm

$$\|x\|_{\mathscr{C}} = \sup_{t\in[0,\mathscr{T}]} |x(t)|,$$

and the space

$$\mathscr{C}^1([0,\mathscr{T}]) = \left\{ x \in \mathscr{C}([0,\mathscr{T}]) : x' \in \mathscr{C}([0,\mathscr{T}]) \right\},$$

which is endowed with the norm $\|x\|_{\mathscr{C}^1} = \|x\|_{\mathscr{C}} + \|x'\|_{\mathscr{C}}$. Finally, the space

$$L^1_{\text{loc}}(0,\infty) = \left\{ x : (0,\infty) \to \mathbb{R} : x \in L^1(0,a) \text{ for every } a > 0 \right\}$$

denotes the set of locally integrable functions on $(0,\infty)$.

Throughout the paper, the notation

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$$

denotes the convolution product of two locally integrable functions. Moreover, for a function x of exponential order, we denote its Laplace transform by

$$X(p) = \mathscr{L}\{x(t)\}(p) = \int_0^\infty e^{-pt}x(t)\,dt.$$

Throughout this work, the one–sided Laplace transform is employed as an operational tool for representing causal zero–past memory kernels and initial–value formulations. Within this setting, the Laplace domain establishes a natural correspondence between the DMG and the associated DMK through inverse transform representations. However, the one–sided Laplace transform should not be interpreted as prescribing the physical initial conditions of a system by itself. In applications, initial conditions, internal states, and prehistory must be determined from the structure and physical state of the modeled system.

Thus, the one–sided Laplace formulation used here corresponds to a causal operational representation under a stated past convention. The general idea of generator–based memory construction is not inherently restricted to this transform and may also be developed through bilateral Laplace, Fourier, Mellin, or distributional transform frameworks whenever such formulations are more appropriate for the considered class of systems.

Unless otherwise stated, all functions considered in the sequel are assumed to be sufficiently smooth so that the involved integrals, derivatives, and Laplace transforms are well defined.

Definition 1 Let $\Phi : (0,\infty)\times\Theta\to(0,\infty)$ be a positive function depending on the Laplace variable p and a collection of modulation parameters Θ. The function $\Phi(p,\Theta)$ is called a DMG if, for the fractional orders used in the paper, the functions $\Phi(p,\Theta)^{-\alpha}$ are Laplace transforms of causal kernels of exponential order. More precisely, we require that their inverse Laplace transforms exist as causal kernels which are locally integrable whenever they are used in the subsequent Volterra operators.

Definition 2 For $\alpha>0$, the DMK associated with the DMG is defined by

$$\Psi_\alpha(t;\Theta)=\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-\alpha}\right\}(t), \tag{1}$$

whenever the inverse transform exists and yields a causal kernel suitable for convolution on finite intervals. In the main analytical results below, this means that $\Psi_\alpha\in L^1_{\mathrm{loc}}(0,\infty)$ and that the involved convolutions are well defined.

Remark 1 The DMG governs the generated memory kernel and thereby determines the hereditary characteristics of the resulting fractional operators.

Remark 2 Different selections of Φ generate different memory structures and therefore different fractional calculus.

Definition 3 Let $\alpha>0$. The DMFI associated with the DMG is defined by

$$\begin{aligned}(\mathscr{I}^{\alpha}_{\Phi}x)(t)&=\int_0^t\Psi_\alpha(t-s;\Theta)x(s)\,ds\\&=\int_0^t\left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-\alpha}\right\}\right](t-s)x(s)\,ds,\qquad t\in[0,\mathscr{T}].\end{aligned} \tag{2}$$

Definition 4 Let $0<\alpha<1$. The RLDMFD associated with the DMG is defined by

$$\begin{aligned}(\mathscr{D}^{\alpha}_{\Phi}x)(t)&=\frac{d}{dt}\int_0^t\Psi_{1-\alpha}(t-s;\Theta)x(s)\,ds\\&=\frac{d}{dt}\int_0^t\left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-(1-\alpha)}\right\}\right](t-s)x(s)\,ds,\qquad t\in[0,\mathscr{T}].\end{aligned} \tag{3}$$

Definition 5 Let $0<\alpha<1$. The CDMFD associated with the DMG is defined by

$$({}^{c}\mathscr{D}^{\alpha}_{\Phi}x)(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds$$

$$= \int_0^t \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-(1-\alpha)}\right\}\right](t-s)x'(s)\,ds, \qquad t \in [0,\mathscr{T}]. \tag{4}$$

The operators introduced above are inherently history dependent. Indeed, both the DMFI and the DMFD are defined through convolution kernels acting over the interval $[0,t]$. Consequently, the value of the operator at time t depends on the accumulated past states of the process rather than solely on its instantaneous value. However, it is important to distinguish between a zero–past assumption and an unspecified past. In the one–sided formulation adopted in this work, the lower terminal 0 represents the starting instant of observation, and the definitions correspond to a causal zero–past convention unless a different prehistory is explicitly prescribed.

More precisely, if the signal is extended to the whole real line by

$$\widetilde{x}(s) = \begin{cases} 0, & s < 0, \\ x(s), & 0 \le s \le T, \end{cases}$$

then the one–sided DMFI can be interpreted as

$$(\mathscr{I}^{\alpha}_{\Phi}x)(t) = \int_{-\infty}^{t} \Psi_{\alpha}(t-s;\Theta)\widetilde{x}(s)\,ds = \int_0^t \Psi_{\alpha}(t-s;\Theta)x(s)\,ds.$$

Thus, the past is not absent, it is rather assumed to be zero before the starting instant.

If a nonzero prehistory is available, say

$$x(s) = \xi(s), \qquad s < 0,$$

then the corresponding prehistory–dependent dynamic–memory fractional integral may be written as

$$(\mathscr{I}^{\alpha,\xi}_{\Phi}x)(t) = \int_{-\infty}^{0} \Psi_{\alpha}(t-s;\Theta)\xi(s)\,ds + \int_0^t \Psi_{\alpha}(t-s;\Theta)x(s)\,ds.$$

In this case, the first integral represents the contribution of the prescribed past before the starting instant, while the second integral represents the contribution of the observed evolution over $[0,t]$.

Similarly, for the Caputo–type dynamic–memory fractional derivative, one may distinguish between the zero–past convention

$$({}^{c}\mathscr{D}^{\alpha}_{\Phi}x)(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds$$

and the prehistory–dependent form

$$({}^{c}\mathscr{D}_{\Phi}^{\alpha,\xi}x)(t) = \int_{-\infty}^{0} \Psi_{1-\alpha}(t-s;\Theta)\xi'(s)\,ds + \int_{0}^{t} \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds,$$

whenever the derivative of the prehistory exists in a suitable sense. Therefore, in applications, the prehistory must either be prescribed explicitly or replaced by a clearly stated zero–past assumption.

This distinction is particularly important for step–type signals. A Heaviside input is not a globally constant function; it is a causal signal with zero past and a transition at the starting instant. Hence, identities such as

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}C = 0$$

apply to genuinely constant functions with compatible constant prehistory, and should not be confused with the dynamic–memory derivative of a causal step signal, where the jump at the origin and the prescribed past contribute to the memory response.

The proposed DMFF possesses several important advantages in comparison with many existing fractional operators available in the literature. In contrast to classical and generalized approaches where the memory kernel is prescribed *a priori*, the present framework generates the memory structure systematically through the DMG, hence providing a unified and flexible mechanism for constructing broad classes of memory–dependent operators. This generator–driven structure allows the memory behavior to be directly modulated through the parameter set Θ, which may encode singular, nonsingular, tempered, logarithmic, oscillatory, adaptive, or hybrid memory effects within a single analytical framework.

Another significant feature of the proposed operator is that it preserves several fundamental properties of fractional calculus and operational analysis. In particular, the DMK satisfy the semigroup relation

$$\Psi_{\alpha} * \Psi_{\beta} = \Psi_{\alpha+\beta},$$

which leads naturally to composition formulas, inverse relations, and a consistent convolution-based operational calculus. Such properties are not automatically guaranteed for many generalized kernel–based fractional operators appearing in the literature and often require additional compatibility assumptions on the kernels. The DMFF provides a unified symbolic mechanism capable of recovering classical convolution–type fractional operators and representing several modern kernel–based memory models through suitable choices of the DMG.

To further demonstrate the unifying capability of the proposed framework, the recovered or represented fractional operators are classified into two categories. It should be emphasized that the basic Laplace–convolution formulation developed in Definitions 2–5 is naturally shift invariant, since the generated kernels depend on time differences of the form $t-s$. Accordingly, Riemann–Liouville, Caputo, tempered, and related convolution–type operators are recovered directly within the present DMFF.

On the other hand, scale–invariant operators, such as Hadamard–type and certain Katugampola–type formulations, are governed by ratios such as t/s or by logarithmic changes of variables. Such operators should therefore be understood as being represented within a logarithmic–time or Mellin–type extension of the generator–based idea, rather than as direct members of the basic Laplace–convolution construction. With this distinction in mind, Table 1 summarizes several classical and foundational operators associated with the proposed generator–based viewpoint.

Furthermore, Table 2 shows how several modern kernel–based memory models can be represented within the generator–based viewpoint. These include tempered memory effects, nonsingular kernels, Mittag–Leffler memory structures, distributed–order behavior, and local reduction cases.

Table 1. Classical and foundational operators directly recovered, structurally represented, or related to the generator–based framework. Direct recovery applies to shift–invariant convolution operators, while scale–invariant and bilateral/symmetric operators are understood as extensions or related representations

Classical Operator	Generator/Kernel Choice	Resulting Kernel	Memory Type/Status
Riemann–Liouville [1]	$\Phi(p,\Theta)=p$	$\Psi_\alpha(t)=\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$	Singular power–law; directly recovered
Caputo [2]	$\Phi(p,\Theta)=p$ in Caputo form	$\Psi_{1-\alpha}(t)=\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$	Singular power–law; directly recovered
Weyl fractional [22]	Fourier/bilateral Laplace extension	$\Psi_\alpha(t)\sim t^{\alpha-1}$	Infinite-memory; extension of the one-sided framework
Riesz fractional [2]	Symmetric power–law structure	$\Psi_\alpha(t)\sim \lvert t\rvert^{\alpha-1}$	Symmetric nonlocal; related Fourier–type representation
Hadamard–type [2]	Logarithmic/Mellin–type memory generator	$\Psi_\alpha(t,s)=\dfrac{1}{\Gamma(\alpha)}\left(\ln\dfrac{t}{s}\right)^{\alpha-1}\dfrac{1}{s}$	Logarithmic memory; scale–invariant extension
Hilfer derivative [29]	Interpolation between RL and Caputo	$\mathscr{D}^{\alpha,\beta}=\mathscr{I}^{\beta(1-\alpha)}\mathscr{D}\mathscr{I}^{(1-\beta)(1-\alpha)}$	Interpolated singular memory; structurally related
Katugampola–type [27]	Power–law generator after variable transformation	$\Psi_\alpha(t,s)=\dfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}(t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}$	Generalized power–law; scale–invariant extension

Table 2. Representation of several modern kernel–based memory models within the generator–based framework. The entries indicate direct recovery only for shift–invariant convolution kernels generated by suitable choices of Φ, while other modern formulations are interpreted as structural representations, related kernel models, or extensions of the generator–based viewpoint

Modern Operator	Generator/Kernel Choice	Resulting Kernel	Memory Type/Status
Tempered fractional [30]	$\Phi(p,\Theta)=p+\lambda$	$\Psi_\alpha(t)=\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t}$	Tempered memory; directly recovered
Prabhakar–type [16]	Mittag–Leffler–type symbolic kernel structure	$\Psi_\alpha(t)=t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^\alpha)$	Generalized ML memory; structurally represented
Sonine–kernel type [12]	General convolution/Sonine–pair structure	$\Psi_\alpha*\Phi_\alpha=\delta$	Sonine memory pair; related convolution framework
Cotangent fractional [31]	$\Phi(p,\Theta)=p+\cot(\theta)$	$\Psi_\alpha(t)=\dfrac{e^{-\cot(\theta)t}t^{\alpha-1}}{\sin^\alpha(\theta)\Gamma(\alpha)}$	Cotangent–tempered memory; directly represented by a shifted generator
Caputo–Fabrizio [5]	Exponential nonsingular kernel	$\Psi_\alpha(t)=c_\alpha e^{-\mu_\alpha t}$	Nonsingular exponential memory; Memory–kernel representation.
Atangana–Baleanu [6]	Mittag–Leffler–type kernel	$\Psi_\alpha(t)=c_\alpha E_\alpha(-\mu_\alpha t^\alpha)$	Nonsingular ML memory; kernel–based representation
Distributed–order [33]	Order–distribution extension of the generator	$\displaystyle\int_0^1 \Phi(p,\Theta)^{-\alpha}\mu(\alpha)\,d\alpha$	Multiscale memory; distributed–order extension
Conformable derivative [23]	Local reduction case	$T_\alpha x(t)=t^{1-\alpha}x'(t)$	Local non–memory case; related limiting/local formulation

Remark 3 Classical fractional operators may possess different invariance properties. Riemann–Liouville and Caputo operators are fundamentally shift invariant because their kernels depend on time differences of the form $t-s$. This is exactly the invariance structure of the basic Laplace–convolution DMFF introduced above, where

$$(\mathscr{I}^{\alpha}_{\Phi}x)(t)=\int_0^t \Psi_\alpha(t-s;\Theta)x(s)\,ds.$$

By contrast, Hadamard–type operators are scale invariant, since their memory structure is governed by ratios such as t/s, or equivalently by differences in the logarithmic variable $\tau=\ln t$. Therefore, Hadamard–type and related scale–

invariant operators should not be viewed as direct Laplace–convolution operators in the original time variable t. Rather, they can be incorporated through a logarithmic–time transformation or through a Mellin–type generator framework.

More precisely, if

$$\tau = \ln t, \qquad \sigma = \ln s,$$

then the ratio t/s becomes

$$\frac{t}{s} = e^{\tau-\sigma},$$

and scale–invariant memory in the original variable is converted into shift–invariant memory in the logarithmic variable. In this transformed setting, the generator–based construction may again be applied, but the corresponding transform framework is no longer the ordinary Laplace–convolution formulation in t; it is a logarithmic–time or Mellin–type analogue.

