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Research Article

(p,q)-Fuzzy $\mathcal{G}$-Open Sets and Associated Functions: A New Framework and Fundamental Properties

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Abstract: This study introduces new classes of fuzzy open sets, namely (p,q)-$F\mathcal{G}$-open (resp. (p,q)-$F\mathcal{G}P$-open, (p,q)-$F\mathcal{G}S$-open, (p,q)-$F\mathcal{G}\alpha$-open, and (p,q)-$F\mathcal{G}\gamma$-open) sets in double fuzzy topological spaces ($DFTSs$) based on the sense of Šostak. We conduct a detailed investigation of the relationships among these classes of open sets, supported by carefully constructed illustrative examples. Also, we propose and characterize the associated $DF\mathcal{G}$-interior and $DF\mathcal{G}$-closure operators. Subsequently, we define and analyze new classes of fuzzy functions based on (p,q)-$F\mathcal{G}$-open sets, referred to as $DF\mathcal{G}$-continuous and $DF\mathcal{G}$-irresolute functions within the framework of $DFTSs$ $(\mathcal{S}, \vartheta, \vartheta^*)$ and $(\mathcal{Z}, \zeta, \zeta^*)$. We also introduce the notions of $DF\mathcal{G}P$-continuous, $DF\mathcal{G}S$-continuous, $DF\mathcal{G}\alpha$-continuous, and $DF\mathcal{G}\gamma$-continuous functions, which constitute weaker forms of $DF\mathcal{G}$-continuity. As an application, we demonstrate that these newly defined continuity concepts generalize, extend, and unify several existing results in the theory of $DFTSs$. Moreover, we propose and discuss the concepts of $DFA\mathcal{G}$-continuity and $DFW\mathcal{G}$-continuity as additional weaker variants of $DF\mathcal{G}$-continuity. Finally, we establish new separation axioms, termed (p,q)-$F\mathcal{G}$-normal and (p,q)-$F\mathcal{G}$-regular spaces, formulated via (p,q)-$F\mathcal{G}$-closed sets.

Keywords: (p,q)-$F\mathcal{G}$-open set, $DF\mathcal{G}$-interior operator, $DF\mathcal{G}$-irresoluteness, $DF\mathcal{G}$-continuity, (p, q)-$F\mathcal{G}$-normal space, (p,q)-$F\mathcal{G}$-regular space

MSC: 03E72, 54A40, 54C05, 54C08, 54D15

1. Introduction and preliminaries

The increasing complexity of practical problems has created a growing demand for mathematical approaches capable of representing incomplete, imprecise, and ambiguous information. To address such situations, Zadeh [1] introduced fuzzy set theory, which extends the classical notion of a set by allowing the degree of membership to be represented explicitly. This framework has subsequently gained considerable importance in mathematics and various applied disciplines because of its ability to characterize and process information affected by vagueness and uncertainty. Its theoretical development and practical relevance have been extensively investigated in the literature [2–4], covering a broad spectrum of theoretical, methodological, and applied contexts. In particular, fuzzy-set-based approaches have been widely studied in decision-

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making and decision-support processes [4–6]. The mathematical foundations of fuzzy set theory have been systematically developed, providing a rigorous theoretical basis for the study and extension of fuzzy mathematical structures [7], while their broader relevance to computational and intelligent-system applications has also been discussed [8, 9]. Moreover, fuzzy set theory has been investigated in conjunction with other mathematical approaches for uncertainty representation, particularly soft set theory and rough set theory [10–12]. The development of fuzzy topology constitutes another important step in extending classical mathematical structures to uncertain environments. In 1968, Chang [13] introduced the concept of fuzzy topological spaces, establishing a basis for transferring fundamental topological notions to the fuzzy setting. This pioneering approach subsequently stimulated extensive investigations into fuzzy counterparts of various classical topological concepts [14, 15]. Nevertheless, the conventional formulation of fuzzy topology does not assign an explicit degree of openness to each fuzzy set. This issue was highlighted by Šostak [16], who observed that regarding a fuzzy topology merely as a crisp family of fuzzy sets leaves an essential aspect of fuzziness unrepresented in the notion of openness. This limitation has motivated the development of more flexible fuzzy topological structures in which the degree of openness can itself be incorporated into the underlying framework, providing a more expressive treatment of uncertainty in topological settings. In [16], Šostak proposed a refined concept of fuzzy topology, interpreting it through the lens of openness in fuzzy sets, thereby extending the earlier definition provided by Chang [13]. Several studies (e.g., [17–19]) subsequently developed and investigated specific topological properties within Šostak's framework, including extensions of compactness, fuzzy bitopological structures, and connectedness.

The notion of intuitionistic fuzzy sets was introduced by Atanassov [20], which is a generalization of fuzzy sets [1]. Coker [21] explored and examined the concept of an intuitionistic fuzzy topology in view of Chang [13]. However, the concept of an intuitionistic fuzzy topology in view of Šostak [16] was displayed by the authors of [22, 23]. Moreover, the name (intuitionistic) was replaced with the name (double) by Garcia and Rodabaugh [24]. The authors of [25–27] introduced and discussed the concepts of (p,q)-fuzzy semi-open ((p,q)-FS-open) sets, (p,q)-fuzzy pre-open ((p,q)-FP-open) sets, (p,q)-fuzzy α-open ((p,q)-$F\alpha$-open) sets, and (p,q)-fuzzy b-open ((p,q)-Fb-open) sets within the framework of double fuzzy topologies in view of Šostak [16]. Here, the concept of (p,q)-fuzzy b-open sets [27] has been renamed (p,q)-fuzzy γ-open sets. Thus, the symbol γ is used solely as a replacement for the symbol b, with no change in the underlying concept or its definitions. Subsequent studies have further investigated specific concepts related to intuitionistic and double fuzzy topological structures, including generalized intuitionistic fuzzy closed sets [28], double fuzzy irresolute functions [29], and generalized semi-compactness in double fuzzy topological spaces [30].

We lay out the remainder of this paper as follows:

• Section 1 contains several notions and results which we use in our study.

• In Section 2, we define new classes of fuzzy open sets, namely (p,q)-$F\mathcal{G}$-open (resp. (p,q)-$F\mathcal{G}P$-open, (p,q)-$F\mathcal{G}S$-open, (p,q)-$F\mathcal{G}\alpha$-open, and (p,q)-$F\mathcal{G}\gamma$-open) sets in $DFTSs$ based on fuzzy topologies in view of Šostak [16]. We conduct a detailed investigation of the relationships among these classes of open sets, supported by carefully constructed illustrative examples. Moreover, we introduce and characterize the associated $DF\mathcal{G}$-interior and $DF\mathcal{G}$-closure operators.

• In Section 3, we explore and discuss new fuzzy functions based on (p,q)-$F\mathcal{G}$-open sets and (p,q)-$F\mathcal{G}$-closed sets, referred to as $DF\mathcal{G}$-continuous (resp. $DF\mathcal{G}$-open and $DF\mathcal{G}$-closed) functions within the context of $DFTSs$. We also display the concepts of $DF\mathcal{G}P$-continuity (resp. $DF\mathcal{G}S$-continuity, $DF\mathcal{G}\alpha$-continuity, and $DF\mathcal{G}\gamma$-continuity), which represent weaker variants of $DF\mathcal{G}$-continuity.

• In Section 4, we introduce and study new fuzzy functions via (p,q)-$F\mathcal{G}$-closed sets and (p,q)-$F\mathcal{G}$-open sets, referred to as $DF\mathcal{G}$-irresolute (resp. $DF\mathcal{G}$-irresolute open, $DF\mathcal{G}$-irresolute closed, and $DF\mathcal{G}$-irresolute homeomorphism) functions within the context of $DFTSs$. However, the relationships between these classes of functions are introduced with the aid of examples.

• In Section 5, we display and discuss the concepts of $DFA\mathcal{G}$-continuity and $DFW\mathcal{G}$-continuity as additional weaker variants of $DF\mathcal{G}$-continuity within the context of $DFTSs$. We also present and examine new fuzzy separation axioms, referred to as (p,q)-$F\mathcal{G}$-regular (resp. (p,q)-$F\mathcal{G}$-normal) spaces based on (p,q)-$F\mathcal{G}$-closed sets.

• In Section 6, we give conclusions and future studies.

In this paper, non empty sets will be denoted by $\mathscr{S}$, $\mathscr{X}$, $\mathscr{Q}$, etc. $I^{\mathscr{S}}$ represents the collection of all fuzzy sets on $\mathscr{S}$ (where $I_\circ = (0,1]$, $I_1 = [0,1)$, and $I = [0,1]$), and for $t \in I$, $\underline{t}(s) = t$, for each $s \in \mathscr{S}$. On $\mathscr{S}$, for each fuzzy set

$\Re \in I^{\mathscr{S}}$, $\Re^c(s) = 1 - \Re(s)$, for any $s \in \mathscr{S}$. A fuzzy point s_t is a fuzzy set, is defined as follows: $s_t(u) = t$ if $u = s$, and $s_t(u) = 0$ for any $u \in \mathscr{S} - \{s\}$. $P_t(S)$ represents the collection of all fuzzy points on $\mathscr{S}$. Also, we say that s_t belongs to $\Re$ ($s_t \in \Re$), if $t \leq \Re(s)$. Moreover, a fuzzy set $\Im$ is a quasi-coincident with a fuzzy set $\Re$ ($\Im \, \Lambda \, \Re$), if there is $s \in S$, such that $\Im(s) + \Re(s) > 1$. Otherwise, $\Im$ is not a quasi-coincident with $\Re$ ($\Im \, \overline{\Lambda} \, \Re$).

Lemma 1 [31] Let $\Im, \Re \in I^S$. Thus, we have the following:

(1) $\Im \, \Lambda \, \Re$ iff there is $s_t \in \Im$ such that $s_t \, \Lambda \, \Re$.

(2) If $\Im \, \Lambda \, \Re$, $\Im \wedge \Re \neq \underline{0}$.

(3) $\Im \, \overline{\Lambda} \, \Re$ iff $\Im \leq \Re^c$.

(4) $\Im \leq \Re$ iff $s_t \in \Im$ implies $s_t \in \Re$ iff $s_t \, \Lambda \, \Im$ implies $s_t \, \Lambda \, \Re$ iff $s_t \, \overline{\Lambda} \, \Re$ implies $s_t \, \overline{\Lambda} \, \Im$.

Definition 1 [22, 27, 29] A double fuzzy topology on S is a pair (ϑ, ϑ^*) of the mappings $\vartheta, \vartheta^* : I^S \longrightarrow I$, which satisfy the following conditions:

(1) $\vartheta(\Re) + \vartheta^*(\Re) \leq 1$, for any $\Re \in I^S$.

(2) $\vartheta(\Re \wedge \partial) \geq \vartheta(\Re) \wedge \vartheta(\partial)$ and $\vartheta^*(\Re \wedge \partial) \leq \vartheta^*(\Re) \vee \vartheta^*(\partial)$, for any $\Re, \partial \in I^S$.

(3) $\vartheta(\bigvee_{i \in \Gamma} \Re_i) \geq \bigwedge_{i \in \Gamma} \vartheta(\Re_i)$ and $\vartheta^*(\bigvee_{i \in \Gamma} \Re_i) \leq \bigvee_{i \in \Gamma} \vartheta^*(\Re_i)$, for any $\{\Re_i\}_{i \in \Gamma} \subset I^S$.

Thus, $(S, \vartheta, \vartheta^*)$ is said to be a double fuzzy topological space ($DFTS$) in view of Šostak [16].

Definition 2 [23, 25, 29] On $(S, \vartheta, \vartheta^*)$, for any $\Re \in I^S$, $p \in I_\circ$, and $q \in I_1$, we introduce DF-operators C_ϑ and I_ϑ : $I^S \times I_\circ \times I_1 \to I^S$ as follows:

$$C_\vartheta(\Re, p, q) = \bigwedge \{\partial \in I^S : \Re \leq \partial, \ \vartheta(\partial^c) \geq p, \vartheta^*(\partial^c) \leq q\}.$$

$$I_\vartheta(\Re, p, q) = \bigvee \{\partial \in I^S : \partial \leq \Re, \ \vartheta(\partial) \geq p, \vartheta^*(\partial) \leq q\}.$$

Proposition 1 [23, 25, 29] On $(S, \vartheta, \vartheta^*)$, for any $\Re \in I^S$, $p \in I_\circ$, and $q \in I_1$, thus

(1) $C_\vartheta(\Re^c, p, q) = (I_\vartheta(\Re, p, q))^c$;

(2) $I_\vartheta(\Re^c, p, q) = (C_\vartheta(\Re, p, q))^c$.

Definition 3 [25, 27] Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$, $p \in I_\circ$, and $q \in I_1$. A fuzzy set $\Re \in I^S$ is called

(1) (p,q)-F-open if $\Re = I_\vartheta(\Re, p, q)$;

(2) (p,q)-FP-open if $\Re \leq I_\vartheta(C_\vartheta(\Re, p, q), p, q)$;

(3) (p,q)-FS-open if $\Re \leq C_\vartheta(I_\vartheta(\Re, p, q), p, q)$;

(4) (p,q)-FR-open if $\Re = I_\vartheta(C_\vartheta(\Re, p, q), p, q)$;

(5) (p,q)-$F\alpha$-open if $\Re \leq I_\vartheta(C_\vartheta(I_\vartheta(\Re, p, q), p, q), p, q)$;

(6) (p,q)-$F\gamma$-open if $\Re \leq C_\vartheta(I_\vartheta(\Re, p, q), p, q) \vee I_\vartheta(C_\vartheta(\Re, p, q), p, q)$.

Definition 4 [25–27] On $(S, \vartheta, \vartheta^*)$, for any $\Re \in I^S$, $p \in I_\circ$, and $q \in I_1$, we introduce DF-operators PI_ϑ, SI_ϑ, αI_ϑ, and $\gamma I_\vartheta : I^S \times I_\circ \times I_1 \to I^S$ as follows:

$$PI_\vartheta(\Re, p, q) = \bigvee \{\partial \in I^S : \partial \leq \Re, \ \partial \text{ is } (p,q)\text{-FP-open}\}.$$

$$SI_\vartheta(\Re, p, q) = \bigvee \{\partial \in I^S : \partial \leq \Re, \ \partial \text{ is } (p,q)\text{-FS-open}\}.$$

$$\alpha I_\vartheta(\Re, p, q) = \bigvee \{\partial \in I^S : \partial \leq \Re, \ \partial \text{ is } (p,q)\text{-F}\alpha\text{-open}\}.$$

$$\gamma I_\vartheta(\Re, p, q) = \bigvee \{\partial \in I^S : \partial \leq \Re, \ \partial \text{ is } (p,q)\text{-F}\gamma\text{-open}\}.$$

Definition 5 [23, 27, 29] A fuzzy function $\varphi : (S, \vartheta, \vartheta^*) \longrightarrow (Z, \zeta, \zeta^*)$ is called
(1) DF-continuous if $\vartheta(\varphi^{-1}(\partial)) \geq \zeta(\partial)$ and $\vartheta^*(\varphi^{-1}(\partial)) \leq \zeta^*(\partial)$, for any $\partial \in I^Z$;
(2) DF-open if $\zeta(\varphi(\Re)) \geq \vartheta(\Re)$ and $\zeta^*(\varphi(\Re)) \leq \vartheta^*(\Re)$, for any $\Re \in I^S$;
(3) DF-closed if $\zeta((\varphi(\Re))^c) \geq \vartheta(\Re^c)$ and $\zeta^*((\varphi(\Re))^c) \leq \vartheta^*(\Re^c)$, for any $\Re \in I^S$.

Definition 6 [25–27] A fuzzy function $\varphi : (S, \vartheta, \vartheta^*) \longrightarrow (Z, \zeta, \zeta^*)$ is called DFP-continuous (resp. DFS-continuous, $DF\alpha$-continuous, and $DF\gamma$-continuous) if $\varphi^{-1}(\partial)$ is (p,q)-FP-open (resp. (p,q)-FS-open, (p,q)-$F\alpha$-open, and (p,q)-$F\gamma$-open) for any $\partial \in I^Z$, with $\zeta(\partial) \geq p$ and $\zeta^*(\partial) \leq q$.

For some basic results and concepts that will be utilized in the subsequent sections, we refer the reader to [25–27].

2. On (p,q)-fuzzy $\mathscr{G}$-open sets

In this section, we propose new classes of fuzzy open sets, namely (p,q)-$F\mathscr{G}$-open (resp. (p,q)-$F\mathscr{G}P$-open, (p,q)-$F\mathscr{G}S$-open, (p,q)-$F\mathscr{G}\alpha$-open, and (p,q)-$F\mathscr{G}\gamma$-open) sets in $DFTSs$ based on the sense of Šostak. We conduct a detailed investigation of the relationships among these classes of open sets, supported by carefully constructed illustrative examples. In addition, we define and characterize the associated $DF\mathscr{G}$-interior and $DF\mathscr{G}$-closure operators.

Definition 7 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$, $\Re \in I^S$, $p \in I_\circ$, and $q \in I_1$. A fuzzy set $\Re$ is called (p,q)-$F\mathscr{G}$-open (resp. (p,q)-$F\mathscr{G}$-closed) if $\Re \leq I_\vartheta((\Re \vee \Im), p, q)$ (resp. $C_\vartheta((\Re \wedge \Im), p, q) \leq \Re$) for each $\Im \in I^S$ with $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$ (resp. $\vartheta(\Im^c) \geq p$ and $\vartheta^*(\Im^c) \leq q$), such that $\underline{1} \neq \Im \neq \underline{0}$.

Remark 1 The complement of (p,q)-$F\mathscr{G}$-open set is (p,q)-$F\mathscr{G}$-closed.

Lemma 2 Each (p,q)-F-open (resp. (p,q)-F-closed) set is (p,q)-$F\mathscr{G}$-open (resp. (p,q)-$F\mathscr{G}$-closed).