Consequently, throughout the paper, shift–invariant operators are recovered directly within the present DMFF, while scale–invariant operators are mentioned as natural extensions of the same generator–based philosophy. This distinction prevents mixing shift–invariant and scale–invariant derivatives within a single operational interpretation.

The following examples show how different choices of the DMG recover classical and generalized fractional operators. For example, the choice $\Phi(p,\Theta) = p$ leads to the classical Riemann–Liouville and Caputo operators, while $\Phi(p,\Theta) = p + \lambda$ produces exponentially tempered memory effects. These examples indicate that different memory structures can be obtained within the same framework by varying the generator.

Example 1 Consider the choice $\Phi(p,\Theta) = p$. Then, by Definition 2, the generalized DMK becomes

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{p^{-\alpha}\}(t).$$

Using the classical Laplace–transform formula

$$\mathscr{L}^{-1}\{p^{-\alpha}\}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ \ \alpha > 0,$$

we obtain $\Psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Consequently, the DMFI reduces to

$$(\mathscr{I}^{\alpha}_{\Phi} x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds,$$

which is precisely the classical Riemann–Liouville fractional integral.

Similarly, the corresponding CDMFD becomes

$${}^{c}\mathscr{D}^{\alpha}_{\Phi} x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'(s)\, ds,$$

which coincides with the classical Caputo fractional derivative.

Example 2 Consider the generator $\Phi(p,\Theta) = p+\lambda,\ \lambda > 0$. Then the associated DMK is given by

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{(p+\lambda)^{-\alpha}\}(t).$$

Using the shifted Laplace–transform identity

$$\mathscr{L}^{-1}\{(p+\lambda)^{-\alpha}\}(t) = e^{-\lambda t}\frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

we obtain $\Psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t}$. Hence, the corresponding DMFI becomes

$$(\mathscr{I}^{\alpha}_{\Phi}x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-s)}x(s)\,ds,$$

which is the usual exponentially tempered fractional integral. Moreover, the corresponding CDMFD takes the form

$${}^{c}\mathscr{D}^{\alpha}_{\Phi}x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}e^{-\lambda(t-s)}x'(s)\,ds.$$

Thus, the choice $\Phi(p,\Theta) = p+\lambda$ generates a CDMFD with an exponentially tempered convolution kernel. This operator corresponds to one common tempered Caputo–type convention based on direct exponential tempering of the Caputo kernel. We do not claim equivalence with all tempered Caputo formulations, since some conventions use exponential changes of variables, shifted differential operators, or additional initialization terms.

From the modeling perspective, the developed framework is particularly suitable for describing complex dynamical systems involving heterogeneous memory interactions, crossover effects, adaptive hereditary mechanisms, and multiscale temporal behavior. At the same time, it combines analytical flexibility with structural consistency while preserving several fundamental properties of fractional calculus.

3. Structural and operational properties of the dynamic–memory fractional calculus

This section develops the main analytical framework underlying the proposed DMFC. The obtained results establish the structural foundations, operational relations, and key analytical properties required for the subsequent theory, including admissibility conditions, inverse formulas, consistency principles, Laplace representations, and generalized memory dynamics.

3.1 *Kernel structure, admissibility, and semigroup properties*

We begin by examining the generated kernels and the conditions under which they produce a mathematically consistent dynamic–memory framework. In particular, we establish admissibility requirements, semigroup relations, and monotonicity properties that form the basis for the operator identities and analytical results developed later.

Theorem 1 Let $\alpha,\beta > 0$ and suppose that $\Phi(p,\Theta)^{-\alpha}$, $\Phi(p,\Theta)^{-\beta}$, and $\Phi(p,\Theta)^{-(\alpha+\beta)}$ are Laplace transforms of causal kernels Ψ_α, Ψ_β, and $\Psi_{\alpha+\beta}$ belonging to $L^1_{\mathrm{loc}}(0,\infty)$ and of exponential order. Assume further that the generated

kernels satisfy the hypotheses required for the one–sided Laplace convolution theorem. Then the following properties hold:

(i) The DMK satisfies $\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta}$.

(ii) The DMFI satisfies the semigroup property $\mathscr{I}_\Phi^\alpha \, \mathscr{I}_\Phi^\beta = \mathscr{I}_\Phi^{\alpha+\beta}$.

(iii) The DMFI is commutative, namely, $\mathscr{I}_\Phi^\alpha \, \mathscr{I}_\Phi^\beta x = \mathscr{I}_\Phi^\beta \, \mathscr{I}_\Phi^\alpha x$.

Proof. By the stated local integrability and exponential–order assumptions, the convolution theorem for the one–sided Laplace transform is applicable. Hence

$$\mathscr{L}\{\Psi_\alpha * \Psi_\beta\}(p) = \Phi(p,\Theta)^{-\alpha}\Phi(p,\Theta)^{-\beta} = \Phi(p,\Theta)^{-(\alpha+\beta)}.$$

Therefore,

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta},$$

which proves (i). Next, using the convolution representation of the DMFI together with (i), we derive

$$\mathscr{I}_\Phi^\alpha \, \mathscr{I}_\Phi^\beta x = \Psi_\alpha * (\Psi_\beta * x) = (\Psi_\alpha * \Psi_\beta) * x = \Psi_{\alpha+\beta} * x = \mathscr{I}_\Phi^{\alpha+\beta} x,$$

which proves (ii).

Finally, since convolution is commutative, we have

$$\Psi_\alpha * \Psi_\beta = \Psi_\beta * \Psi_\alpha.$$

Hence, $\mathscr{I}_\Phi^\alpha \, \mathscr{I}_\Phi^\beta x = \mathscr{I}_\Phi^\beta \, \mathscr{I}_\Phi^\alpha x$, which establishes (iii). □

Remark 4 The semigroup identity in Theorem 1 should be understood under the hypotheses stated in the theorem. In particular, the existence of the inverse Laplace transforms alone is not sufficient. The generated functions must define causal kernels belonging to $L^1_{\mathrm{loc}}(0,\infty)$, be of suitable exponential order, and satisfy the assumptions required for the one–sided convolution theorem. Under these conditions, the identity

$$\mathscr{L}\{\Psi_\alpha * \Psi_\beta\}(p) = \mathscr{L}\{\Psi_\alpha\}(p)\mathscr{L}\{\Psi_\beta\}(p) = \Phi(p,\Theta)^{-\alpha}\Phi(p,\Theta)^{-\beta} = \Phi(p,\Theta)^{-(\alpha+\beta)}$$

implies

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta}.$$

Thus, the semigroup relation is derived from the Laplace–domain product rule only within an admissible convolution setting; it is not claimed to follow merely from formal inverse–transform existence.

Remark 5 The semigroup property established above should be distinguished from the notions of causality and time invariance. The semigroup relation is an algebraic property describing the composition of fractional operators,

whereas causality requires that the present state depend only on past values of the process, and time invariance refers to invariance under temporal translations. Although these concepts are related, they are not equivalent. Indeed, certain classical fractional operators remain causal even when a complete semigroup structure is not available. In the present framework, causality follows naturally from the convolution representation of the generated kernels, while the semigroup property provides an additional level of operational consistency.

Remark 6 The semigroup relation established in Theorem 1 is primarily a kernel and integral semigroup relation. More precisely, it asserts that

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta}$$

and consequently

$$\mathscr{I}_\Phi^\alpha \mathscr{I}_\Phi^\beta = \mathscr{I}_\Phi^{\alpha+\beta}.$$

This property should not be confused with an unrestricted composition rule for fractional derivatives. In general, one cannot assert that

$$\mathscr{D}_\Phi^\alpha \mathscr{D}_\Phi^\beta x = D_\Phi^{\alpha+\beta} x \quad \text{or} \quad {}^c\mathscr{D}_\Phi^\alpha {}^c\mathscr{D}_\Phi^\beta x = {}^c\mathscr{D}_\Phi^{\alpha+\beta} x$$

without imposing additional assumptions on the function space, the prescribed past, and the initial or compatibility conditions.

Indeed, in one–sided formulations, the composition of two derivative operators may generate additional terms depending on the initialization, boundary contribution, or prescribed prehistory of the signal. Therefore, derivative composition must be interpreted together with the assumed past convention. Under zero–state or compatible–prehistory assumptions, certain composition identities may simplify; however, in the general causal initial–value setting, extra terms may appear and should not be ignored.

Thus, within the present DMFF, the relation

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta}$$

provides an operationally consistent memory–kernel structure, while the composition of DMFD operators requires separate analysis. This distinction is important for causal modeling, system–theoretic interpretation, and engineering applications.

The flexibility of the DMFF depends fundamentally on the selection of the DMG. However, not every choice necessarily produces a mathematically admissible DMK or a well–defined operational structure. Therefore, it is important to identify suitable conditions under which the generated kernels possess appropriate analytical and structural properties.

Definition 6 A DMG is called admissible on a given set of orders if the following conditions hold:

(A1) For every $p > 0$, $\Phi(p,\Theta) > 0$;

(A2) For each required order $\rho > 0$, the function $\Phi(p,\Theta)^{-\rho}$ is the Laplace transform of a causal kernel $\Psi_\rho(\cdot;\Theta)$;

(A3) The generated kernel satisfies $\Psi_\rho \in L^1_{\mathrm{loc}}(0,\infty)$ and is of exponential order, so that the corresponding Volterra convolution operators are well defined on the function spaces under consideration;

(A4) For the orders $\rho = \alpha, \beta, \alpha+\beta$, the generated kernels satisfy the hypotheses required for the one–sided Laplace convolution theorem. Under these conditions, the semigroup identity

$$\Psi_\alpha * \Psi_\beta = \Psi_{\alpha+\beta}$$

is obtained as a consequence of the Laplace convolution theorem rather than imposed as an independent assumption.

Remark 7 Conditions (A1)–(A4) separate the analytic requirements on the generated kernels from the algebraic consequence of the Laplace–domain product rule. Thus, admissibility first guarantees that the kernels and Volterra operators are well defined; the semigroup relation then follows when the convolution theorem applies. Inverse relations involving derivatives still require the additional normalization or compatibility conditions stated later.

Definition 7 A DMK is called completely monotone if

$$(-1)^n \frac{d^n}{dt^n} \Psi_\alpha(t;\Theta) \geq 0, \qquad t > 0, \quad n = 0,1,2,\ldots,$$

whenever the derivatives exist.

Theorem 2 Suppose that the DMG is a Bernstein function. Then, for every $\alpha > 0$, the function $\Phi(p,\Theta)^{-\alpha}$ is completely monotone. Consequently, $\Phi(p,\Theta)^{-\alpha}$ is the Laplace transform of a nonnegative measure. If this measure admits a locally integrable density Ψ_α, then the generated DMK is nonnegative. Additional regularity assumptions on the generator may be imposed to ensure that the density Ψ_α itself is completely monotone.

Proof. Since Φ is a Bernstein function, classical properties of Bernstein functions imply that $\Phi(p,\Theta)^{-\alpha}$ is completely monotone for every $\alpha > 0$. By Bernstein's theorem for Laplace transforms, there exists a nonnegative measure μ_α on $[0,\infty)$ such that

$$\Phi(p,\Theta)^{-\alpha} = \int_0^\infty e^{-pt}\, d\mu_\alpha(t).$$

Thus, $\Phi(p,\Theta)^{-\alpha}$ is the Laplace transform of a nonnegative measure. If μ_α is absolutely continuous with respect to the Lebesgue measure and admits a locally integrable density $\Psi_\alpha(t;\Theta)$, then

$$d\mu_\alpha(t) = \Psi_\alpha(t;\Theta)\, dt,$$

and hence

$$\Phi(p,\Theta)^{-\alpha} = \mathscr{L}\{\Psi_\alpha(t;\Theta)\}(p).$$

Since μ_α is nonnegative, its density Ψ_α is nonnegative almost everywhere. Additional regularity assumptions on Φ may be imposed if complete monotonicity of the kernel Ψ_α itself is required. This completes the proof. □

Remark 8 Completely monotone kernels play an important role in the modeling of hereditary phenomena, anomalous diffusion, relaxation processes, and nonlocal dynamical systems. The above result therefore establishes an important connection between the DMFF and modern theories of generalized fractional calculus generated by Bernstein and Sonine–type structures.

Related admissibility concepts arise in general fractional derivatives associated with Sonine kernels, see [13]. The relationship between Bernstein functions and completely monotone Kernels follows from classical results in [32].

Remark 9 If there exists a locally integrable kernel $\widetilde{\Psi}_\alpha$ such that

$$(\widetilde{\Psi}_\alpha * \Psi_\alpha * x)(t) = x(t)$$

for sufficiently smooth functions x, then $(\Psi_\alpha, \widetilde{\Psi}_\alpha)$ forms a Sonine pair and generates mutually inverse convolution operators. Consequently, the proposed DMFF admits inverse relations analogous to those in Sonine kernel fractional calculus.

Remark 10 The asymptotic behavior of the DMG determines the qualitative nature of the generated DMK. For example, power–law generators typically produce long–memory effects, tempered generators lead to exponentially attenuated memory, while Mittag–Leffler–type generators may induce crossover behavior between distinct temporal phases. Thus, the DMG provides a mechanism for classifying memory structures through the generator itself.

3.2 *Generalized dynamic–memory operators and consistency principles*

The DMFO introduced previously are naturally left–sided, since their definitions depend on the past history of the state variable over the interval $[0,t]$. However, several analytical settings, including variational formulations, adjoint systems, terminal–value problems, and integration–by–parts identities, motivate the introduction of corresponding right–sided operators. Moreover, to ensure analytical coherence and compatibility with existing theories, it is important to investigate reduction and consistency principles under suitable choices of the DMG. Therefore, this subsection develops generalized dynamic–memory operators and establishes consistency relations linking the resulting framework with several classical fractional formulations.

Let $0 \le a < t < b < \infty$. We first define the left–sided DMFI associated with the DMG.