Proof. Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$ and $\Re \in I^S$ with $\vartheta(\Re) \geq p$ and $\vartheta^*(\Re) \leq q$. Therefore, $\Re = I_\vartheta(\Re, p, q) \leq I_\vartheta((\Re \vee \Im), p, q)$ for each $\Im \in I^S$ with $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$, such that $\underline{1} \neq \Im \neq \underline{0}$. Thus, $\Re$ is (p,q)-$F\mathscr{G}$-open. The other case also follows similar lines. □

Remark 2 The converse of Lemma 2 fails, as demonstrated in Example 1.

Example 1 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.5}, \frac{s_2}{0.7}, \frac{s_3}{0.9}\}$, $\Re = \{\frac{s_1}{0.4}, \frac{s_2}{0.3}, \frac{s_3}{0.9}\}$. Define $\vartheta, \vartheta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, $\Re$ is an $(\frac{1}{3}, \frac{1}{3})$-$F\mathscr{G}$-open set, but it is not $(\frac{1}{3}, \frac{1}{3})$-$F$-open. Also, $\Re^c$ is an $(\frac{1}{3}, \frac{1}{3})$-$F\mathscr{G}$-closed set, but it is not $(\frac{1}{3}, \frac{1}{3})$-$F$-closed.

Corollary 1 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$, $p \in I_\circ$, and $q \in I_1$. Thus,
(1) the union of (p,q)-$F\mathscr{G}$-open sets is (p,q)-$F\mathscr{G}$-open;
(2) the intersection of (p,q)-$F\mathscr{G}$-closed sets is (p,q)-$F\mathscr{G}$-closed.

Proof. This can be readily established using Definition 7. □

Definition 8 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$, $\Re \in I^S$, $p \in I_\circ$, and $q \in I_1$. A fuzzy set $\Re$ is called (p,q)-$F\mathscr{G}P$-open (resp. (p,q)-$F\mathscr{G}S$-open, (p,q)-$F\mathscr{G}\alpha$-open, and (p,q)-$F\mathscr{G}\gamma$-open) if $\Re \leq PI_\vartheta((\Re \vee \Im), p, q)$ (resp. $\Re \leq SI_\vartheta((\Re \vee \Im), p, q)$, $\Re \leq \alpha I_\vartheta((\Re \vee \Im), p, q)$, and $\Re \leq \gamma I_\vartheta((\Re \vee \Im), p, q)$) for each $\Im \in I^S$ with $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$, such that $\underline{1} \neq \Im \neq \underline{0}$.

Remark 3 The complement of (p,q)-$F\mathscr{G}P$-open (resp. (p,q)-$F\mathscr{G}S$-open, (p,q)-$F\mathscr{G}\alpha$-open, and (p,q)-$F\mathscr{G}\gamma$-open) set is (p,q)-$F\mathscr{G}P$-closed (resp. (p,q)-$F\mathscr{G}S$-closed, (p,q)-$F\mathscr{G}\alpha$-closed, and (p,q)-$F\mathscr{G}\gamma$-closed).

Remark 4 Based on the definitions outlined earlier, the following diagram is established.

$$\begin{array}{ccccc}
(\mathbf{p},\mathbf{q})\text{-}\mathbf{F}\mathscr{G}\text{-}\mathbf{open} & \longrightarrow & (\mathbf{p},\mathbf{q})\text{-}\mathbf{F}\mathscr{G}\mathbf{ff}\text{-}\mathbf{open} & \longrightarrow & (\mathbf{p},\mathbf{q})\text{-}\mathbf{F}\mathscr{G}\mathbf{S}\text{-}\mathbf{open} \\
 & & \downarrow & & \downarrow \\
 & & (\mathbf{p},\mathbf{q})\text{-}\mathbf{F}\mathscr{G}\mathbf{P}\text{-}\mathbf{open} & \longrightarrow & (\mathbf{p},\mathbf{q})\text{-}\mathbf{F}\mathscr{G}\mathbf{fl}\text{-}\mathbf{open}
\end{array}$$

Remark 5 Examples 2-6 illustrate that the converse of Remark 4 fails.

Example 2 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define ∂, $\Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.2}, \frac{s_2}{0.7}, \frac{s_3}{0.4}\}$, $\Re = \{\frac{s_1}{0.7}, \frac{s_2}{0.9}, \frac{s_3}{0.1}\}$. Define $\vartheta, \vartheta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, $\Re$ is an $(\frac{1}{3}, \frac{1}{3})$-$F\mathscr{G}\alpha$-open set, but it is not $(\frac{1}{3}, \frac{1}{3})$-$F\mathscr{G}$-open.

Example 3 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define ∂, $\Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.2}, \frac{s_3}{0.2}\}$, $\Re = \{\frac{s_1}{0.2}, \frac{s_2}{0.3}, \frac{s_3}{0.7}\}$. Define $\vartheta, \vartheta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, $\Re$ is an $(\frac{1}{4}, \frac{1}{4})$-$F\mathscr{G}S$-open set, but it is neither $(\frac{1}{4}, \frac{1}{4})$-$F\mathscr{G}\alpha$-open nor $(\frac{1}{4}, \frac{1}{4})$-$F\mathscr{G}P$-open.

Example 4 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define ∂, $\Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.3}, \frac{s_3}{0.4}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.8}, \frac{s_3}{0.7}\}$. Define $\vartheta, \vartheta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, $\Re$ is an $(\frac{1}{2}, \frac{1}{2})$-$F\mathscr{G}P$-open set, but it is neither $(\frac{1}{2}, \frac{1}{2})$-$F\mathscr{G}\alpha$-open nor $(\frac{1}{2}, \frac{1}{2})$-$F\mathscr{G}S$-open.

Remark 6 From Examples 3 and 4, (p,q)-$F\mathscr{G}P$-open sets and (p,q)-$F\mathscr{G}S$-open sets are independent of each other.

Example 5 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define ∂, $\Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.3}, \frac{s_3}{0.1}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.5}, \frac{s_3}{0.7}\}$. Define $\vartheta, \vartheta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla)=\begin{cases}1, & \text{if} \quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{5}, & \text{if} \quad \nabla=\partial,\\ 0, & \text{otherwise,}\end{cases} \qquad \vartheta^*(\nabla)=\begin{cases}0, & \text{if} \quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{5}, & \text{if} \quad \nabla=\partial,\\ 1, & \text{otherwise.}\end{cases}$$

Thus, $\Re$ is an $(\frac{1}{5},\frac{1}{5})$-$F\mathcal{G}\gamma$-open set, but it is not $(\frac{1}{5},\frac{1}{5})$-$F\mathcal{G}P$-open.

Example 6 Let $\mathcal{S}=\{s_1,s_2,s_3\}$ and define ∂, $\Re\in I^S$ as follows: $\partial=\{\frac{s_1}{0.2},\frac{s_2}{0.3},\frac{s_3}{0.4}\}$, $\Re=\{\frac{s_1}{0.3},\frac{s_2}{0.8},\frac{s_3}{0.8}\}$. Define $\vartheta,\vartheta^*:I^S\longrightarrow I$ as follows:

$$\vartheta(\nabla)=\begin{cases}1, & \text{if} \quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \nabla=\partial,\\ 0, & \text{otherwise,}\end{cases} \qquad \vartheta^*(\nabla)=\begin{cases}0, & \text{if} \quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{4}, & \text{if} \quad \nabla=\partial,\\ 1, & \text{otherwise.}\end{cases}$$

Thus, $\Re$ is an $(\frac{1}{4},\frac{1}{4})$-$F\mathcal{G}\gamma$-open set, but it is not $(\frac{1}{4},\frac{1}{4})$-$F\mathcal{G}S$-open.

Definition 9 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an *DFTS*, $\Re\in I^S$, $p\in I_\circ$, and $q\in I_1$. We introduce *DF*$\mathcal{G}$-closure operator $\mathcal{G}C_\vartheta:I^S\times I_\circ\times I_1\to I^S$ as follows:

$$\mathcal{G}C_\vartheta(\Re,p,q)=\bigwedge\{\Im\in I^S:\Re\leq\Im,\ \Im \text{ is } (p,q)\text{-F}\mathcal{G}\text{-closed}\}.$$

Proposition 2 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an *DFTS*, $\Re\in I^S$, $p\in I_\circ$, and $q\in I_1$. A fuzzy set $\Re$ is (p,q)-$F\mathcal{G}$-closed iff $\mathcal{G}C_\vartheta(\Re,p,q)=\Re$.

Proof. This can be readily established using Definition 9. □

Theorem 1 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an *DFTS* and $\Re,\Im\in I^S$, and let $p\in I_\circ$, $q\in I_1$. An *DF*$\mathcal{G}$-closure operator $\mathcal{G}C_\vartheta:I^S\times I_\circ\times I_1\to I^S$ possesses the following characteristics:

(1) $\mathcal{G}C_\vartheta(\underline{0},p,q)=\underline{0}$.

(2) $\Re\leq\mathcal{G}C_\vartheta(\Re,p,q)\leq C_\vartheta(\Re,p,q)$.

(3) $\mathcal{G}C_\vartheta(\Re,p,q)\leq\mathcal{G}C_\vartheta(\Im,p,q)$ if $\Re\leq\Im$.

(4) $\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)=\mathcal{G}C_\vartheta(\Re,p,q)$.

(5) $\mathcal{G}C_\vartheta(\Re\vee\Im,p,q)\geq\mathcal{G}C_\vartheta(\Re,p,q)\vee\mathcal{G}C_\vartheta(\Im,p,q)$.

(6) $\mathcal{G}C_\vartheta(C_\vartheta(\Re,p,q),p,q)=C_\vartheta(\Re,p,q)$.

Proof. Statements (1), (2), and (3) can be readily established using Definition 9.

(4) From (2) and (3), we have $\mathcal{G}C_\vartheta(\Re,p,q)\leq\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)$. Now, we show that

$$\mathcal{G}C_\vartheta(\Re,p,q)\geq\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q).$$

If $\mathcal{G}C_\vartheta(\Re,p,q)$ does not contain $\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)$, there are $s\in S$ and $t\in(0,1)$ with

$$\mathcal{G}C_\vartheta(\Re,p,q)(s)<t<\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)(s). \qquad (W)$$

Since $\mathcal{G}C_\vartheta(\Re,p,q)(s)<t$, by Definition 9, there is $\partial\in I^S$ as (p,q)-$F\mathcal{G}$-closed and $\Re\leq\partial$ with $\mathcal{G}C_\vartheta(\Re,p,q)(s)\leq\partial(s)<t$. Since $\Re\leq\partial$, then $\mathcal{G}C_\vartheta(\Re,p,q)\leq\partial$. Again, this is by the definition of $\mathcal{G}C_\vartheta$, $\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)\leq\partial$. Hence, $\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)(s)\leq\partial(s)<t$, which is a contradiction for (W). Thus, $\mathcal{G}C_\vartheta(\Re,p,q)\geq\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)$. Therefore, $\mathcal{G}C_\vartheta(\mathcal{G}C_\vartheta(\Re,p,q),p,q)=\mathcal{G}C_\vartheta(\Re,p,q)$.

(5) Since $\Re\leq\Re\vee\Im$ and $\Im\leq\Re\vee\Im$, hence by (3), $\mathcal{G}C_\vartheta(\Re,p,q)\leq\mathcal{G}C_\vartheta(\Re\vee\Im,p,q)$, and $\mathcal{G}C_\vartheta(\Im,p,q)\leq\mathcal{G}C_\vartheta(\Re\vee\Im,p,q)$. Thus, $\mathcal{G}C_\vartheta(\Re\vee\Im,p,q)\geq\mathcal{G}C_\vartheta(\Re,p,q)\vee\mathcal{G}C_\vartheta(\Im,p,q)$.

(6) Utilizing Proposition 2 and the property that $C_\vartheta(\Re,p,q)$ is (p,q)-$F\mathcal{G}$-closed, we deduce that

$$\mathcal{G}C_\vartheta(C_\vartheta(\Re,p,q),p,q)=C_\vartheta(\Re,p,q).$$

□

Definition 10 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an $DFTS$, $\Re\in I^S$, $p\in I_\circ$, and $q\in I_1$. We introduce $DF\mathcal{G}$-interior operator $\mathcal{G}I_\vartheta:I^S\times I_\circ\times I_1\to I^S$ as follows:

$$\mathcal{G}I_\vartheta(\Re,p,q)=\bigvee\{\Im\in I^S:\Im\leq\Re,\ \Im\text{ is }(p,q)\text{-F}\mathcal{G}\text{-open}\}.$$

Proposition 3 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an $DFTS$, $\Re\in I^S$, $p\in I_\circ$, and $q\in I_1$. Thus, we have:

(1) $(\mathcal{G}I_\vartheta(\Re,p,q))^c=\mathcal{G}C_\vartheta(\Re^c,p,q)$;

(2) $(\mathcal{G}C_\vartheta(\Re,p,q))^c=\mathcal{G}I_\vartheta(\Re^c,p,q)$.

Proof. (1) For each $\Re\in I^S$, $(\mathcal{G}I_\vartheta(\Re,p,q))^c=[\bigvee\{\Im^c\in I^S:\Im^c\leq\Re,\ \Im^c\text{ is }(p,q)\text{-F}\mathcal{G}\text{-open}\}]^c=\bigwedge\{\Im\in I^S:\Re^c\leq\Im,\ \Im\text{ is }(p,q)\text{-F}\mathcal{G}\text{-closed}\}=\mathcal{G}C_\vartheta(\Re^c,p,q)$.

(2) The argument is analogous to that presented in (1). □

Proposition 4 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an $DFTS$, $\Re\in I^S$, $p\in I_\circ$, and $q\in I_1$. A fuzzy set $\Re$ is (p,q)-$F\mathcal{G}$-open iff $\mathcal{G}I_\vartheta(\Re,p,q)=\Re$.

Proof. This can be readily established using Definition 10. □

Theorem 2 Let $(\mathcal{S},\vartheta,\vartheta^*)$ be an $DFTS$ and $\Re,\Im\in I^S$, and let $p\in I_\circ$, $q\in I_1$. An $DF\mathcal{G}$-interior operator $\mathcal{G}I_\vartheta:I^S\times I_\circ\times I_1\to I^S$ possesses the following characteristics:

(1) $\mathcal{G}I_\vartheta(\underline{1},p,q)=\underline{1}$.

(2) $I_\vartheta(\Re,p,q)\leq\mathcal{G}I_\vartheta(\Re,p,q)\leq\Re$.

(3) $\mathcal{G}I_\vartheta(\Re,p,q)\leq\mathcal{G}I_\vartheta(\Im,p,q)$ if $\Re\leq\Im$.

(4) $\mathcal{G}I_\vartheta(\mathcal{G}I_\vartheta(\Re,p,q),p,q)=\mathcal{G}I_\vartheta(\Re,p,q)$.

(5) $\mathcal{G}I_\vartheta(\Re,p,q)\wedge\mathcal{G}I_\vartheta(\Im,p,q)\geq\mathcal{G}I_\vartheta(\Re\wedge\Im,p,q)$.

Proof. The proof follows by applying the same method used in Theorem 1. □

3. Double fuzzy $\mathcal{G}$-continuity

In this section, we explore and discuss new fuzzy functions via (p,q)-$F\mathcal{G}$-open sets and (p,q)-$F\mathcal{G}$-closed sets, referred to as $DF\mathcal{G}$-continuous (resp. $DF\mathcal{G}$-open and $DF\mathcal{G}$-closed) functions within the context of $DFTSs$ $(\mathcal{S},\vartheta,\vartheta^*)$ and $(\mathcal{Z},\zeta,\zeta^*)$ based on the sense of Šostak. Moreover, we display the concepts of $DF\mathcal{G}P$-continuity (resp. $DF\mathcal{G}S$-continuity, $DF\mathcal{G}\alpha$-continuity, and $DF\mathcal{G}\gamma$-continuity), which represent weaker variants of $DF\mathcal{G}$-continuity. The relationships between these classes of functions are introduced with the aid of examples.

Definition 11 Let $(\mathscr{S},\vartheta,\vartheta^*)$ and $(\mathscr{Z},\zeta,\zeta^*)$ be DFTSs, $\Re\in I^S$, $\Im\in I^Z$, $p\in I_\circ$, and $q\in I_1$. A fuzzy function $\varphi: I^S\longrightarrow I^Z$ is called

(1) $DF\mathscr{G}$-continuous if $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-open, for each $\Im$ with $\zeta(\Im)\geq p$ and $\zeta^*(\Im)\leq q$;

(2) $DF\mathscr{G}$-open if $\varphi(\Re)$ is (p,q)-$F\mathscr{G}$-open, for each $\Re$ with $\vartheta(\Re)\geq p$ and $\vartheta^*(\Re)\leq q$;

(3) $DF\mathscr{G}$-closed if $\varphi(\Re)$ is (p,q)-$F\mathscr{G}$-closed, for each $\Re$ with $\vartheta(\Re^c)\geq p$ and $\vartheta^*(\Re^c)\leq q$.

Lemma 3 (1) Each DF-continuous function is $DF\mathscr{G}$-continuous.

(2) Each DF-open function is $DF\mathscr{G}$-open.

(3) Each DF-closed function is $DF\mathscr{G}$-closed.

Proof. This can be readily established using Lemma 2. □

Remark 7 The converse of Lemma 3 fails, as demonstrated in Example 7.