Definition 8 Let $\alpha > 0$. The left–sided DMFI is defined by

$$(\mathscr{I}^{\alpha}_{\Phi,a+}x)(t) = \int_a^t \Psi_\alpha(t-s;\Theta)x(s)\,ds = \int_a^t \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-\alpha}\right\}\right](t-s)x(s)\,ds.$$

Definition 9 Let $\alpha > 0$. The right–sided DMFI is defined by

$$(\mathscr{I}^{\alpha}_{\Phi,b-}x)(t) = \int_t^b \Psi_\alpha(s-t;\Theta)x(s)\,ds = \int_t^b \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-\alpha}\right\}\right](s-t)x(s)\,ds.$$

The left– and right–sided DMFI introduced above generalize the corresponding classical fractional integrals through the DMK

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{\Phi(p,\Theta)^{-\alpha}\}(t).$$

Definition 10 Let $0 < \alpha < 1$. The left–sided RLDMFD is defined by

$$(\mathscr{D}^{\alpha}_{\Phi,a+}x)(t) = \frac{d}{dt}\int_a^t \Psi_{1-\alpha}(t-s;\Theta)x(s)\,ds.$$

Definition 11 Let $0 < \alpha < 1$. The right–sided RLDMFD is defined by

$$(\mathscr{D}^{\alpha}_{\Phi,b-}x)(t) = -\frac{d}{dt}\int_t^b \Psi_{1-\alpha}(s-t;\Theta)x(s)\,ds = -\frac{d}{dt}\int_t^b \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-(1-\alpha)}\right\}\right](s-t)x(s)\,ds.$$

Definition 12 Let $0 < \alpha < 1$. The left–sided CDMFD is defined by

$$({}^{c}\mathscr{D}^{\alpha}_{\Phi,a+}x)(t) = \int_a^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds = \int_a^t \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-(1-\alpha)}\right\}\right](t-s)x'(s)\,ds.$$

Definition 13 Let $0 < \alpha < 1$. The right–sided CDMFD is defined by

$$({}^{c}\mathscr{D}^{\alpha}_{\Phi,b-}x)(t) = -\int_t^b \Psi_{1-\alpha}(s-t;\Theta)x'(s)\,ds = -\int_t^b \left[\mathscr{L}^{-1}\left\{\Phi(p,\Theta)^{-(1-\alpha)}\right\}\right](s-t)x'(s)\,ds.$$

Remark 11 The left–sided operators represent past memory effects, while the right–sided operators account for future or terminal memory interactions. Therefore, both types of hereditary behavior can be treated within the same framework.

Remark 12 When the generated kernel depends only on the time difference $t-s$, the corresponding dynamic–memory operators are time–translation invariant. More general choices of the generator may lead to memory structures with different invariance properties. Therefore, causality, semigroup structure, and time invariance should be viewed as distinct characteristics of the generated fractional operators.

Remark 13 The right–sided operators may be useful in future studies on variational problems, adjoint systems, optimal control, boundary–value problems, and integration–by–parts formulas.

Theorem 3 The proposed DMFF consistently recovers the classical Riemann–Liouville and Caputo fractional operators when the generator is chosen as $\Phi(p,\Theta) = p$. In particular:

- the DMFI reduces to the classical Riemann–Liouville fractional integral;
- the RLDMFD reduces to the classical Riemann–Liouville fractional derivative;
- the CDMFD reduces to the classical Caputo fractional derivative.

Proof. If $\Phi(p,\Theta) = p$, then

$$\Psi_{\alpha}(t;\Theta) = \mathscr{L}^{-1}\{p^{-\alpha}\}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Substituting this kernel into Definitions 3–5 immediately yields the classical Riemann–Liouville and Caputo operators. □

Theorem 4 Suppose that the family $\Psi_{1-\alpha}(\cdot;\Theta)$ forms a one–sided approximate identity on the finite interval $[0,\mathscr{T}]$ as $\alpha \to 1^{-}$. More precisely, assume that for every $f \in C([0,\mathscr{T}])$ and every $t \in (0,\mathscr{T}]$,

$$\int_0^t \Psi_{1-\alpha}(t-s;\Theta)f(s)\,ds \longrightarrow f(t) \qquad \text{as } \alpha \to 1^{-}.$$

Then, for every $x \in C^{1}([0,\mathscr{T}])$,

$$\lim_{\alpha\to 1^-} {}^{c}\mathscr{D}_{\Phi}^{\alpha}x(t) = x'(t), \qquad t \in (0, \mathscr{T}].$$

Proof. From Definition 5, we have

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}x(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds.$$

Since $x \in C^1([0,\mathscr{T}])$, we have $x' \in C([0,\mathscr{T}])$. Applying the finite–interval approximate–identity assumption with $f = x'$ gives

$$\lim_{\alpha\to 1^-} \int_0^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds = x'(t).$$

Therefore,

$$\lim_{\alpha\to 1^-} {}^{c}\mathscr{D}_{\Phi}^{\alpha}x(t) = x'(t), \qquad t \in (0, \mathscr{T}].$$

□

Theorem 5 Suppose that the $\Psi_{\alpha}(\cdot;\Theta)$ form an approximate identity as $\alpha \to 0^+$, namely,

$$\int_0^{\infty} \Psi_{\alpha}(s;\Theta)f(t-s)\,ds \to f(t)$$

for every bounded continuous function f. Then, for every $x \in \mathscr{C}([0,\mathscr{T}])$,

$$\lim_{\alpha\to 0^+} \mathscr{I}_{\Phi}^{\alpha}x(t) = x(t).$$

Proof. From the definition of the DMFI and since x is continuous on $[0,\mathscr{T}]$, the approximate identity property implies

$$\lim_{\alpha\to 0^+} \int_0^t \Psi_{\alpha}(t-s;\Theta)x(s)\,ds = x(t).$$

Hence, $\lim_{\alpha\to 0^+} \mathscr{I}_{\Phi}^{\alpha}x(t) = x(t)$.

□

Remark 14 The previous assumptions require the DMK to behave as approximate identities as $\alpha \to 1^-$ or $\alpha \to 0^+$. Such behavior occurs for classical choices of the DMG, including $\Phi(p,\Theta) = p$, hence recovering the standard limiting relations of fractional calculus.

Theorem 6 Assume that the $\Psi_{1-\alpha}(t;\Theta)$ is nonnegative for all $t > 0$. Let $x \in \mathscr{C}^1([0,\mathscr{T}])$ be a nondecreasing function. Then

$$ {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) \geq 0, \;\; t \in [0, \mathscr{T}]. $$

Similarly, if x is nonincreasing, then ${}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) \leq 0$.

Proof. Suppose that x is nondecreasing. Then $x'(t) \geq 0, \;\; t \in [0, \mathscr{T}]$. Since the generated kernel is nonnegative, we obtain

$$ \Psi_{1-\alpha}(t-s;\Theta) x'(s) \geq 0, \qquad 0 \leq s \leq t. $$

Therefore,

$$ {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) = \int_{0}^{t} \Psi_{1-\alpha}(t-s;\Theta) x'(s)\, ds \geq 0. $$

The proof for nonincreasing functions follows analogously. □

Remark 15 The previous theorem shows that the proposed DMFD preserves the qualitative monotonicity behavior of the underlying function whenever the generated memory kernel is nonnegative. This property is important in the analysis of stability, positivity, comparison principles, and memory–dependent dynamical systems.

3.3 *Comparative analysis and modeling advantages*

This subsection compares the proposed DMFO with several classes of generalized kernel–based memory operators, including the Caputo–Fabrizio operator, the Atangana–Baleanu operator, and fractional operators associated with Sonine kernels. These operators correspond to different memory structures, such as nonsingular exponential kernels, Mittag–Leffler kernels, and general convolution formulations. It should be emphasized that the inclusion of such operators in the present comparison is structural rather than terminological. In particular, representing a given kernel within the DMFF does not necessarily mean that the corresponding operator is fractional in the strict Liouville frequency–domain sense

$$ H_{\alpha}(\omega) = (\mathrm{i}\omega)^{\alpha}. $$

Rather, such operators are treated here as memory–kernel or filter–type models that can be analyzed within the same generator–based operational setting.

As shown in Table 3, the proposed DMFO differs from these approaches because the memory kernels are generated through the DMG instead of being prescribed in advance. Different choices of the generator may therefore lead to different memory behaviors within the same framework. The generated operators also retain several operational properties developed earlier, including semigroup relations, inverse formulas, and Laplace representations. This allows different classes of memory effects to be studied using the same analytical setting.

Remark 16 The Caputo–Fabrizio and Atangana–Baleanu operators are included in Table 3 as kernel–based memory models. This inclusion should not be interpreted as a claim that they are fractional derivatives in the strict classical sense associated with the ideal engineering frequency response

$$ H_{\alpha}(\omega) = (\mathrm{i}\omega)^{\alpha}. $$

The Caputo–Fabrizio operator is associated with a nonsingular exponential kernel and may therefore be viewed, from a systems perspective, as an exponential memory or high-pass filter-type model. Similarly, the Atangana–Baleanu operator is associated with a Mittag–Leffler–type nonsingular kernel and may be interpreted as a crossover memory model. Thus, within the present DMFF, these operators are considered as particular kernel–generated memory structures, while their classification as genuine fractional derivatives depends on the adopted mathematical or engineering criteria.

Beyond the comparisons presented above, the proposed DMFF allows different memory behaviors to be considered within the same framework. In practical models, memory effects may combine short–time exponential decay, crossover behavior, long–time power–law effects, and multiscale or adaptive interactions rather than being determined by a single fixed kernel.

Such behaviors arise naturally in viscoelastic materials, anomalous diffusion processes, biological memory systems, heat transfer in complex media, and relaxation phenomena in heterogeneous environments. However, most existing fractional operators are constructed using a prescribed fixed kernel. For instance:

• the classical Caputo operator is associated with a singular power–law kernel;

• the Caputo–Fabrizio operator employs a nonsingular exponential kernel;

• the Atangana–Baleanu operator uses a Mittag–Leffler-type memory kernel.

Although these operators are useful in many applications, each of them is typically associated with a single dominant memory structure. Consequently, combining several hereditary mechanisms within a unified operator often becomes difficult.

In contrast, the proposed DMFF generates the memory structure through the DMG which allows different memory behaviors to coexist simultaneously within the same operator. To illustrate this advantage, consider the dynamic–memory fractional relaxation equation

$$ {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) = -\kappa x(t), \;\; x(0) = x_0, \tag{5} $$

where $\kappa > 0$ is a relaxation parameter. Suppose now that the DMG is chosen in the form

$$ \Phi(p,\Theta) = (p+\lambda)^{\mu} + \eta p^{\nu}, \tag{6} $$

where $\lambda > 0,\ \eta > 0,\ 0 < \mu, \nu < 1$. The generator (6) combines two different hereditary mechanisms:

• the term $(p+\lambda)^{\mu}$ produces exponentially tempered memory effects associated with short-memory relaxation;

• the term p^{ν} generates long–memory power–law hereditary behavior.

Consequently, the generated kernel

$$ \Psi_{\alpha}(t;\Theta) = \mathscr{L}^{-1}\left\{ \left((p+\lambda)^{\mu} + \eta p^{\nu} \right)^{-\alpha} \right\}(t) $$

contains contributions associated with both the shifted term $(p+\lambda)^{\mu}$ and the power–law term p^{ν}. Its precise short–time and long-time behavior depends on the asymptotic behavior of the generator in the Laplace domain. In particular, short–time behavior is related to the regime $p \to \infty$, whereas long–time behavior is related to the regime $p \to 0^{+}$. The dominant contribution is therefore determined by the relative values of μ and ν, as well as by the parameters λ and η.

Thus, the generator should be interpreted as a symbolic mechanism capable of combining tempered and power–law memory components, rather than as giving a parameter–independent dominance rule. Under suitable parameter choices, this structure may produce crossover or multiscale memory behavior, but the precise asymptotic profile of the generated kernel requires a separate analysis of the corresponding inverse Laplace transform.

Therefore, the DMFF naturally captures the transition between short–memory and long–memory dynamics within a single symbolic operator. By contrast, classical fractional operators usually require changing the kernel itself or introducing hybrid coupled models in order to describe such crossover phenomena.

Conversely, many existing generalized fractional operators rely on prescribed memory kernels, thereby restricting the resulting hereditary behavior to a predetermined form. For example, the classical Caputo operator captures power–law memory, the Caputo–Fabrizio operator reflects exponentially decaying memory, while the Atangana–Baleanu operator is governed by a Mittag–Leffler–type kernel. Therefore, these operators do not directly accommodate the combined symbolic memory structure

$$\Phi(p,\Theta) = (p+\lambda)^{\mu} + \eta p^{\nu},$$

within their original formulations, since this generator simultaneously incorporates exponentially tempered and power–law hereditary mechanisms through a single generator–based construction. In particular, the corresponding DMK

$$\Psi_{\alpha}(t;\Theta) = \mathscr{L}^{-1}\left\{((p+\lambda)^{\mu} + \eta p^{\nu})^{-\alpha}\right\}(t)$$

cannot generally be represented through a single fixed exponential kernel, a single Mittag–Leffler kernel, or a pure power–law kernel. Consequently, standard generalized fractional operators would typically require:

- switching between different fractional operators in different temporal regimes;
- constructing coupled hybrid fractional models;
- introducing additional empirical correction terms

In order to reproduce the same crossover memory behavior generated naturally by the proposed framework.

By contrast, the present dynamic–memory formulation incorporates these heterogeneous hereditary effects directly through the structure of the generator itself. This provides a more unified, flexible, and analytically consistent mechanism for modeling multiscale memory dynamics.

Moreover, different choices of the DMG allow the dynamic memory structure to be tuned continuously according to the physical characteristics of the modeled system. Consequently, the proposed framework provides significantly greater flexibility for modeling heterogeneous and adaptive memory processes.

Figure 1 shows the qualitative behavior of several memory kernels corresponding to different choices of the DMG. The figure illustrates how different generators produce transitions between exponentially decaying and power–law memory behaviors.

Remark 17 The previous example shows why suitable choices of the DMG lead to different combinations of memory effects within the same framework. Indeed, the generated kernel is determined by the symbolic relation

$$\mathscr{L}\{\Psi_{\alpha}(t;\Theta)\}(p) = \Phi(p,\Theta)^{-\alpha}.$$

Consequently, each component of a composite generator contributes a corresponding Laplace-domain memory component. After inverse transformation, these components appear as convolution kernels with different decay rates, regularity properties, or scaling behavior. For instance, a power–law part of the generator produces persistent hereditary behavior, a shifted term such as $p+\lambda$ produces exponentially tempered memory, and a mixed generator such as $(p+\lambda)^{\mu} + \eta p^{\nu}$ combines distinct temporal scales through the interaction of its two symbolic components. Therefore, multiscale and heterogeneous memory processes are obtained by modifying the generator Φ and not by changing the definition of the operator itself. In this way, the same convolution–based construction can describe power–law, tempered, hybrid, and crossover memory effects while retaining the same operational form. The semigroup and composition

properties remain valid whenever the generated kernels satisfy the admissibility assumptions stated in Definition 6. This explains the mathematical basis of the statement that heterogeneous and multiscale memory processes can be modeled within the same dynamic–memory framework while preserving the underlying operational structure.

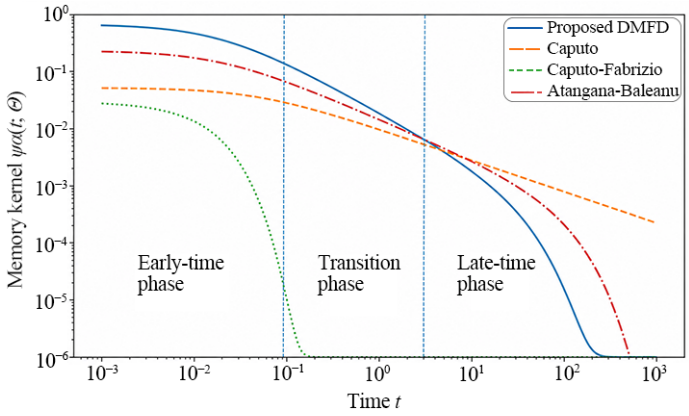


Figure 1. Qualitative comparison of generated memory kernels associated with representative choices of the DMG. The plotted generators are $\Phi_1(p) = p$, $\Phi_2(p) = p + \lambda$, and $\Phi_3(p) = (p + \lambda)^{\mu} + \eta p^{\nu}$, where $\lambda > 0$, $\eta > 0$, and $0 < \mu, \nu < 1$. The curves provide a qualitative illustration of the transition between power–law, tempered, and hybrid memory behavior

3.4 *Analytical, differential, and nonlocal properties*

This subsection establishes several fundamental analytical and calculus–type properties of the proposed dynamic–memory fractional operators, including inverse relations, boundedness, linearity, integration formulas, and nonlocal effects. These results illustrate how the developed framework preserves important structures of classical analysis while introducing new hereditary and nonlocal behaviors.

Remark 18 For a general generator Φ, the first–order generated integral I^1_{Φ} should be regarded as an operator generated by the kernel Ψ_1, and not as the ordinary first–order integral unless an additional normalization is imposed. Hence, in general,

$$\mathscr{I}^1_{\Phi} f(t) = \int_0^t \Psi_1(t-s;\Theta) f(s)\, ds$$

does not necessarily coincide with the ordinary first–order integral $\int_0^t f(s)\, ds$. Therefore, whenever inverse relations are reduced to the classical forms

$$\mathscr{D}^{\alpha}_{\Phi}\, \mathscr{I}^{\alpha}_{\Phi} x = x \quad or \quad \mathscr{I}^{\alpha}_{\Phi}\, {}^{c}\mathscr{D}^{\alpha}_{\Phi} x = x - x(0),$$

we assume the normalization condition

$$\Psi_1(t;\Theta) = 1, \qquad t > 0,$$

or equivalently,

$$\Phi(p,\Theta) = p.$$

More generally, these reductions are valid whenever the selected DMG is such that $\mathscr{I}_{\Phi}^{1}$ acts as the ordinary first–order integral on the considered function space. Without this normalization, the corresponding identities should be written in the generated form

$$\mathscr{D}_{\Phi}^{\alpha}\ \mathscr{I}_{\Phi}^{\alpha} x = \frac{d}{dt}\mathscr{I}_{\Phi}^{1} x$$

and

$$\mathscr{I}_{\Phi}^{\alpha\,c}\mathscr{D}_{\Phi}^{\alpha} x = \mathscr{I}_{\Phi}^{1} x'.$$

This distinction is important because the semigroup relation of the generated kernels does not by itself guarantee that the first–order generated integral coincides with the classical integral.

Consequently, throughout the remainder of the paper, inverse formulas are interpreted in the generated sense unless the normalization $\Psi_1(t) = 1$ (or equivalently $\Phi(p) = p$) is explicitly assumed.

Theorem 7 (Fundamental properties of the DMFI and DMFD) Let $0 < \alpha < 1$. Suppose that the generated kernels are admissible, the involved operators are well defined on the considered function spaces, and the semigroup relation

$$\mathscr{I}_{\Phi}^{\rho}\ \mathscr{I}_{\Phi}^{\sigma} = \mathscr{I}_{\Phi}^{\rho+\sigma}$$

holds for the required orders. Then the following properties hold:

(i) For every x for which the expressions are well defined, the RLDMFD satisfies

$$\mathscr{D}_{\Phi}^{\alpha}\ \mathscr{I}_{\Phi}^{\alpha} x = \frac{d}{dt}\mathscr{I}_{\Phi}^{1} x.$$

In particular, if the first–order generated integral is normalized so that

$$\Psi_1(t;\Theta) = 1,\ \ t > 0,$$

or equivalently $\mathscr{I}_{\Phi}^{1}$ coincides with the ordinary first–order integral on the considered function space, then

$$\mathscr{D}_{\Phi}^{\alpha}\ \mathscr{I}_{\Phi}^{\alpha} x = x.$$

(ii) For sufficiently smooth functions, the CDMFD satisfies

$$\mathscr{I}_{\Phi}^{\alpha\,c}\mathscr{D}_{\Phi}^{\alpha} x = \mathscr{I}_{\Phi}^{1} x'.$$

In particular, if $\mathscr{I}_{\Phi}^{1}$ coincides with the ordinary first–order integral, then

$$\mathscr{I}_{\Phi}^{\alpha}\,{}^{c}\mathscr{D}_{\Phi}^{\alpha}x = x(t) - x(0).$$

(iii) The operator $\mathscr{I}_{\Phi}^{\alpha}$ is linear.
(iv) If $\Psi_{\alpha}(\cdot;\Theta) \in L^{1}(0,\mathscr{T})$, then $\mathscr{I}_{\Phi}^{\alpha} : L^{\infty}(0,\mathscr{T}) \to L^{\infty}(0,\mathscr{T})$ is bounded and

$$\|\mathscr{I}_{\Phi}^{\alpha}x\|_{\infty} \leq \|\Psi_{\alpha}\|_{L^{1}(0,\mathscr{T})}\|x\|_{\infty}.$$

(v) Assume that the kernels $\Psi_{1-\alpha}(\cdot;\Theta)$ form a one–sided approximate identity on $[0,\mathscr{T}]$ as $\alpha \to 1^{-}$, that is,

$$\int_{0}^{t} \Psi_{1-\alpha}(t-s;\Theta)f(s)\,ds \longrightarrow f(t), \qquad t \in (0,\mathscr{T}],$$

for every $f \in C([0,\mathscr{T}])$. Then, for every $x \in C^{1}([0,\mathscr{T}])$,

$$\lim_{\alpha\to 1^{-}} {}^{c}\mathscr{D}_{\Phi}^{\alpha}x(t) = x'(t), \qquad t \in (0,\mathscr{T}].$$

Proof. For property (i), by the definition of RLDMFD, we have

$$\mathscr{D}_{\Phi}^{\alpha}\ \mathscr{I}_{\Phi}^{\alpha}x = \frac{d}{dt}\mathscr{I}_{\Phi}^{1-\alpha}\ \mathscr{I}_{\Phi}^{\alpha}x.$$

Using the semigroup property of the $\mathscr{DMFI}$, we obtain

$$\mathscr{I}_{\Phi}^{1-\alpha}\ \mathscr{I}_{\Phi}^{\alpha}x = \mathscr{I}_{\Phi}^{1}x.$$

Hence,

$$\mathscr{D}_{\Phi}^{\alpha}\ \mathscr{I}_{\Phi}^{\alpha}x = \frac{d}{dt}\mathscr{I}_{\Phi}^{1}x.$$

This is the correct inverse–type identity for a general generator Φ. It reduces to the classical formula only when $\mathscr{I}_{\Phi}^{1}$ is the ordinary first–order integral. Indeed, if

$$\Psi_{1}(t;\Theta) = 1, \qquad t > 0,$$

then

$$\mathscr{I}_{\Phi}^{1}x(t) = \int_{0}^{t} x(s)\,ds,$$

and therefore

$$\mathscr{D}_{\Phi}^{\alpha}\,\mathscr{I}_{\Phi}^{\alpha}x = \frac{d}{dt}\int_{0}^{t} x(s)\,ds = x(t).$$

For property (ii), by the definition of the CDMFD, we have

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}x = \mathscr{I}_{\Phi}^{1-\alpha}x'.$$

Applying $\mathscr{I}_{\Phi}^{\alpha}$ to both sides and using the semigroup property gives

$$\mathscr{I}_{\Phi}^{\alpha}\,{}^{c}\mathscr{D}_{\Phi}^{\alpha}x = \mathscr{I}_{\Phi}^{\alpha}\,\mathscr{I}_{\Phi}^{1-\alpha}x' = \mathscr{I}_{\Phi}^{1}x'.$$

Thus, for a general generator Φ, the correct identity is

$$\mathscr{I}_{\Phi}^{\alpha}\,{}^{c}\mathscr{D}_{\Phi}^{\alpha}x = \mathscr{I}_{\Phi}^{1}x',$$

not necessarily $x(t) - x(0)$. If, in addition, $\mathscr{I}_{\Phi}^{1}$ coincides with the ordinary first–order integral, then

$$\mathscr{I}_{\Phi}^{1}x'(t) = \int_{0}^{t} x'(s)\,ds = x(t) - x(0).$$

Consequently,

$$\mathscr{I}_{\Phi}^{\alpha}\,{}^{c}\mathscr{D}_{\Phi}^{\alpha}x = x(t) - x(0)$$

under this normalization condition.

Property (iii) follows directly from the linearity of the convolution integral defining $\mathscr{I}_{\Phi}^{\alpha}$.

To prove property (iv), let $x \in L^{\infty}(0,\mathscr{T})$. Then

$$|(\mathscr{I}_{\Phi}^{\alpha}x)(t)| \le \int_{0}^{t} |\Psi_{\alpha}(t-s;\Theta)|\,|x(s)|\,ds \le \|x\|_{\infty}\int_{0}^{t} |\Psi_{\alpha}(t-s;\Theta)|\,ds.$$

Hence,

$$|(\mathscr{I}_{\Phi}^{\alpha}x)(t)| \le \|x\|_{\infty}\|\Psi_{\alpha}\|_{L^{1}(0,\mathscr{T})}.$$

Taking the supremum over $t \in [0, \mathscr{T}]$, we obtain

$$\|\mathscr{I}_{\Phi}^{\alpha} x\|_{\infty} \leq \|\Psi_{\alpha}\|_{L^1(0,\mathscr{T})} \|x\|_{\infty}.$$

This proves the boundedness of $\mathscr{I}_{\Phi}^{\alpha}$ from $L^{\infty}(0, \mathscr{T})$ into itself.
Finally, for property (v), from the definition of CDMFD, we have

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta) x'(s)\, ds.$$

Since $x \in C^1([0, \mathscr{T}])$, it follows that $x' \in C([0, \mathscr{T}])$. Applying the one–sided approximate–identity assumption with $f = x'$, we get

$$\lim_{\alpha \to 1^-} \int_0^t \Psi_{1-\alpha}(t-s;\Theta) x'(s)\, ds = x'(t), \qquad t \in (0, \mathscr{T}].$$

Therefore,

$$\lim_{\alpha \to 1^-} {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) = x'(t), \qquad t \in (0, \mathscr{T}].$$

This completes the proof. □

Theorem 8 (Linearity of the CDMFD) Let $0 < \alpha < 1$ and let x, y be sufficiently smooth functions. For any constants $a, b \in \mathbb{R}$, we have

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}(ax + by)(t) = a\, {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) + b\, {}^{c}\mathscr{D}_{\Phi}^{\alpha} y(t).$$

Proof. By the definition of the CDMFD and the linearity of the ordinary derivative and the integral, we have

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}(ax + by)(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)(ax'(s) + by'(s))\, ds = a\, {}^{c}\mathscr{D}_{\Phi}^{\alpha} x(t) + b\, {}^{c}\mathscr{D}_{\Phi}^{\alpha} y(t).$$

□

Theorem 9 (Derivative of Constants) Let C be a constant. Then ${}^{c}\mathscr{D}_{\Phi}^{\alpha} C = 0$.
This follows immediately from the definition of the CDMFD, because the derivative of a constant is zero.

Remark 19 The previous result applies to a genuinely constant function with compatible constant prehistory. It should not be confused with the case of a causal step signal. In particular, the Heaviside function is not a constant function on the whole time axis, it rather represents a signal with zero past and a transition at the starting instant. Thus, while

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} C = 0$$

for a true constant C, the dynamic–memory fractional derivative of a step–type signal must take into account the prescribed past and the jump occurring at the origin. Consequently, in applications, the statement that the derivative of a constant is zero should always be understood together with the assumed prehistory of the signal.

Theorem 10 (Product Rule) Let $x, y \in \mathscr{C}^1([0,\mathscr{T}])$. Then

$$ {}^c\mathscr{D}_{\Phi}^{\alpha}(xy)(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)\big[x'(s)y(s)+x(s)y'(s)\big]\,ds. $$

Equivalently,

$$ {}^c\mathscr{D}_{\Phi}^{\alpha}(xy)(t) = \mathscr{I}_{\Phi}^{1-\alpha}(x'y)(t)+\mathscr{I}_{\Phi}^{1-\alpha}(xy')(t). $$

Remark 20 In general, the classical product rule does not hold for the proposed DMFD, namely,

$$ {}^c\mathscr{D}_{\Phi}^{\alpha}(xy)(t) \neq x(t)\,{}^c\mathscr{D}_{\Phi}^{\alpha}y(t)+y(t)\,{}^c\mathscr{D}_{\Phi}^{\alpha}x(t). $$

The reason is that the operator is nonlocal: it does not act only on the instantaneous values $x(t)$ and $y(t)$, but on the whole past history of the product. Although $(xy)' = x'y+xy'$ inside the integral, the factors are evaluated at the integration variable s, and therefore the convolution cannot generally be rewritten as a classical Leibniz rule at the terminal time t. Consequently, product dynamics involve coupled hereditary effects and cannot generally be separated as in classical local calculus.