Example 7 Let $\mathscr{S}=\{s_1,s_2,s_3\}$ and define ∂, $\Re\in I^S$ as follows: $\partial=\{\frac{s_1}{0.5},\frac{s_2}{0.7},\frac{s_3}{0.9}\}$, $\Re=\{\frac{s_1}{0.4},\frac{s_2}{0.3},\frac{s_3}{0.9}\}$. Define $\vartheta,\vartheta^*,\zeta,\zeta^*:I^S\longrightarrow I$ as follows:

$$\vartheta(\nabla)=\begin{cases}1, & \text{if}\quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if}\quad \nabla=\partial,\\ 0, & \text{otherwise},\end{cases}\qquad \vartheta^*(\nabla)=\begin{cases}0, & \text{if}\quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if}\quad \nabla=\partial,\\ 1, & \text{otherwise},\end{cases}$$

$$\zeta(\nabla)=\begin{cases}1, & \text{if}\quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if}\quad \nabla=\Re,\\ 0, & \text{otherwise},\end{cases}\qquad \zeta^*(\nabla)=\begin{cases}0, & \text{if}\quad \nabla\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if}\quad \nabla=\Re,\\ 1, & \text{otherwise}.\end{cases}$$

Thus, the identity function $\varphi:(\mathscr{S},\vartheta,\vartheta^*)\longrightarrow(\mathscr{S},\zeta,\zeta^*)$ is $DF\mathscr{G}$-continuous, but it is not DF-continuous. Also, the identity function $\varphi:(\mathscr{S},\zeta,\zeta^*)\longrightarrow(\mathscr{S},\vartheta,\vartheta^*)$ is $DF\mathscr{G}$-open (resp. $DF\mathscr{G}$-closed), but it is not DF-open (resp. DF-closed).

Theorem 3 A fuzzy function $\varphi:(\mathscr{S},\vartheta,\vartheta^*)\longrightarrow(\mathscr{Z},\zeta,\zeta^*)$ is $DF\mathscr{G}$-continuous iff for each $s_t\in P_t(S)$ and each $\Im\in I^Z$ with $\zeta(\Im)\geq p$ and $\zeta^*(\Im)\leq q$ containing $\varphi(s_t)$, there is $\Re\in I^S$ that is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re)\leq\Im$.

Proof. ($\Rightarrow$) Let $s_t\in P_t(S)$ and $\Im\in I^Z$, with $\zeta(\Im)\geq p$ and $\zeta^*(\Im)\leq q$ containing $\varphi(s_t)$, and then $\varphi^{-1}(\Im)\leq\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q)$. Since $s_t\in\varphi^{-1}(\Im)$, then we obtain $s_t\in\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q)=\Re$ (say). Hence, $\Re\in I^S$ is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re)\leq\Im$.

($\Leftarrow$) Let $s_t\in P_t(S)$ and $\Im\in I^Z$, with $\zeta(\Im)\geq p$, $\zeta^*(\Im)\leq q$, and $s_t\in\varphi^{-1}(\Im)$, there is $\Re\in I^S$ that is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re)\leq\Im$. Thus, $s_t\in\Re\leq\varphi^{-1}(\Im)$, and $s_t\in\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q)$. Then, $\varphi^{-1}(\Im)\leq\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q)$, so $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-open. Therefore, φ is $DF\mathscr{G}$-continuous. □

Theorem 4 Let $\varphi:(\mathscr{S},\vartheta,\vartheta^*)\longrightarrow(\mathscr{Z},\zeta,\zeta^*)$ be a fuzzy function, $\Re\in I^S$, $\Im\in I^Z$, $p\in I_\circ$, and $q\in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathscr{G}$-continuous.

(2) $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-closed for each $\Im$ with $\zeta(\Im^c)\geq p$ and $\zeta^*(\Im^c)\leq q$.

(3) $\varphi(\mathscr{G}C_\vartheta(\Re,p,q))\leq C_\zeta(\varphi(\Re),p,q)$.

(4) $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im),p,q)\leq\varphi^{-1}(C_\zeta(\Im,p,q))$.

(5) $\varphi^{-1}(I_\zeta(\Im,p,q))\leq\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q)$.

Proof. (1) $\Leftrightarrow$ (2) The proof can be readily established using Definition 11 and $\varphi^{-1}(\Im^c)=(\varphi^{-1}(\Im))^c$.

(2) $\Rightarrow$ (3) Let $\Re\in I^S$. By (2), $\varphi^{-1}(C_\zeta(\varphi(\Re),p,q))$ is (p,q)-$F\mathscr{G}$-closed. Thus,

$$\mathscr{G}C_\vartheta(\Re,p,q)\leq\mathscr{G}C_\vartheta(\varphi^{-1}(\varphi(\Re)),p,q)\leq\mathscr{G}C_\vartheta(\varphi^{-1}(C_\zeta(\varphi(\Re),p,q)),p,q)=\varphi^{-1}(C_\zeta(\varphi(\Re),p,q)).$$

Therefore, $\varphi(\mathscr{G}C_\vartheta(\Re,p,q)) \leq C_\zeta(\varphi(\Re),p,q)$.

(3) $\Rightarrow$ (4) Let $\Im \in I^Z$. By (3), $\varphi(\mathscr{G}C_\vartheta(\varphi^{-1}(\Im),p,q)) \leq C_\zeta(\varphi(\varphi^{-1}(\Im)),p,q) \leq C_\zeta(\Im,p,q)$. Thus, $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im),p,q) \leq \varphi^{-1}(\varphi(\mathscr{G}C_\vartheta(\varphi^{-1}(\Im),p,q))) \leq \varphi^{-1}(C_\zeta(\Im,p,q))$.

(4) $\Leftrightarrow$ (5) The proof can be readily established using Proposition 3 and $\varphi^{-1}(\Im^c) = (\varphi^{-1}(\Im))^c$.

(5) $\Rightarrow$ (1) Let $\Im \in I^Z$ with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$. By (5), we obtain $\varphi^{-1}(\Im) = \varphi^{-1}(I_\zeta(\Im,p,q)) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q) \leq \varphi^{-1}(\Im)$. Then, $\mathscr{G}I_\vartheta(\varphi^{-1}(\Im),p,q) = \varphi^{-1}(\Im)$. Thus, $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-open. Therefore, φ is $DF\mathscr{G}$-continuous. □

Definition 12 Let $(\mathscr{S},\vartheta,\vartheta^*)$ and $(\mathscr{Z},\zeta,\zeta^*)$ be DFTSs, $p \in I_\circ$, and $q \in I_1$. A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is called $DF\mathscr{G}P$-continuous (resp. $DF\mathscr{G}S$-continuous, $DF\mathscr{G}\alpha$-continuous, and $DF\mathscr{G}\gamma$-continuous) if $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}P$-open (resp. (p,q)-$F\mathscr{G}S$-open, (p,q)-$F\mathscr{G}\alpha$-open, and (p,q)-$F\mathscr{G}\gamma$-open) for each $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$.

Remark 8 Based on the definitions outlined earlier, the following diagram is established.

$$\begin{array}{ccccc}
\textbf{DF}\mathscr{G}\textbf{-continuity} & \longrightarrow & \textbf{DF}\mathscr{G}\textbf{ff-continuity} & \longrightarrow & \textbf{DF}\mathscr{G}\textbf{S-continuity} \\
 & & \downarrow & & \downarrow \\
 & & \textbf{DF}\mathscr{G}\textbf{P-continuity} & \longrightarrow & \textbf{DF}\mathscr{G}\textbf{fl-continuity}
\end{array}$$

Remark 9 Examples 8-12 illustrate that the converse of Remark 8 does not hold.

Example 8 Let $\mathscr{S} = \{s_1,s_2,s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.2},\frac{s_2}{0.7},\frac{s_3}{0.4}\}$, $\Re = \{\frac{s_1}{0.7},\frac{s_2}{0.9},\frac{s_3}{0.1}\}$. Define $\vartheta,\vartheta^*,\zeta,\zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{2}{3}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{S},\zeta,\zeta^*)$ is $DF\mathscr{G}\alpha$-continuous, but it is not $DF\mathscr{G}$-continuous.

Example 9 Let $\mathscr{S} = \{s_1,s_2,s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1},\frac{s_2}{0.2},\frac{s_3}{0.2}\}$, $\Re = \{\frac{s_1}{0.2},\frac{s_2}{0.3},\frac{s_3}{0.7}\}$. Define $\vartheta,\vartheta^*,\zeta,\zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is $DF\mathscr{G}S$-continuous, but it is neither $DF\mathscr{G}\alpha$-continuous nor $DF\mathscr{G}P$-continuous.

Example 10 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.3}, \frac{s_3}{0.4}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.8}, \frac{s_3}{0.7}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is $DF\mathscr{G}P$-continuous, but it is neither $DF\mathscr{G}\alpha$-continuous nor $DF\mathscr{G}S$-continuous.

Remark 10 From Examples 9 and 10, $DF\mathscr{G}P$-continuity and $DF\mathscr{G}S$-continuity are independent of each other.

Example 11 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.3}, \frac{s_3}{0.1}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.5}, \frac{s_3}{0.7}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is $DF\mathscr{G}\gamma$-continuous, but it is not $DF\mathscr{G}P$-continuous.

Example 12 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.2}, \frac{s_2}{0.3}, \frac{s_3}{0.4}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.8}, \frac{s_3}{0.8}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ is $DF\mathscr{G}\gamma$-continuous, but it is not $DF\mathscr{G}S$-continuous.

Theorem 5 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathscr{G}$-open.

(2) $\varphi(I_\vartheta(\Re, p, q)) \leq \mathscr{G}I_\zeta(\varphi(\Re), p, q)$.

(3) $I_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\mathscr{G}I_\zeta(\Im, p, q))$.