Remark 21 A characteristic feature of classical differentiation is that trigonometric functions remain within the same family under repeated differentiation. In contrast, memory–dependent fractional operators generally modify both the amplitude and phase of oscillatory signals through the accumulated hereditary effects encoded in the memory kernel. Consequently, the image of a sinusoidal function under a dynamic–memory fractional derivative need not remain purely sinusoidal. This behavior is not a deficiency of the operator; rather, it reflects the presence of memory and is consistent with the dynamics of many physical systems exhibiting hereditary effects.

For example, viscoelastic materials, anomalous diffusion processes, and relaxation systems with memory typically do not preserve ideal sinusoidal responses. Therefore, the loss of sinusoidal invariance may be regarded as a natural consequence of memory rather than a violation of the principles of calculus.

Theorem 11 (Chain Rule) Let $x \in \mathscr{C}^1([0,\mathscr{T}])$ and let $F \in \mathscr{C}^1(\mathbb{R})$. Then

$$ {}^c\mathscr{D}_{\Phi}^{\alpha}(F\circ x)(t) = \int_0^t \Psi_{1-\alpha}(t-s;\Theta)F'(x(s))x'(s)\,ds. $$

Equivalently,

$$ {}^c\mathscr{D}_{\Phi}^{\alpha}(F(x))(t) = \mathscr{I}_{\Phi}^{1-\alpha}\big(F'(x)x'\big)(t). $$

Proof. Since

$$ \frac{d}{ds}F(x(s)) = F'(x(s))x'(s), $$

the result follows directly from the definition of the CDMFD . □

Remark 22 Unlike the classical conformable derivative, the above chain rule remains nonlocal because the memory kernel $\Psi_{1-\alpha}$ acts on the whole history of $F'(x)x'$ over the interval $[0,t]$.

Theorem 12 (Integration by Parts Formula) Let $0<\alpha<1$, and let $x,y\in\mathscr{C}^1([0,\mathscr{T}])$. Then

$$\int_0^{\mathscr{T}} y(t)\,{}^c\mathscr{D}_\Phi^\alpha x(t)\,dt=\int_0^{\mathscr{T}} x'(s)\left(\int_s^{\mathscr{T}} y(t)\Psi_{1-\alpha}(t-s;\Theta)\,dt\right)ds.$$

Proof. Using the definition of the CDMFD, we have

$$\int_0^{\mathscr{T}} y(t)\,{}^c\mathscr{D}_\Phi^\alpha x(t)\,dt=\int_0^{\mathscr{T}} y(t)\left(\int_0^t \Psi_{1-\alpha}(t-s;\Theta)x'(s)\,ds\right)dt.$$

By changing the order of integration over the triangular region $0\le s\le t\le\mathscr{T}$, we obtain

$$\int_0^{\mathscr{T}} y(t)\,{}^c\mathscr{D}_\Phi^\alpha x(t)\,dt=\int_0^{\mathscr{T}} x'(s)\left(\int_s^{\mathscr{T}} y(t)\Psi_{1-\alpha}(t-s;\Theta)\,dt\right)ds.$$

□

Remark 23 The inner integral in the above formula represents a right–sided DMFI acting on y. Thus, the integration by parts formula connects the left–sided CDMFD of x with a right–sided memory action on y.

Theorem 13 (Dynamic–Memory Taylor–Type Formula) Let $0<\alpha<1$, and let x be sufficiently smooth on $[0,\mathscr{T}]$. Assume, in addition, that the normalization condition $\Psi_1(t;\Theta)=1$ holds. Then

$$x(t)=x(0)+\mathscr{I}_\Phi^\alpha\,{}^c\mathscr{D}_\Phi^\alpha x(t),\qquad t\in[0,\mathscr{T}].$$

This formula is stated under the normalization condition above; otherwise the generated inverse relation gives $\mathscr{I}_\Phi^1 x'$ instead of the ordinary increment $x(t)-x(0)$.

Proof. The result follows directly from the Caputo–type inverse formula

$$\mathscr{I}_\Phi^\alpha\,{}^c\mathscr{D}_\Phi^\alpha x=x-x(0).$$

Therefore,

$$x(t)=x(0)+\mathscr{I}_\Phi^\alpha\,{}^c\mathscr{D}_\Phi^\alpha x(t),$$

which gives the stated representation. □

Remark 24 The above representation can be regarded as a first–order dynamic–memory fractional Taylor formula. In contrast to the classical local expansion, the correction term incorporates accumulated hereditary effects through the DMK, thus preserving the nonlocal nature of the underlying dynamics.

Consequently, certain classical local properties, including the preservation of sinusoidal responses under differentiation, need not remain valid in the presence of hereditary memory effects, since the operator reflects accumulated past dynamics rather than instantaneous local behavior alone.

Remark 25 (Dynamic–memory interpretation of Rolle–type behavior) Let $x \in C^1([a,b])$ satisfy $x(a) = x(b)$. Then, by the classical Rolle theorem, there exists $\xi \in (a,b)$ such that $x'(\xi) = 0$. However, the CDMFD does not depend only on the value of x' at a single point. Rather, for a fixed $t \in (a,b]$, it depends on the accumulated hereditary contribution of x' over the interval $[a,t]$ through the generated kernel $\Psi_{1-\alpha}$. Consequently, the vanishing of the classical derivative at an isolated point does not necessarily determine the value of the dynamic–memory fractional derivative. Instead, the operator reflects a history–dependent balance of the past dynamics. This provides a memory-based interpretation of Rolle–type behavior within the proposed DMFF.

Remark 26 (Dynamic–memory mean value formula) Let $x \in C^1([a,b])$, and assume that, for each fixed $t \in (a,b]$, the kernel

$$s \longrightarrow \Psi_{1-\alpha}(t-s;\Theta)$$

is nonnegative and integrable on $[a,t]$, with

$$\int_a^t \Psi_{1-\alpha}(t-s;\Theta)\,ds > 0.$$

Then, by the weighted mean value theorem for integrals, there exists $\xi_t \in [a,t]$ such that

$${}^c\mathscr{D}^{\alpha}_{\Phi,a+}x(t) = x'(\xi_t)\int_a^t \Psi_{1-\alpha}(t-s;\Theta)\,ds.$$

Equivalently,

$${}^c\mathscr{D}^{\alpha}_{\Phi,a+}x(t) = x'(\xi_t)\,\mathscr{I}^{1-\alpha}_{\Phi,a+}1(t).$$

Thus, the CDMFD can be interpreted as a memory–weighted mean value of the classical derivative over $[a,t]$. Unlike the classical mean value theorem, the weight is determined by the generated kernel $\Psi_{1-\alpha}$, and therefore depends on the selected dynamic memory generator Φ.

In particular, if $\Phi(p,\Theta) = p$, then

$$\Psi_{1-\alpha}(t-s;\Theta) = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Hence, the above formula reduces to

$${}^c\mathscr{D}^{\alpha}_{a+}x(t) = x'(\xi_{t,\alpha})\frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)},$$

for some $\xi_{t,\alpha} \in [a,t]$. Moreover, under the usual approximate–identity behavior of the classical Caputo kernel,

$$\lim_{\alpha \to 1^-} {}^{c}\mathscr{D}^{\alpha}_{a+} x(t) = x'(t).$$

Therefore, the dynamic–memory mean value formula is consistent with the classical local behavior when the generator is reduced to the standard choice $\Phi(p,\Theta) = p$ and $\alpha \to 1^-$.

3.5 *Dynamic–memory fractional derivatives of power and polynomial functions*

Power functions play a fundamental role in fractional calculus and numerical approximation, since they frequently appear in consistency analysis, interpolation procedures, spectral methods, predictor–corrector schemes, and benchmark problems. Therefore, it is important to characterize the action of the proposed DMFO on such functions.

Theorem 14 (DMFD of power functions) Let

$$x(t) = (t-a)^b, \qquad b > 0,$$

and suppose that the $\Psi_{1-\alpha}(\cdot;\Theta)$ belongs to $L^1_{\mathrm{loc}}(0,\infty)$. Then, the CDMFD satisfies

$${}^{c}\mathscr{D}^{\alpha}_{\Phi}(t-a)^b = b\int_a^t \Psi_{1-\alpha}(t-s;\Theta)(s-a)^{b-1}\,ds.$$

Equivalently,

$${}^{c}\mathscr{D}^{\alpha}_{\Phi}(t-a)^b = b\Big(\Psi_{1-\alpha} * (\cdot - a)^{b-1}\Big)(t).$$

Proof. For $x(t) = (t-a)^b$ and by Definition 5, we have

$${}^{c}\mathscr{D}^{\alpha}_{\Phi}(t-a)^b = b\int_a^t \Psi_{1-\alpha}(t-s;\Theta)(s-a)^{b-1}\,ds.$$

The convolution form follows directly from the definition of convolution. □

Remark 27 The previous result recovers several known formulas through suitable choices of the DMG. If $\Phi(p,\Theta) = p$, then

$$\Psi_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

and consequently

$${}^{c}\mathscr{D}^{\alpha}(t-a)^b = \frac{\Gamma(b+1)}{\Gamma(b+1-\alpha)}(t-a)^{b-\alpha},$$

which is precisely the classical Caputo fractional derivative of power functions.

Similarly, alternative selections of Φ generate tempered, Mittag–Leffler, nonsingular, or other generalized dynamic–memory power laws. Therefore, the proposed DMFF extends the classical power–law formulas through the DMK.

Remark 28 Unlike the classical fractional derivative, where the derivative of $(t-a)^b$ remains a pure power function, the proposed DMFD depends on the generated kernel $\Psi_{1-\alpha}$. Consequently, the resulting derivative reflects the interaction between the local growth of the polynomial term and the hereditary structure induced by the DMG. Different choices of the generator therefore lead to different memory–modulated power laws.

The preceding formula for power functions extends naturally to polynomial functions, as stated in the next theorem.

Theorem 15 (DMFD of polynomial functions) Let

$$P_n(t) = \sum_{k=0}^{n} a_k (t-a)^k,$$

where $a_k \in \mathbb{R},\ k = 0, 1, \ldots, n$. Then

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} P_n(t) = \sum_{k=1}^{n} k a_k \int_a^t \Psi_{1-\alpha}(t-s;\Theta)(s-a)^{k-1}\, ds.$$

Equivalently, we get

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} P_n(t) = \sum_{k=1}^{n} k a_k \Big(\Psi_{1-\alpha} * (\cdot - a)^{k-1}\Big)(t).$$

Proof. Using the linearity property established in Theorem 8 together with Theorem 14 applied term–by–term, we obtain

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} P_n(t) = \sum_{k=0}^{n} a_k\, {}^{c}\mathscr{D}_{\Phi}^{\alpha} (t-a)^k.$$

Since a_0 is constant, Theorem 9 implies that ${}^{c}\mathscr{D}_{\Phi}^{\alpha}(a_0) = 0$. Hence,

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} P_n(t) = \sum_{k=1}^{n} k a_k \int_a^t \Psi_{1-\alpha}(t-s;\Theta)(s-a)^{k-1}\, ds.$$

The proof is complete. □

Remark 29 For $\Phi(p,\Theta) = p$, the previous theorem reduces to

$${}^{c}\mathscr{D}^{\alpha} P_n(t) = \sum_{k=1}^{n} a_k \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (t-a)^{k-\alpha},$$

which coincides with the classical Caputo derivative of polynomial functions.

Remark 30 The obtained formulas for power and polynomial functions are useful in numerical analysis because polynomial and power functions are the building blocks of local interpolation, collocation, and quadrature approximations.

For example, in a predictor–corrector or collocation method, the nonlinear term is often replaced locally by a polynomial interpolant. Applying the DMFD to such basis functions gives explicit consistency defects and therefore helps in deriving local truncation errors, convergence estimates, and benchmark solutions for validating numerical approximations [36–38].

Remark 31 The previous results also explain why orthogonal polynomial families, including Legendre, Chebyshev, Bernstein, and Jacobi polynomials, may be adapted to the present dynamic–memory framework. Such polynomials are finite linear combinations of powers after an affine transformation of the interval. Since the action of the DMFD on powers is explicitly described above, one can compute the action of the generated operator on each polynomial basis function term by term. This provides a natural route for constructing operational matrices and spectral or pseudospectral methods adapted to heterogeneous and multiscale memory effects.

3.6 *Laplace operational calculus*

For simplicity, throughout this subsection we write

$$X(p) = \mathscr{L}\{x(t)\}(p).$$

Since

$$\mathscr{L}\{\Psi_\alpha(t;\Theta)\}(p) = \Phi(p,\Theta)^{-\alpha},$$

the DMFI satisfies the following Laplace representation.

Theorem 16 (Laplace Transform of the DMFI) The Laplace transform of the DMFI is

$$\mathscr{L}\{\mathscr{I}_\Phi^\alpha x\}(p) = \Phi(p,\Theta)^{-\alpha} X(p).$$

Proof. From the definition of the DMFI, we have

$$\mathscr{I}_\Phi^\alpha x = \Psi_\alpha * x.$$

Taking the Laplace transform and using the convolution theorem give

$$\mathscr{L}\{\mathscr{I}_\Phi^\alpha x\}(p) = \mathscr{L}\{\Psi_\alpha\}(p)\mathscr{L}\{x\}(p).$$

Since $\mathscr{L}\{\Psi_\alpha\}(p) = \Phi(p,\Theta)^{-\alpha}$ and $\mathscr{L}\{x\}(p) = X(p)$, it follows that

$$\mathscr{L}\{\mathscr{I}_\Phi^\alpha x\}(p) = \Phi(p,\Theta)^{-\alpha} X(p).$$

□

Theorem 17 (Laplace Transform of the CDMFD) Let x be of exponential order. Then

$$\mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha}x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big).$$

Proof. From the definition of the CDMFD, we get

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}x = \Psi_{1-\alpha} * x'.$$

Taking the Laplace transform yields

$$\mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha}x\}(p) = \mathscr{L}\{\Psi_{1-\alpha}\}(p)\mathscr{L}\{x'\}(p).$$

Since $\mathscr{L}\{\Psi_{1-\alpha}\}(p) = \Phi(p,\Theta)^{\alpha-1}$ and $\mathscr{L}\{x'\}(p) = pX(p) - x(0)$, we obtain

$$\mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha}x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big).$$

□

Remark 32 The Laplace symbols in the previous result show that the convolution–defined dynamic–memory fractional derivatives should not be confused with a purely symbolic fractional operator having multiplier $\Phi(p,\Theta)^{\alpha}$. In particular, because the RLDMFD is defined by applying the ordinary derivative d/dt after the generated fractional integral of order $1-\alpha$, its Laplace representation involves the factor $p\Phi(p,\Theta)^{\alpha-1}$, together with the corresponding initial contribution. Similarly, the CDMFD defined through convolution with x' satisfies

$$\mathscr{L}\{{}^{c}D_{\Phi}^{\alpha}x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big).$$

Therefore, the multiplier $\Phi(p,\Theta)^{\alpha}$ is not used in this work as the Laplace symbol of the convolution–defined CDMFD. If such a multiplier is considered, it must be interpreted as a separate auxiliary symbolic operator and not as the operator introduced in Definition 5.

The next remark distinguishes the CDMFD from the auxiliary symbolic operator.

Remark 33 The Laplace representation in Theorem 17 is associated with the CDMFD introduced in Definition 5. In particular, since

$${}^{c}\mathscr{D}_{\Phi}^{\alpha}x = \Psi_{1-\alpha} * x',$$

its Laplace transform is given by

$$\mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha}x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big).$$

Thus, in this Caputo–type formulation, the ordinary differentiation symbol p is not completely replaced by the generator $\Phi(p,\Theta)$. Rather, the classical derivative contributes the factor

$$pX(p) - x(0),$$

while the generated memory kernel contributes the multiplier $\Phi(p,\Theta)^{\alpha-1}$.

If one adopts instead a purely symbolic dynamic–memory derivative whose Laplace–domain multiplier is $\Phi(p,\Theta)^{\alpha}$, then a different operator is obtained. To avoid confusion with the RLDMFD and the CDMFD defined in this paper, we denote such an auxiliary symbolic operator by

$$\mathscr{D}^{\alpha}_{\Phi,\mathrm{sym}}.$$

In that case, its Laplace representation would be of the form

$$\mathscr{L}\{\mathscr{D}^{\alpha}_{\Phi,\mathrm{sym}}x\}(p) = \Phi(p,\Theta)^{\alpha}X(p) - \Phi(p,\Theta)^{\alpha-1}x(0).$$

Therefore, $\mathscr{D}^{\alpha}_{\Phi,\mathrm{sym}}$ should be understood as a different symbolic generator–based derivative, and not as the Caputo–type dynamic–memory derivative ${}^{c}\mathscr{D}^{\alpha}_{\Phi}$ introduced in Definition 5. Throughout the present work, all results involving the CDMFD are based on the Laplace formula established in Theorem 17.

Remark 34 (Role of the one–sided Laplace transform) The Laplace representations obtained in this subsection should be understood as operational identities associated with the adopted one–sided causal formulation. In particular, the term $x(0)$ appearing in

$$\mathscr{L}\{{}^{c}\mathscr{D}^{\alpha}_{\Phi}x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big)$$

is not meant to imply that the one–sided Laplace transform determines the physical initial state of the system. Rather, the initial value, possible internal variables, and prescribed prehistory must be specified from the modeled system itself.

Consequently, the one–sided Laplace transform is used here as a symbolic tool for deriving causal memory kernels, operational identities, and zero–past transfer relations. If a nonzero prehistory, an infinite–past formulation, or a stationary frequency–response interpretation is required, then the corresponding analysis may be more naturally formulated using a prehistory–dependent convolution, a bilateral Laplace transform, a Fourier transform, or another suitable operational framework.

Therefore, the present DMFF should not be viewed as depending essentially on the one–sided Laplace transform. Rather, this transform provides one realization of the generator–based memory construction, particularly suitable for causal zero–past initial–value settings. Other transform settings may generate analogous DMK structures adapted to different physical or engineering interpretations.

3.7 *Engineering interpretation and frequency response*

In engineering applications, fractional differentiators and integrators are often interpreted through their frequency response. In the classical shift–invariant case, an ideal fractional differentiator of order α is characterized by the transfer function

$$H_{\alpha}(\omega) = (\mathrm{i}\omega)^{\alpha},$$

where ω denotes the angular frequency and $\mathrm{i}^2 = -1$. This representation leads to a straight–line Bode magnitude diagram with slope 20α dB/decade and constant phase $\alpha\pi/2$ for positive frequencies.

The proposed dynamic–memory framework contains this classical engineering interpretation as a special case. From Theorem 16, the DMFI satisfies

$$\mathscr{L}\{\mathscr{I}_{\Phi}^{\alpha} x\}(p) = \Phi(p,\Theta)^{-\alpha} X(p).$$

Therefore, in the zero–state shift–invariant frequency–response interpretation, where the Laplace variable is evaluated on the imaginary axis $p = \mathrm{i}\omega$ and initial/prehistory contributions are not included in the transfer relation, the corresponding frequency response of the DMFI is formally given by

$$G_{\Phi,\alpha}(\omega) = \Phi(\mathrm{i}\omega,\Theta)^{-\alpha}.$$

In particular, if $\Phi(p,\Theta) = p$, then

$$G_{\Phi,\alpha}(\omega) = (\mathrm{i}\omega)^{-\alpha},$$

which is precisely the classical fractional integrator of order α.

Similarly, from Theorem 17, the CDMFD satisfies

$$\mathscr{L}\{{}^{c}D_{\Phi}^{\alpha} x\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p) - x(0)\big).$$

Consequently, under the zero–state assumption, or equivalently when the initial/prehistory contribution is omitted from the transfer relation, we obtain

$$\mathscr{L}\{{}^{c}D_{\Phi}^{\alpha} x\}(p) = p\,\Phi(p,\Theta)^{\alpha-1} X(p).$$

Thus, the corresponding frequency response of the CDMFD is

$$H_{\Phi,\alpha}(\omega) = \mathrm{i}\omega\,\Phi(\mathrm{i}\omega,\Theta)^{\alpha-1}.$$

When the DMG is chosen as $\Phi(p,\Theta) = p$, we obtain

$$H_{\Phi,\alpha}(\omega) = \mathrm{i}\omega\,(\mathrm{i}\omega)^{\alpha-1} = (\mathrm{i}\omega)^{\alpha}.$$

Hence, the proposed CDMFD recovers the classical Liouville–type fractional differentiator in the shift–invariant frequency–response interpretation.

For positive frequencies $\omega > 0$, the classical choice $\Phi(p,\Theta) = p$ gives

$$|H_{\Phi,\alpha}(\omega)| = \omega^{\alpha}, \qquad \arg H_{\Phi,\alpha}(\omega) = \frac{\alpha\pi}{2}.$$

Accordingly,

$$20\log_{10}|H_{\Phi,\alpha}(\omega)| = 20\alpha\log_{10}\omega,$$

which corresponds to a straight line in the Bode magnitude diagram with slope 20α dB/decade. Therefore, the standard engineering interpretation of fractional differentiation is fully recovered when the DMG is reduced to the classical symbol p.

For a general DMG, the frequency response

$$H_{\Phi,\alpha}(\omega) = \mathrm{i}\omega\,\Phi(\mathrm{i}\omega,\Theta)^{\alpha-1}$$

does not necessarily produce a straight Bode line. Instead, it describes a generalized dynamic–memory differentiator or memory–filter operator whose amplitude and phase characteristics are governed by the selected generator. Hence, the proposed framework should be understood as a generator–based extension of the classical shift–invariant fractional differentiator: it preserves the standard engineering response

$$H_{\alpha}(\omega) = (\mathrm{i}\omega)^{\alpha}$$

when $\Phi(p,\Theta) = p$, while allowing more general frequency responses when the memory generator differs from the classical Laplace symbol p.

In this sense, the DMG acts as a frequency–domain memory symbol. If $\Phi(\mathrm{i}\omega,\Theta)$ behaves like $\mathrm{i}\omega$ over a prescribed frequency range, then the corresponding DMFD behaves like the ideal engineering fractional differentiator over that range. Conversely, deviations of $\Phi(\mathrm{i}\omega,\Theta)$ from $\mathrm{i}\omega$ encode non–classical memory, filtering, attenuation, crossover, or multiscale effects.

To illustrate the frequency–domain interpretation of the proposed framework, Figure 2 compares the Bode magnitude responses associated with several choices of the DMG. For the classical generator

$$\Phi(p,\Theta) = p,$$

the response is

$$H_{\Phi,\alpha}(\omega) = (\mathrm{i}\omega)^{\alpha},$$

and hence the Bode magnitude is a straight line with slope 20α dB/decade. For the tempered generator

$$\Phi(p,\Theta) = p + \lambda,$$

the corresponding response becomes

$$H_{\Phi,\alpha}(\omega) = \mathrm{i}\omega\,(\mathrm{i}\omega+\lambda)^{\alpha-1},$$

which produces a modified low-frequency behavior while approaching the classical fractional slope at high frequencies. For the hybrid generator

$$\Phi(p,\Theta) = (p+\lambda)^{\mu} + \eta p^{\nu},$$

the frequency response is given by

$$H_{\Phi,\alpha}(\omega) = \mathrm{i}\omega\,((\mathrm{i}\omega+\lambda)^{\mu} + \eta(\mathrm{i}\omega)^{\nu})^{\alpha-1}.$$

This response may exhibit crossover behavior between different frequency regimes, reflecting the multiscale memory structure encoded in the generator.

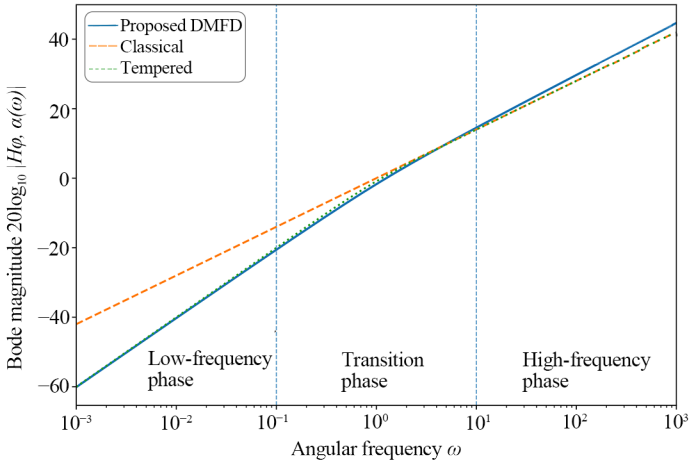


Figure 2. Bode magnitude comparison of the DMFD frequency response for different choices of the DMG. The classical generator $\Phi(p,\Theta) = p$ yields the ideal straight–line fractional differentiator, while tempered and hybrid generators produce modified frequency responses associated with non-classical memory effects

3.8 *Dynamic–memory Mittag–Leffler functions*

Mittag–Leffler functions play a fundamental role in fractional calculus and memory–dependent dynamical systems, similarly to the role of the exponential function in classical differential equations. In the proposed DMFF, the generalized Mittag–Leffler functions arise naturally through the DMG and provide a unified representation of generalized relaxation and hereditary phenomena.

Definition 14 The generalized DMMLF associated with the CDMFD is defined through the Laplace resolvent

$$E_{\Phi,\alpha}(\lambda,t) = \mathscr{L}^{-1}\left\{\frac{\Phi(p,\Theta)^{\alpha-1}}{p\Phi(p,\Theta)^{\alpha-1}-\lambda}\right\}(t),$$

whenever the inverse Laplace transform exists.

Theorem 18 Assume that the generalized DMMLF is normalized by $E_{\Phi,\alpha}(\lambda,0)=1$, and that the involved Laplace transforms are well defined. Then the function $E_{\Phi,\alpha}(\lambda,t)$ satisfies

$$ {}^{c}\mathscr{D}_{\Phi}^{\alpha} E_{\Phi,\alpha}(\lambda,t) = \lambda E_{\Phi,\alpha}(\lambda,t). $$

Proof. Let

$$ X(p) = \frac{\Phi(p,\Theta)^{\alpha-1}}{p\Phi(p,\Theta)^{\alpha-1}-\lambda}. $$

Using the Laplace–transform formula for the CDMFD, we obtain

$$ \mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha} E_{\Phi,\alpha}(\lambda,t)\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p)-E_{\Phi,\alpha}(\lambda,0)\big). $$

From the assumption $E_{\Phi,\alpha}(\lambda,0)=1$, we get

$$ \mathscr{L}\{{}^{c}\mathscr{D}_{\Phi}^{\alpha} E_{\Phi,\alpha}(\lambda,t)\}(p) = \Phi(p,\Theta)^{\alpha-1}\big(pX(p)-1\big). $$

Substituting the expression for $X(p)$ and simplifying, we get

$$ \Phi(p,\Theta)^{\alpha-1}\left(\frac{p\Phi(p,\Theta)^{\alpha-1}}{p\Phi(p,\Theta)^{\alpha-1}-\lambda}-1\right) = \lambda\frac{\Phi(p,\Theta)^{\alpha-1}}{p\Phi(p,\Theta)^{\alpha-1}-\lambda}. $$

Hence,

$$ \mathscr{L}\{{}^{c}D_{\Phi}^{\alpha}E_{\Phi,\alpha}(\lambda,t)\}(p) = \lambda\mathscr{L}\{E_{\Phi,\alpha}(\lambda,t)\}(p). $$

Applying the inverse Laplace transform completes the proof. □

Remark 35 The above definition of $E_{\Phi,\alpha}(\lambda,t)$ is associated with the CDMFD introduced in Definition 5 and with the Laplace representation established in Theorem 17. Therefore, the denominator

$$ p\Phi(p,\Theta)^{\alpha-1}-\lambda $$

is used in the relaxation resolvent. The alternative denominator

$$ \Phi(p,\Theta)^{\alpha}-\lambda $$

would correspond to a different auxiliary symbolic operator, not to the CDMFD considered in this paper.

Theorem 19 (Reduction to the Classical Mittag–Leffler Function) If $\Phi(p,\Theta) = p$, then

$$E_{\Phi,\alpha}(\lambda,t) = E_\alpha(\lambda t^\alpha),$$

where E_α denotes the classical one–parameter Mittag–Leffler function.