(4) For each $\Im$ and each $\Re$ with $\vartheta(\Re^c) \geq p$, $\vartheta^*(\Re^c) \leq q$, and $\varphi^{-1}(\Im) \leq \Re$, there is $\partial \in I^Z$ that is (p,q)-$F\mathscr{G}$-closed with $\Im \leq \partial$ such that $\varphi^{-1}(\partial) \leq \Re$.

Proof. (1) $\Rightarrow$ (2) Since $\varphi(I_\vartheta(\Re, p, q)) \leq \varphi(\Re)$, by (1), $\varphi(I_\vartheta(\Re, p, q))$ is (p,q)-$F\mathscr{G}$-open. Therefore, $\varphi(I_\vartheta(\Re, p, q)) \leq \mathscr{G}I_\zeta(\varphi(\Re), p, q)$.

(2) $\Rightarrow$ (3) Set $\Re = \varphi^{-1}(\Im)$ and by (2), $\varphi(I_\vartheta(\varphi^{-1}(\Im), p, q)) \leq \mathscr{G}I_\zeta(\varphi(\varphi^{-1}(\Im)), p, q) \leq \mathscr{G}I_\zeta(\Im, p, q)$. Then, $I_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\mathscr{G}I_\zeta(\Im, p, q))$.

(3) $\Rightarrow$ (4) Let $\Re \in I^S$ and $\Im \in I^Z$, with $\vartheta(\Re^c) \geq p$ and $\vartheta^*(\Re^c) \leq q$ such that $\varphi^{-1}(\Im) \leq \Re$. Since $\Re^c \leq \varphi^{-1}(\Im^c)$, $\Re^c = I_\vartheta(\Re^c, p, q) \leq I_\vartheta(\varphi^{-1}(\Im^c), p, q)$. Then by (3), $\Re^c \leq I_\vartheta(\varphi^{-1}(\Im^c), p, q) \leq \varphi^{-1}(\mathscr{G}I_\zeta(\Im^c, p, q))$. Thus, we have $\Re \geq (\varphi^{-1}(\mathscr{G}I_\zeta(\Im^c, p, q)))^c = \varphi^{-1}(\mathscr{G}C_\zeta(\Im, p, q))$. Hence, there is $\mathscr{G}C_\zeta(\Im, p, q) \in I^Z$ that is (p,q)-$F\mathscr{G}$-closed and $\Im \leq \mathscr{G}C_\zeta(\Im, p, q)$ with $\varphi^{-1}(\mathscr{G}C_\zeta(\Im, p, q)) \leq \Re$.

(4) $\Rightarrow$ (1) Let $\hbar \in I^S$ with $\vartheta(\hbar) \geq p$ and $\vartheta^*(\hbar) \leq q$. Set $\Im = (\varphi(\hbar))^c$ and $\Re = \hbar^c$, $\varphi^{-1}(\Im) = \varphi^{-1}((\varphi(\hbar))^c) \leq \Re$. Then by (4), there is $\partial \in I^Z$ that is (p,q)-$F\mathscr{G}$-closed and $\Im \leq \partial$ with $\varphi^{-1}(\partial) \leq \Re = \hbar^c$. Thus, $\varphi(\hbar) \leq \varphi(\varphi^{-1}(\partial^c)) \leq \partial^c$. Since $\Im \leq \partial$, then $\varphi(\hbar) = \Im^c \geq \partial^c$. Hence, $\varphi(\hbar) = \partial^c$ and $\varphi(\hbar)$ is (p,q)-$F\mathscr{G}$-open. Therefore, φ is $DF\mathscr{G}$-open. □

Theorem 6 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathscr{G}$-closed.

(2) $\mathscr{G}C_\zeta(\varphi(\Re), p, q) \leq \varphi(C_\vartheta(\Re, p, q))$.

(3) $\varphi^{-1}(\mathscr{G}C_\zeta(\Im, p, q)) \leq C_\vartheta(\varphi^{-1}(\Im), p, q)$.

(4) For each $\Im$ and each $\Re$ with $\vartheta(\Re) \geq p$, $\vartheta^*(\Re) \leq q$, and $\varphi^{-1}(\Im) \leq \Re$, there is $\partial \in I^Z$ that is (p,q)-$F\mathscr{G}$-open with $\Im \leq \partial$ such that $\varphi^{-1}(\partial) \leq \Re$.

Proof. The proof follows by applying the same method used in Theorem 5. □

4. Double fuzzy $\mathscr{G}$-irresoluteness

In this section, we introduce and study new fuzzy functions via (p,q)-$F\mathscr{G}$-closed sets and (p,q)-$F\mathscr{G}$-open sets, referred to as $DF\mathscr{G}$-irresolute (resp. $DF\mathscr{G}$-irresolute open, $DF\mathscr{G}$-irresolute closed, and $DF\mathscr{G}$-irresolute homeomorphism) functions within the context of $DFTSs$ $(\mathscr{S}, \vartheta, \vartheta^*)$ and $(\mathscr{Z}, \zeta, \zeta^*)$ based on the sense of Šostak. Also, the relationships between these classes of functions are introduced with the aid of examples.

Definition 13 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ and $(\mathscr{Z}, \zeta, \zeta^*)$ be DFTSs, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is called

(1) $DF\mathscr{G}$-irresolute if $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-open, for each (p,q)-$F\mathscr{G}$-open set $\Im$;

(2) $DF\mathscr{G}$-irresolute open if $\varphi(\Re)$ is (p,q)-$F\mathscr{G}$-open, for each (p,q)-$F\mathscr{G}$-open set $\Re$;

(3) $DF\mathscr{G}$-irresolute closed if $\varphi(\Re)$ is (p,q)-$F\mathscr{G}$-closed, for each (p,q)-$F\mathscr{G}$-closed set $\Re$.

Lemma 4 (1) Each $DF\mathscr{G}$-irresolute function is $DF\mathscr{G}$-continuous.

(2) Each $DF\mathscr{G}$-irresolute open function is $DF\mathscr{G}$-open.

(3) Each $DF\mathscr{G}$-irresolute closed function is $DF\mathscr{G}$-closed.

Proof. This can be readily established using Lemma 2. □

Remark 11 The converse of Lemma 4 fails, as demonstrated in Example 13

Example 13 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.3}, \frac{s_2}{0.2}, \frac{s_3}{0.5}\}$, $\Re = \{\frac{s_1}{0.1}, \frac{s_2}{0.2}, \frac{s_3}{0.3}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is $DF\mathscr{G}$-continuous, but it is not $DF\mathscr{G}$-irresolute. Also, the identity function $\varphi : (\mathscr{S}, \zeta, \zeta^*) \longrightarrow (\mathscr{S}, \vartheta, \vartheta^*)$ is $DF\mathscr{G}$-open (resp. $DF\mathscr{G}$-closed), but it is not $DF\mathscr{G}$-irresolute open (resp. $DF\mathscr{G}$-irresolute closed).

Theorem 7 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathscr{G}$-irresolute.

(2) $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-closed for each (p,q)-$F\mathscr{G}$-closed set $\Im$.

(3) $\varphi(\mathscr{G}C_\vartheta(\Re, p, q)) \leq \mathscr{G}C_\zeta(\varphi(\Re), p, q)$.

(4) $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\mathscr{G}C_\zeta(\Im, p, q))$.

(5) $\varphi^{-1}(\mathscr{G}I_\zeta(\Im, p, q)) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q)$.

Proof. (1) $\Leftrightarrow$ (2) The proof can be readily established using Definition 13 and $\varphi^{-1}(\Im^c) = (\varphi^{-1}(\Im))^c$.

(2) $\Rightarrow$ (3) Let $\Re \in I^S$. By (2), $\varphi^{-1}(\mathscr{G}C_\zeta(\varphi(\Re), p, q))$ is (p,q)-$F\mathscr{G}$-closed. Thus,

$$\mathscr{G}C_\vartheta(\Re, p, q) \leq \mathscr{G}C_\vartheta(\varphi^{-1}(\varphi(\Re)), p, q) \leq \mathscr{G}C_\vartheta(\varphi^{-1}(\mathscr{G}C_\zeta(\varphi(\Re), p, q)), p, q) = \varphi^{-1}(\mathscr{G}C_\zeta(\varphi(\Re), p, q)).$$

Therefore, $\varphi(\mathscr{G}C_\vartheta(\Re, p, q)) \leq \mathscr{G}C_\zeta(\varphi(\Re), p, q)$.

(3) $\Rightarrow$ (4) Let $\Im \in I^Z$. By (3), $\varphi(\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q)) \leq \mathscr{G}C_\zeta(\varphi(\varphi^{-1}(\Im)), p, q) \leq \mathscr{G}C_\zeta(\Im, p, q)$. Thus, $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\varphi(\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q))) \leq \varphi^{-1}(\mathscr{G}C_\zeta(\Im, p, q))$.

(4) $\Leftrightarrow$ (5) The proof can be readily established using Proposition 3 and $\varphi^{-1}(\Im^c) = (\varphi^{-1}(\Im))^c$.