Proof. If $\Phi(p,\Theta) = p$, then

$$E_{\Phi,\alpha}(\lambda,t) = \mathscr{L}^{-1}\left\{\frac{p^{\alpha-1}}{p^\alpha - \lambda}\right\}(t),$$

which is precisely the classical Laplace–transform representation of the Mittag–Leffler function $E_\alpha(\lambda t^\alpha)$. □

Remark 36 The converse of Theorem 19 does not hold in general without additional uniqueness assumptions on the admissible generator. The theorem shows that the particular choice $\Phi(p,\Theta) = p$ is sufficient to recover the classical Mittag–Leffler function. However, the recovery of the classical Mittag–Leffler transform representation does not by itself force the generator to be exactly $\Phi(p,\Theta) = p$. Different admissible symbols may lead to equivalent transform representations on restricted classes of functions, after normalization, or under additional structural constraints. Therefore, a converse statement would require a separate uniqueness theorem for the Laplace–domain generator within a prescribed admissible class.

Remark 37 Different choices of the DMG produce different classes of generalized relaxation functions because the relaxation kernel is determined by the resolvent factor in the Laplace domain. Thus, changing Φ changes the poles, branch behavior, decay rate, and possible crossover patterns of the inverse transform. Power–law generators yield slowly decaying relaxation, tempered generators produce exponentially attenuated memory, and hybrid generators may combine several temporal behaviors. This is why the proposed framework can represent a wide spectrum of hereditary and multiscale dynamical behaviors.

4. Application to a nonlinear DMFLS

In this section, we illustrate the applicability of the DMFF through a class of nonlinear DMFLS involving coupled memory interactions, nonlinear forcing terms, and nonlocal effects. The considered model extends several existing fractional Langevin equations by incorporating generator–driven memory structures together with mixed dynamic–memory fractional operators.

4.1 *Model formulation and problem description*

We consider the nonlinear DMFLS of the form

$$ {}^c\mathscr{D}^\alpha_\Phi\Big({}^c\mathscr{D}^\beta_\Phi x(t) + \lambda(t)x(t)\Big) = \mathscr{A}x(t) + \mathscr{B}x(\sigma(t)) + \mathscr{F}\big(t,x(t),x(\sigma(t))\big), \quad t \in [0,\mathscr{T}], \tag{7}$$

subject to the initial conditions

$$x(0) = x_0, \qquad \mathscr{Y}(0) = \mathscr{Y}_0, \tag{8}$$

where $\mathscr{Y}(t) = {}^{c}\mathscr{D}_{\Phi}^{\beta} x(t) + \lambda(t)x(t)$. This formulation avoids assigning an independent point value to ${}^{c}\mathscr{D}_{\Phi}^{\beta} x$ at $t = 0$, which may be ambiguous when the generated kernel is singular at the origin. If the Caputo–type derivative admits a finite right limit at the origin, then one may equivalently write $\mathscr{Y}_0 = x_1 + \lambda(0)x_0$, where $x_1 = \lim_{t\to 0^+} {}^{c}\mathscr{D}_{\Phi}^{\beta} x(t)$. The unknown function $x : [0, \mathscr{T}] \to \mathbb{R}^d$ denotes the state vector, while $\alpha, \beta \in (0,1)$ represent the dynamic–memory fractional orders. The function $\lambda : [0, \mathscr{T}] \to \mathbb{R}$ is a continuous damping coefficient, and $\mathscr{A}, \mathscr{B} \in \mathbb{R}^{d\times d}$ are constant matrices describing linear interactions. Moreover, the mapping $\sigma : [0, \mathscr{T}] \to [0, \mathscr{T}]$ denotes a deviating argument that may account for delay, proportional delay, or more general hybrid memory effects, whereas $\mathscr{F} : [0, \mathscr{T}] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ represents a nonlinear forcing term. Fractional Langevin equations have been studied in connection with anomalous diffusion, relaxation processes, and systems with memory effects; see, for example, [34, 35].

The proposed model combines several features, including nested DMFO, memory effects generated through the DMG, Langevin–type damping terms, nonlinear forcing functions, and deviating arguments representing delayed or hybrid interactions. Therefore, system (7) includes several existing fractional Langevin models as particular cases, such as models with classical, tempered, nonsingular, or other generalized memory structures.

4.2 *Theoretical analysis*

To study solvability, we first rewrite the proposed DMFLS as an equivalent nonlinear integral equation. Since the general generated inverse identities established in Section 3 do not necessarily reduce to their classical forms, the following additional condition is imposed throughout this subsection.

Assumption 1 For each $\rho \in \{\alpha, \beta\}$, the selected admissible generator and the considered function class satisfy the left–inverse relation

$$\mathscr{I}_{\Phi}^{\rho}\, {}^{c}\mathscr{D}_{\Phi}^{\rho} z = z - z(0)$$

for every admissible function z.

Remark 38 Assumption 1 is an additional structural condition required only for the Volterra reduction developed in this subsection. It is satisfied in the classical normalized case $\Phi(p, \Theta) = p$ and, more generally, whenever the selected generated operators possess the stated left–inverse property on the considered function class. Without this condition, the general inverse identity remains

$$\mathscr{I}_{\Phi}^{\rho}\, {}^{c}\mathscr{D}_{\Phi}^{\rho} z = \mathscr{I}_{\Phi}^{1} z',$$

as established in Section 3.

Let

$$\mathscr{Y}(t) = {}^{c}\mathscr{D}_{\Phi}^{\beta} x(t) + \lambda(t)x(t).$$

Then system (7) can be written as

$${}^{c}\mathscr{D}_{\Phi}^{\alpha} \mathscr{Y}(t) = \mathscr{A}x(t) + \mathscr{B}x(\sigma(t)) + \mathscr{F}\big(t, x(t), x(\sigma(t))\big).$$

Under Assumption 1, applying $\mathscr{I}_{\Phi}^{\alpha}$ to both sides gives

$$\mathscr{Y}(t) = \mathscr{Y}_0 + \mathscr{I}_{\Phi}^{\alpha}\left(\mathscr{A}x + \mathscr{B}x(\sigma) + \mathscr{F}\left(\cdot, x, x(\sigma)\right)\right)(t),$$

where $\mathscr{Y}_0 := \mathscr{Y}(0)$. Equivalently,

$$^{c}\mathscr{D}_{\Phi}^{\beta}x(t) + \lambda(t)x(t) = \mathscr{Y}_0 + \mathscr{I}_{\Phi}^{\alpha}\left(\mathscr{A}x + \mathscr{B}x(\sigma) + \mathscr{F}\left(\cdot, x, x(\sigma)\right)\right)(t).$$

Applying $\mathscr{I}_{\Phi}^{\beta}$ and using Assumption 1, together with the semigroup relation for the generated fractional integrals, yields

$$x(t) - x_0 + \mathscr{I}_{\Phi}^{\beta}(\lambda x)(t) = \mathscr{Y}_0 \mathscr{I}_{\Phi}^{\beta} 1(t) + \mathscr{I}_{\Phi}^{\alpha+\beta}\left(\mathscr{A}x + \mathscr{B}x(\sigma) + \mathscr{F}\left(\cdot, x, x(\sigma)\right)\right)(t).$$

Therefore, the corresponding nonlinear Volterra–type integral equation is

$$\begin{aligned} x(t) = x_0 - \mathscr{I}_{\Phi}^{\beta}(\lambda x)(t) + \mathscr{Y}_0 \mathscr{J}_{\Phi}^{\beta}(t) \\ + \mathscr{I}_{\Phi}^{\alpha+\beta}\left(\mathscr{A}x + \mathscr{B}x(\sigma) + \mathscr{F}\left(\cdot, x, x(\sigma)\right)\right)(t), \end{aligned} \tag{9}$$

where

$$\mathscr{J}_{\Phi}^{\beta}(t) = \mathscr{I}_{\Phi}^{\beta} 1(t) = \int_0^t \Psi_{\beta}(t - s; \Theta)\, ds.$$

Equation (9) rewrites the original DMFLS as a nonlinear Volterra–type integral equation for the class of admissible generators satisfying Assumption 1. Within this class, the generated memory kernels can be treated through a common fixed–point framework.

Since

$$\mathscr{Y}(t) = {}^{c}\mathscr{D}_{\Phi}^{\beta}x(t) + \lambda(t)x(t),$$

the initial value is imposed on the Langevin expression itself. This formulation is appropriate for general generated kernels because the point value of ${}^{c}\mathscr{D}_{\Phi}^{\beta}x$ at the origin may not exist unless an additional right–limit or compatibility condition is imposed. If this right limit exists and is denoted by x_1, then

$$\mathscr{Y}_0 = x_1 + \lambda(0)x_0,$$

which connects the present formulation with the alternative notation used in some classical Langevin models.

To investigate existence and uniqueness, we formulate the problem as a fixed–point equation in the Banach space

$$\mathscr{X} = C([0,\mathscr{T}],\mathbb{R}^d),$$

equipped with the norm

$$\|x\|_{\mathscr{X}} = \sup_{t\in[0,\mathscr{T}]} \|x(t)\|.$$

Define $\mathbb{T} : \mathscr{X} \to \mathscr{X}$ by

$$\begin{aligned}(\mathbb{T}x)(t) &= x_0 - \mathscr{I}_{\Phi}^{\beta}(\lambda x)(t) + \mathscr{Y}_0 \mathscr{J}_{\Phi}^{\beta}(t) \\ &\quad + \mathscr{I}_{\Phi}^{\alpha+\beta}\left(\mathscr{A}x + \mathscr{B}x(\sigma) + \mathscr{F}\left(\cdot,x,x(\sigma)\right)\right)(t).\end{aligned} \tag{10}$$

Thus, mild solutions of system (7) are precisely the fixed points of $\mathbb{T}$.

Theorem 20 (Existence and uniqueness under Lipschitz–type conditions) Assume that Assumption 1 holds and that:
(H1) The function $\mathscr{F} : [0,\mathscr{T}] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is continuous.
(H2) There exists $L > 0$ such that

$$\|\mathscr{F}(t,u_1,v_1) - \mathscr{F}(t,u_2,v_2)\| \le L\left(\|u_1-u_2\| + \|v_1-v_2\|\right)$$

for every $t \in [0,\mathscr{T}]$ and $u_1,u_2,v_1,v_2 \in \mathbb{R}^d$.
(H3) The deviating argument $\sigma : [0,\mathscr{T}] \longrightarrow [0,\mathscr{T}]$ is continuous.
(H4) The generated integral operators $\mathscr{I}_{\Phi}^{\beta}$ and $\mathscr{I}_{\Phi}^{\alpha+\beta}$ are bounded on $\mathscr{X}$.
(H5) The following contraction condition holds:

$$\mathfrak{q} := \left\|\mathscr{I}_{\Phi}^{\beta}\right\| \|\lambda\|_{\infty} + \left\|\mathscr{I}_{\Phi}^{\alpha+\beta}\right\| \left(\|\mathscr{A}\| + \|\mathscr{B}\| + 2L\right) < 1.$$

Then system (7) admits a unique mild solution on $[0,\mathscr{T}]$.

Proof. Let $x,y \in \mathscr{X}$. From (10), we have

$$\begin{aligned}\|(\mathbb{T}x)(t) - (\mathbb{T}y)(t)\| &\le \left\|\mathscr{I}_{\Phi}^{\beta}\left(\lambda(x-y)\right)(t)\right\| \\ &\quad + \left\|\mathscr{I}_{\Phi}^{\alpha+\beta}\left(\mathscr{A}(x-y) + \mathscr{B}\big(x(\sigma)-y(\sigma)\big) + \mathscr{F}\left(\cdot,x,x(\sigma)\right) - \mathscr{F}\left(\cdot,y,y(\sigma)\right)\right)(t)\right\|.\end{aligned}$$

Since

$$\|x(\sigma(t)) - y(\sigma(t))\| \le \|x-y\|_{\mathscr{X}},$$

the Lipschitz condition gives

$$\left\|\mathscr{F}\left(t,x(t),x(\sigma(t))\right)-\mathscr{F}\left(t,y(t),y(\sigma(t))\right)\right\|\leq 2L\|x-y\|_{\mathscr{X}}.$$

Consequently,

$$\|\mathbb{T}x-\mathbb{T}y\|_{\mathscr{X}}\leq \mathfrak{q}\,\|x-y\|_{\mathscr{X}}.$$

Since $\mathfrak{q}<1$, the operator $\mathbb{T}$ is a contraction on $\mathscr{X}$. Banach's contraction principle therefore yields a unique fixed point of $\mathbb{T}$, which is the unique mild solution of system (7). □

Theorem 21 (Existence of mild solutions) Assume that Assumption 1 holds and that:
(G1) The function $\mathscr{F}:[0,\mathscr{T}]\times\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is continuous.
(G2) There exists a constant $\mathscr{M}>0$ such that

$$\|\mathscr{F}(t,u,v)\|\leq\mathscr{M}\,(1+\|u\|+\|v\|)$$

for every $t\in[0,\mathscr{T}]$ and $u,v\in\mathbb{R}^d$.
(G3) The generated kernels Ψ_β and $\Psi_{\alpha+\beta}$ belong to $L^1(0,\mathscr{T})$.
(G4) The deviating argument $\sigma:[0,\mathscr{T}]\longrightarrow[0,\mathscr{T}]$ is continuous.
(G5) The constant

$$\mathfrak{c}_1:=\left\|\mathscr{I}_\Phi^\beta\right\|\|\lambda\|_\infty+\left\|\mathscr{I}_\Phi^{\alpha+\beta}\right\|(\|\mathscr{A}\|+\|\mathscr{B}\|+2\mathscr{M})$$

satisfies

$$\mathfrak{c}_1<1.$$

Then system (7) admits at least one mild solution on $[0,\mathscr{T}]$.

Proof. Define $\mathfrak{c}_0$ by

$$\mathfrak{c}_0:=\|x_0\|+\|\mathscr{Y}_0\|\|\mathscr{J}_\Phi^\beta\|_\infty+\mathscr{M}\left\|\mathscr{I}_\Phi^{\alpha+\beta}\right\|.$$

Choose

$$R\geq\frac{\mathfrak{c}_0}{1-\mathfrak{c}_1},$$

and consider the closed, bounded, and convex set

$$B_R = \{x \in \mathscr{X} : \|x\|_{\mathscr{X}} \leq R\}.$$

For every $x \in B_R$, the growth condition in (G2) gives

$$\left\|\mathscr{F}\left(t, x(t), x(\sigma(t))\right)\right\| \leq \mathscr{M}(1+2R).$$

Hence,

$$\begin{aligned}\|\mathbb{T}x\|_{\mathscr{X}} &\leq \|x_0\| + \|\mathscr{Y}_0\|\|\mathscr{J}_{\Phi}^{\beta}\|_{\infty} + \left\|\mathscr{J}_{\Phi}^{\beta}\right\| \|\lambda\|_{\infty} R \\ &\quad + \left\|\mathscr{J}_{\Phi}^{\alpha+\beta}\right\| [(\|\mathscr{A}\| + \|\mathscr{B}\|)R + \mathscr{M}(1+2R)] \\ &= \mathfrak{c}_0 + \mathfrak{c}_1 R \leq R.\end{aligned}$$

Therefore,

$$\mathbb{T}(B_R) \subseteq B_R.$$

The continuity of $\mathscr{F}$, λ, and σ, together with the boundedness of the generated integral operators, implies that $\mathbb{T}$ is continuous on B_R.