(5) $\Rightarrow$ (1) Let $\Im \in I^Z$ be (p,q)-$F\mathscr{G}$-open. By (5), we obtain $\varphi^{-1}(\Im) = \varphi^{-1}(\mathscr{G}I_\zeta(\Im, p, q)) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\Im)$. Then, $\mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q) = \varphi^{-1}(\Im)$. Thus, $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-open. Therefore, φ is $DF\mathscr{G}$-irresolute. □

Proposition 5 Let $(\mathscr{S}, \vartheta, \vartheta^*)$, $(\mathscr{Q}, \tau, \tau^*)$, and $(\mathscr{Z}, \zeta, \zeta^*)$ be *DFTSs*, and let $\phi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Q}, \tau, \tau^*)$ and $\varphi : (\mathscr{Q}, \tau, \tau^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be fuzzy functions. Then, $\varphi \circ \phi$ is $DF\mathscr{G}$-continuous (resp. $DF\mathscr{G}$-irresolute) if ϕ is $DF\mathscr{G}$-irresolute, and φ is $DF\mathscr{G}$-continuous (resp. $DF\mathscr{G}$-irresolute).

Proof. This can be readily established using Definitions 11 and 13. □

Theorem 8 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathscr{G}$-irresolute open.

(2) $\varphi(\mathscr{G}I_\vartheta(\Re, p, q)) \leq \mathscr{G}I_\zeta(\varphi(\Re), p, q)$.

(3) $\mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(\mathscr{G}I_\zeta(\Im, p, q))$.

(4) For each $\Im$ and each (p,q)-$F\mathscr{G}$-closed set $\Re$ with $\varphi^{-1}(\Im) \leq \Re$, there is $\partial \in I^Z$ that is (p,q)-$F\mathscr{G}$-closed with $\Im \leq \partial$ such that $\varphi^{-1}(\partial) \leq \Re$.

Proof. (1) $\Rightarrow$ (2) Since $\varphi(\mathcal{G}I_\vartheta(\Re,p,q)) \leq \varphi(\Re)$, by (1), $\varphi(\mathcal{G}I_\vartheta(\Re,p,q))$ is (p,q)-$F\mathcal{G}$-open. Therefore, $\varphi(\mathcal{G}I_\vartheta(\Re,p,q)) \leq \mathcal{G}I_\zeta(\varphi(\Re),p,q)$.

(2) $\Rightarrow$ (3) Set $\Re = \varphi^{-1}(\Im)$ and by (2), $\varphi(\mathcal{G}I_\vartheta(\varphi^{-1}(\Im),p,q)) \leq \mathcal{G}I_\zeta(\varphi(\varphi^{-1}(\Im)),p,q) \leq \mathcal{G}I_\zeta(\Im,p,q)$. Then, $\mathcal{G}I_\vartheta(\varphi^{-1}(\Im),p,q) \leq \varphi^{-1}(\mathcal{G}I_\zeta(\Im,p,q))$.

(3) $\Rightarrow$ (4) Let $\Re$ be (p,q)-$F\mathcal{G}$-closed and $\Im \in I^Z$ such that $\varphi^{-1}(\Im) \leq \Re$. Since $\Re^c \leq \varphi^{-1}(\Im^c)$, $\Re^c = \mathcal{G}I_\vartheta(\Re^c,p,q) \leq \mathcal{G}I_\vartheta(\varphi^{-1}(\Im^c),p,q)$. Then by (3), $\Re^c \leq \mathcal{G}I_\vartheta(\varphi^{-1}(\Im^c),p,q) \leq \varphi^{-1}(\mathcal{G}I_\zeta(\Im^c,p,q))$. Thus, $\Re \geq (\varphi^{-1}(\mathcal{G}I_\zeta(\Im^c,p,q)))^c = \varphi^{-1}(\mathcal{G}C_\zeta(\Im,p,q))$. Hence, there is $\mathcal{G}C_\zeta(\Im,p,q) \in I^Z$ that is (p,q)-$F\mathcal{G}$-closed and $\Im \leq \mathcal{G}C_\zeta(\Im,p,q)$ with $\varphi^{-1}(\mathcal{G}C_\zeta(\Im,p,q)) \leq \Re$.

(4) $\Rightarrow$ (1) Let $\hbar \in I^S$ be (p,q)-$F\mathcal{G}$-open. Set $\Im = (\varphi(\hbar))^c$ and $\Re = \hbar^c$, $\varphi^{-1}(\Im) = \varphi^{-1}((\varphi(\hbar))^c) \leq \Re$. Then by (4), there is $\partial \in I^Z$ that is (p,q)-$F\mathcal{G}$-closed and $\Im \leq \partial$ with $\varphi^{-1}(\partial) \leq \Re = \hbar^c$. Thus, $\varphi(\hbar) \leq \varphi(\varphi^{-1}(\partial^c)) \leq \partial^c$. Since $\Im \leq \partial$, then $\varphi(\hbar) = \Im^c \geq \partial^c$. Hence, $\varphi(\hbar) = \partial^c$ and $\varphi(\hbar)$ is (p,q)-$F\mathcal{G}$-open. Therefore, φ is $DF\mathcal{G}$-irresolute open. □

Theorem 9 Let $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{Z},\zeta,\zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathcal{G}$-irresolute closed.

(2) $\mathcal{G}C_\zeta(\varphi(\Re),p,q) \leq \varphi(\mathcal{G}C_\vartheta(\Re,p,q))$.

(3) $\varphi^{-1}(\mathcal{G}C_\zeta(\Im,p,q)) \leq \mathcal{G}C_\vartheta(\varphi^{-1}(\Im),p,q)$.

(4) For each $\Im$ and each (p,q)-$F\mathcal{G}$-open set $\Re$ with $\varphi^{-1}(\Im) \leq \Re$, there is $\partial \in I^Z$ that is (p,q)-$F\mathcal{G}$-open with $\Im \leq \partial$ such that $\varphi^{-1}(\partial) \leq \Re$.

Proof. The proof follows by applying the same method used in Theorem 8. □

Proposition 6 Let $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{Z},\zeta,\zeta^*)$ be a bijective. Thus, φ is $DF\mathcal{G}$-irresolute open iff φ is $DF\mathcal{G}$-irresolute closed.

Proof. This can be readily established using:
$\varphi^{-1}(\mathcal{G}C_\zeta(\Im,p,q)) \leq \mathcal{G}C_\vartheta(\varphi^{-1}(\Im),p,q) \Longleftrightarrow \varphi^{-1}(\mathcal{G}I_\zeta(\Im^c,p,q)) \geq \mathcal{G}I_\vartheta(\varphi^{-1}(\Im^c),p,q)$. □

Definition 14 A bijective fuzzy function $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{Z},\zeta,\zeta^*)$ is called $DF\mathcal{G}$-irresolute homeomorphism if φ and φ^{-1} are $DF\mathcal{G}$-irresolute.

Corollary 2 Let $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{Z},\zeta,\zeta^*)$ be a fuzzy function, $\Re \in I^S$, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DF\mathcal{G}$-irresolute homeomorphism.

(2) φ is $DF\mathcal{G}$-irresolute and $DF\mathcal{G}$-irresolute open.

(3) φ is $DF\mathcal{G}$-irresolute and $DF\mathcal{G}$-irresolute closed.

(4) $\varphi(\mathcal{G}I_\vartheta(\Re,p,q)) = \mathcal{G}I_\zeta(\varphi(\Re),p,q)$.

(5) $\varphi(\mathcal{G}C_\vartheta(\Re,p,q)) = \mathcal{G}C_\zeta(\varphi(\Re),p,q)$.

(6) $\mathcal{G}I_\vartheta(\varphi^{-1}(\Im),p,q) = \varphi^{-1}(\mathcal{G}I_\zeta(\Im,p,q))$.

(7) $\mathcal{G}C_\vartheta(\varphi^{-1}(\Im),p,q) = \varphi^{-1}(\mathcal{G}C_\zeta(\Im,p,q))$.

Proof. This can be readily established using the previous results and definitions. □

5. Further results

In this section, we display and discuss the concepts of $DFA\mathcal{G}$-continuity and $DFW\mathcal{G}$-continuity as additional weaker variants of $DF\mathcal{G}$-continuity within the context of $DFTSs$ $(\mathscr{S},\vartheta,\vartheta^*)$ and $(\mathscr{Z},\zeta,\zeta^*)$ based on the sense of Šostak. We also present and examine new fuzzy separation axioms, referred to as (p,q)-$F\mathcal{G}$-regular (resp. (p,q)-$F\mathcal{G}$-normal) spaces via (p,q)-$F\mathcal{G}$-closed sets.

Definition 15 Let $(\mathscr{S},\vartheta,\vartheta^*)$ and $(\mathscr{Z},\zeta,\zeta^*)$ be DFTSs, and let $p \in I_\circ$ and $q \in I_1$. A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is called $DFA\mathcal{G}$-continuous if $\varphi^{-1}(\Im) \leq \mathcal{G}I_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im,p,q),p,q)),p,q)$ for each $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$.

Lemma 5 Each $DF\mathcal{G}$-continuous function is $DFA\mathcal{G}$-continuous.