Moreover, the functions appearing under the generated Volterra integrals are uniformly bounded for $x \in B_R$. Since $\Psi_{\beta}, \Psi_{\alpha+\beta} \in L^1(0, \mathscr{T})$, their zero extensions have translations that are continuous in the L^1–norm. It follows that the family $\mathbb{T}(B_R)$ is equicontinuous on $[0, \mathscr{T}]$. The preceding estimate also shows that this family is uniformly bounded. Hence, by the Arzelà–Ascoli theorem, $\mathbb{T}(B_R)$ is relatively compact in $\mathscr{X}$.

Therefore, $\mathbb{T}$ is a continuous compact self–map of B_R. Schauder's fixed point theorem yields at least one fixed point of $\mathbb{T}$, which is a mild solution of system (7). □

Remark 39 The hypotheses of the preceding existence theorem provide sufficient, but not necessary, conditions for the existence of a mild solution. In particular, the existence of a solution for a specific problem does not imply that all the stated continuity, growth, compactness, inverse, or kernel admissibility assumptions must hold. The conditions are imposed to ensure that the fixed–point operator is well defined, maps a suitable closed ball into itself, and possesses the compactness properties required by Schauder's fixed point theorem.

4.3 *Numerical approximation scheme*

In this subsection, we develop a predictor–corrector numerical scheme for approximating solutions of the system (7). The proposed approach adapts the classical Adams–Bashforth–Moulton methodology to the generator–based dynamic–memory setting. Classical predictor-corrector methods for fractional differential equations, such as [36], are typically formulated for prescribed fractional kernels, while the broader numerical literature [39] likewise develops numerical schemes around specified fractional operators. The present formulation instead accommodates memory effects generated through the DMG.

The numerical approximation of dynamic–memory fractional systems is generally more challenging than that of classical fractional equations, since the generated kernels

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{\Phi(p,\Theta)^{-\alpha}\}(t)$$

may exhibit singular, nonsingular, tempered, oscillatory, or multiscale memory behaviors. Consequently, numerical schemes designed for fixed power–law kernels cannot always be applied directly.

To address this issue, we develop a predictor–corrector approach based on the generated DMK rather than on a prescribed memory law. The proposed scheme extends the classical Adams–Bashforth–Moulton framework to generator–based DMFO , allowing different classes of memory effects to be treated within a unified computational procedure. In contrast to many existing methods, which often require separate formulations for different kernels, the present approach naturally accommodates singular, nonsingular, tempered, Mittag–Leffler, hybrid, and multiscale memory structures while preserving the convolution and semigroup properties of the continuous model.

Let $0 = t_0 < t_1 < \cdots < t_N = \mathscr{T}$, where

$$t_n = nh, \qquad h = \frac{\mathscr{T}}{N}.$$

Using the equivalent integral representation (9), we obtain

$$\begin{aligned} x(t_{n+1}) = x_0 &- \int_0^{t_{n+1}} \Psi_\beta(t_{n+1}-s;\Theta)\lambda(s)x(s)\,ds \\ &+ \int_0^{t_{n+1}} \Psi_{\alpha+\beta}(t_{n+1}-s;\Theta)\mathscr{G}(s,x(s))\,ds \\ &+ \mathscr{Y}_0\,\mathscr{J}_\Phi^\beta(t_{n+1}), \end{aligned} \tag{11}$$

where

$$\mathscr{G}(s,x(s)) = \mathscr{A}x(s) + \mathscr{B}x(\sigma(s)) + \mathscr{F}\big(s,x(s),x(\sigma(s))\big).$$

Since the generated kernels may be singular at the origin, the quadrature rule is formulated through integrated cell weights rather than through pointwise evaluations of $\Psi_\rho(0;\Theta)$. For $\rho \in \{\beta, \alpha+\beta\}$, define the predictor weights by

$$b_{n+1,j}^{(\rho)} := \int_{t_j}^{t_{j+1}} \Psi_\rho(t_{n+1}-s;\Theta)\,ds, \qquad j = 0,1,\ldots,n.$$

These weights are well defined whenever $\Psi_\rho \in L^1(0,\mathscr{T})$, including the case in which the kernel possesses an integrable singularity at the origin.

Let

$$\lambda_j := \lambda(t_j), \qquad x_j \approx x(t_j),$$

and set

$$\mathscr{G}_j := \mathscr{A} x_j + \mathscr{B}\widehat{x}_{\sigma,j} + \mathscr{F}\left(t_j, x_j, \widehat{x}_{\sigma,j}\right),$$

where $\widehat{x}_{\sigma,j}$ denotes an approximation of $x(\sigma(t_j))$. If $\sigma(t_j)$ is not a mesh point, its value may be obtained by interpolation from the previously computed states.

Using a piecewise constant product–integration approximation on each subinterval, the predictor is defined by

$$x^P_{n+1} = x_0 - \sum_{j=0}^{n} b^{(\beta)}_{n+1,j}\lambda_j x_j + \sum_{j=0}^{n} b^{(\alpha+\beta)}_{n+1,j}\mathscr{G}_j + \mathscr{Y}_0\, \mathscr{J}^{\beta}_{\Phi}(t_{n+1}). \tag{12}$$

To construct the corrector, the functions appearing under the Volterra integrals are approximated linearly on every subinterval. For $\rho \in \{\beta, \alpha+\beta\}$, define

$$a^{(\rho)}_{n+1,0} := \frac{1}{h}\int_{t_0}^{t_1} \Psi_\rho(t_{n+1}-s;\Theta)(t_1-s)\,ds,$$

$$a^{(\rho)}_{n+1,j} := \frac{1}{h}\int_{t_{j-1}}^{t_j} \Psi_\rho(t_{n+1}-s;\Theta)(s-t_{j-1})\,ds + \frac{1}{h}\int_{t_j}^{t_{j+1}} \Psi_\rho(t_{n+1}-s;\Theta)(t_{j+1}-s)\,ds,$$

for $j = 1, 2, \ldots, n$, and

$$a^{(\rho)}_{n+1,n+1} := \frac{1}{h}\int_{t_n}^{t_{n+1}} \Psi_\rho(t_{n+1}-s;\Theta)(s-t_n)\,ds.$$

At the new time level, define

$$\widehat{x}^P_{\sigma,n+1} \approx x\big(\sigma(t_{n+1})\big)$$

by interpolation, using the predictor x^P_{n+1} whenever the interpolation interval contains t_{n+1}, and set

$$\mathscr{G}^P_{n+1} = \mathscr{A} x^P_{n+1} + \mathscr{B}\widehat{x}^P_{\sigma,n+1} + \mathscr{F}\left(t_{n+1}, x^P_{n+1}, \widehat{x}^P_{\sigma,n+1}\right).$$

The corrected approximation is then given by

$$x_{n+1} = x_0 - \sum_{j=0}^{n} a_{n+1,j}^{(\beta)} \lambda_j x_j - a_{n+1,n+1}^{(\beta)} \lambda_{n+1} x_{n+1}^{P}$$

$$+ \sum_{j=0}^{n} a_{n+1,j}^{(\alpha+\beta)} \mathscr{G}_j + a_{n+1,n+1}^{(\alpha+\beta)} \mathscr{G}_{n+1}^{P}$$

$$+ \mathscr{Y}_0 \, \mathscr{J}_{\Phi}^{\beta}(t_{n+1}). \tag{13}$$

The numerical solution is computed successively. At each time level, the predictor (12) is first evaluated using the previously computed states. The predicted value is then employed in the endpoint contribution of the corrector (13). The resulting procedure is a product–integration predictor–corrector scheme whose weights are determined by the generated kernels.

Unlike a direct trapezoidal discretization, the present formulation does not require the pointwise value $\Psi_\rho(0;\Theta)$. Instead, the possible near–origin singularity is incorporated through the integrated weights $b_{n+1,j}^{(\rho)}$ and $a_{n+1,j}^{(\rho)}$. Consequently, the scheme can be applied to a broad class of admissible locally integrable kernels, including power–law, tempered, and hybrid generated memory kernels.

The proposed discretization provides a general computational template. Its consistency, stability, and convergence depend on the regularity of the solution and on the analytical behavior of the generated kernels, particularly near the origin, and require a separate analysis for each admissible kernel class.

Remark 40 Unlike many existing predictor–corrector methods designed for prescribed memory kernels, the proposed approach is driven by the generated memory structure induced by the DMG. As a result, the same numerical framework can treat diverse memory behaviors within a unified computational setting, without requiring kernel–specific reformulations.

Remark 41 A rigorous analysis of the convergence and stability of the proposed predictor–corrector scheme depends on the admissibility properties of the generated kernels and the regularity assumptions imposed on the DMG. The investigation of these properties is left for future work.

4.4 *Recovery and generation of existing fractional Langevin models*

The proposed DMFF can recover several existing fractional Langevin equations as special cases through appropriate choices of the DMG. It also allows the construction of more general memory–dependent Langevin systems, including models involving classical, tempered, nonsingular, Mittag–Leffler, and hybrid memory effects, within a common framework.

To illustrate this mechanism, we consider several representative choices of the DMG and examine the corresponding reductions of the proposed DMFLS. These examples demonstrate how different memory generators induce distinct hereditary laws and therefore produce different classes of Langevin dynamics while preserving a common analytical structure.

Example 3 (Recovery of the classical fractional Langevin model) Consider the DMG

$$\Phi(p,\Theta) = p.$$

Then the generated DMK becomes

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{p^{-\alpha}\}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Hence, the DMFI reduces to the classical Riemann–Liouville fractional integral while the associated DMFD recovers the standard Riemann–Liouville and Caputo fractional derivatives. Consequently, the proposed DMFLS recovers classical fractional Langevin dynamics with power–law hereditary effects; see [40].

Example 4 (Recovery of exponentially tempered Langevin models) Let

$$\Phi(p,\Theta) = p+\lambda, \ \ \lambda > 0.$$

Then

$$\Psi_\alpha(t;\Theta) = \mathscr{L}^{-1}\{(p+\lambda)^{-\alpha}\}(t) = \frac{e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)}.$$

Therefore,

$$(\mathscr{I}^{\alpha}_{\Phi}x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}x(s)\,ds.$$

Thus, the resulting Langevin model exhibits exponentially attenuated hereditary effects, similar to those arising in generalized and tempered fractional Langevin dynamics; see [41].

Example 5 (Recovery of generalized Mittag–Leffler memory) Suppose that the DMG generates kernels satisfying

$$\Psi_\alpha(t;\Theta) = t^{\alpha-1}E_{\mu,\alpha}(-\lambda t^{\mu}),$$

where $E_{\mu,\alpha}$ denotes the two–parameter Mittag–Leffler function. Then the corresponding dynamic–memory operators generate nonlocal interactions involving crossover and multiscale memory effects.

Consequently, the resulting DMFLS extends fractional Langevin models governed by Mittag–Leffler hereditary structures; see [42].

The above examples demonstrate that different selections of the DMG lead naturally to distinct classes of fractional Langevin models, including classical power–law memory, exponentially tempered memory, and generalized crossover hereditary effects. Therefore, the proposed DMFF provides a unified mechanism for constructing and analyzing broad families of fractional Langevin systems through the structure of the generated memory kernel itself.

5. Conclusion

This work established a generator–based dynamic–memory framework for fractional calculus in which hereditary effects are produced through a dynamic memory generator rather than imposed through a prescribed fixed kernel. The proposed approach offers a unified symbolic mechanism for constructing broad classes of memory–dependent fractional

operators while retaining a coherent analytical and operational structure. In this sense, the framework provides a flexible alternative to several existing generalized fractional formulations whose memory laws are fixed in advance.

Within this setting, dynamic–memory fractional integrals and the corresponding Riemann–Liouville and Caputo–type dynamic–memory fractional derivatives were introduced by means of inverse Laplace representations. The developed theory established the main structural properties of these operators, including admissibility conditions, semigroup relations for generated integrals, boundedness results, qualified inverse formulas, composition identities, and Laplace–domain operational representations. The same construction also led naturally to dynamic–memory Mittag–Leffler functions associated with the selected generator, thereby linking generated kernels with relaxation–type behavior in a systematic way.

Explicit formulas for the action of the proposed derivatives on power and polynomial functions were also derived. These formulas extend the classical fractional power laws to the generated–kernel setting and provide useful analytical tools for consistency analysis, approximation procedures, predictor–corrector methods, and future numerical schemes adapted to heterogeneous memory structures.

Another important feature of the developed theory is its ability to directly recover classical shift–invariant convolution operators, such as the Riemann–Liouville, Caputo, and tempered forms, and to represent several modern kernel–based memory models at the level of generated kernels or suitable transform extensions. Claims concerning non–shift–invariant operators are therefore understood in this qualified representational sense rather than as literal recoveries inside the basic Laplace–convolution setting.

The applicability of the framework was demonstrated through a nonlinear dynamic–memory fractional Langevin system involving coupled memory effects, nonlinear interactions, and generalized hereditary behavior. The resulting Volterra formulation, solvability analysis, and numerical approximation scheme illustrate how different memory mechanisms, including crossover and multiscale effects, can be incorporated within a single symbolic structure without the need to redefine the operator for each separate kernel.

Several directions remain open for future investigation. These include stochastic dynamic–memory models, variable–order and adaptive generators, numerical methods for generator–dependent operators, fractional optimal control problems, and partial differential equations with dynamic memory effects. Further developments may also address spectral theory, inverse problems, deeper admissibility criteria, and applications to anomalous diffusion, viscoelasticity, biological systems, and other complex media where memory behavior is heterogeneous or evolving.

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Data availability

No external datasets were used in this study. Any numerical results presented in the article were generated from the mathematical models, parameter settings, and numerical procedures described in the manuscript.

Conflict of interest

The author declares no competing financial interest.

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