Proof. This can be readily established using Definitions 11 and 15. □

Remark 12 The converse of Lemma 5 fails, as demonstrated in Example 14.

Example 14 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.1}, \frac{s_2}{0.4}, \frac{s_3}{0.1}\}$, $\Re = \{\frac{s_1}{0.3}, \frac{s_2}{0.5}, \frac{s_3}{0.8}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is *DFA*$\mathscr{G}$-continuous, but it is not *DF*$\mathscr{G}$-continuous.

Theorem 10 A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is *DFA*$\mathscr{G}$-continuous iff for each $s_t \in P_t(S)$ and each $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$ containing $\varphi(s_t)$, there is $\Re \in I^S$ that is (p,q)-*F*$\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq I_\zeta(C_\zeta(\Im, p, q), p, q)$.

Proof. ($\Rightarrow$) Let $s_t \in P_t(S)$ and $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$ containing $\varphi(s_t)$, and then $\varphi^{-1}(\Im) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q)), p, q)$. Since $s_t \in \varphi^{-1}(\Im)$, then we obtain

$$s_t \in \mathscr{G}I_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q)), p, q) = \Re \text{ (say)}.$$

Hence, $\Re \in I^S$ is (p,q)-*F*$\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq I_\zeta(C_\zeta(\Im, p, q), p, q)$.

($\Leftarrow$) Let $s_t \in P_t(S)$ and $\Im \in I^Z$, with $\zeta(\Im) \geq p$, $\zeta^*(\Im) \leq q$, and $s_t \in \varphi^{-1}(\Im)$, there is $\Re \in I^S$ that is (p,q)-*F*$\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq I_\zeta(C_\zeta(\Im, p, q), p, q)$. Thus, $s_t \in \Re \leq \varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q))$, and $s_t \in \mathscr{G}I_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q)), p, q)$. Then,

$$\varphi^{-1}(\Im) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q)), p, q).$$

Therefore, φ is *DFA*$\mathscr{G}$-continuous. □

Theorem 11 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is *DFA*$\mathscr{G}$-continuous.

(2) $\varphi^{-1}(\Im)$ is (p,q)-*F*$\mathscr{G}$-open for each (p,q)-*FR*-open set $\Im$.

(3) $\varphi^{-1}(\Im)$ is (p,q)-*F*$\mathscr{G}$-closed for each (p,q)-*FR*-closed set $\Im$.

(4) $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(C_\zeta(\Im, p, q))$ for each (p,q)-*F*γ-open set $\Im$.

(5) $\mathscr{G}C_\vartheta(\varphi^{-1}(\Im), p, q) \leq \varphi^{-1}(C_\zeta(\Im, p, q))$ for each (p,q)-*FS*-open set $\Im$.

Proof. (1) $\Rightarrow$ (2) Let $s_t \in P_t(S)$ and $\Im \in I^Z$ be (p,q)-*FR*-open, with $s_t \in \varphi^{-1}(\Im)$. Hence, by (1), there is $\Re \in I^S$ that is (p,q)-*F*$\mathscr{G}$-open, with $s_t \in \Re$ and $\varphi(\Re) \leq I_\zeta(C_\zeta(\Im, p, q), p, q)$. Thus, $\Re \leq \varphi^{-1}(I_\zeta(C_\zeta(\Im, p, q), p, q)) = \varphi^{-1}(\Im)$, and $s_t \in \mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q)$. Therefore, $\varphi^{-1}(\Im) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(\Im), p, q)$, so $\varphi^{-1}(\Im)$ is (p,q)-*F*$\mathscr{G}$-open.

(2) $\Rightarrow$ (3) If $\Im \in I^Z$ is (p,q)-*FR*-closed, by (2), $\varphi^{-1}(\Im^c) = (\varphi^{-1}(\Im))^c$ is (p,q)-*F*$\mathscr{G}$-open. Therefore, $\varphi^{-1}(\Im)$ is (p,q)-*F*$\mathscr{G}$-closed.

(3) $\Rightarrow$ (4) If $\Im \in I^Z$ is (p,q)-*F*γ-open, and since $C_\zeta(\Im, p, q)$ is (p,q)-*FR*-closed, then by (3), $\varphi^{-1}(C_\zeta(\Im, p, q))$ is (p,q)-*F*$\mathscr{G}$-closed. Since $\varphi^{-1}(\Im) \leq \varphi^{-1}(C_\zeta(\Im, p, q))$, hence,

$$\mathscr{G}C_{\vartheta}(\varphi^{-1}(\Im),p,q) \leq \varphi^{-1}(C_{\zeta}(\Im,p,q)).$$

(4) $\Rightarrow$ (5) The proof can be readily established via the fact that each (p,q)-FS-open set is (p,q)-$F\gamma$-open.

(5) $\Rightarrow$ (3) If $\Im \in I^Z$ is (p,q)-FR-closed, then $\Im$ is (p,q)-FS-open. Hence by (5), $\mathscr{G}C_{\vartheta}(\varphi^{-1}(\Im),p,q) \leq \varphi^{-1}(C_{\zeta}(\Im,p,q)) = \varphi^{-1}(\Im)$. Therefore, $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-closed.

(3) $\Rightarrow$ (1) If $s_t \in P_t(S)$ and $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$ such that $s_t \in \varphi^{-1}(\Im)$, then $s_t \in \varphi^{-1}(I_{\zeta}(C_{\zeta}(\Im,p,q), p,q))$. Since $[I_{\zeta}(C_{\zeta}(\Im,p,q),p,q)]^c$ is (p,q)-FR-closed, then by (3),

$$\varphi^{-1}([I_{\zeta}(C_{\zeta}(\Im,p,q),p,q)]^c) \text{ is } (p,q)\text{-}F\mathscr{G}\text{-closed.}$$

Hence, $\varphi^{-1}(I_{\zeta}(C_{\zeta}(\Im,p,q),p,q))$ is (p,q)-$F\mathscr{G}$-open, and $s_t \in \mathscr{G}I_{\vartheta}(\varphi^{-1}(I_{\zeta}(C_{\zeta}(\Im,p,q),p,q)),p,q)$. Thus, $\varphi^{-1}(\Im) \leq \mathscr{G}I_{\vartheta}(\varphi^{-1}(I_{\zeta}(C_{\zeta}(\Im,p,q),p,q)),p,q)$. Therefore, φ is $DFA\mathscr{G}$-continuous. □

Proposition 7 Let $(\mathscr{S},\vartheta,\vartheta^*)$, $(\mathscr{Q},\tau,\tau^*)$, and $(\mathscr{Z},\zeta,\zeta^*)$ be $DFTSs$, and let $\phi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{Q},\tau,\tau^*)$ and $\varphi : (\mathscr{Q},\tau,\tau^*) \longrightarrow (\mathscr{Z},\zeta,\zeta^*)$ be fuzzy functions. Then, $\varphi \circ \phi$ is $DFA\mathscr{G}$-continuous if ϕ is $DF\mathscr{G}$-continuous (resp. $DF\mathscr{G}$-irresolute) and φ is DF-continuous (resp. $DFA\mathscr{G}$-continuous).

Proof. This can be readily established using the previous definitions and Theorem 11. □

Definition 16 Let $(\mathscr{S},\vartheta,\vartheta^*)$ and $(\mathscr{Z},\zeta,\zeta^*)$ be DFTSs, and let $p \in I_{\circ}$ and $q \in I_1$. A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is called $DFW\mathscr{G}$-continuous if $\varphi^{-1}(\Im) \leq \mathscr{G}I_{\vartheta}(\varphi^{-1}(C_{\zeta}(\Im,p,q)),p,q)$ for each $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$.

Lemma 6 Each $DF\mathscr{G}$-continuous function is $DFW\mathscr{G}$-continuous.

Proof. This can be readily established using Definitions 11 and 16. □

Remark 13 The converse of Lemma 6 fails, as demonstrated in Example 15.

Example 15 Let $\mathscr{S} = \{s_1,s_2,s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.2}, \frac{s_2}{0.4}, \frac{s_3}{0.3}\}$, $\Re = \{\frac{s_1}{0.4}, \frac{s_2}{0.5}, \frac{s_3}{0.8}\}$. Define $\vartheta,\vartheta^*,\zeta,\zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise,} \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise,} \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise,} \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1},\underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S},\vartheta,\vartheta^*) \longrightarrow (\mathscr{S},\zeta,\zeta^*)$ is $DFW\mathscr{G}$-continuous, but it is not $DF\mathscr{G}$-continuous.

Theorem 12 A fuzzy function $\varphi : I^S \longrightarrow I^Z$ is $DFW\mathscr{G}$-continuous iff for each $s_t \in P_t(S)$ and each $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$ containing $\varphi(s_t)$, there is $\Re \in I^S$ that is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq C_{\zeta}(\Im,p,q)$.

Proof. ($\Rightarrow$) Let $s_t \in P_t(S)$ and $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$ containing $\varphi(s_t)$, and then $\varphi^{-1}(\Im) \leq \mathscr{G}I_{\vartheta}(\varphi^{-1}(C_{\zeta}(\Im,p,q)),p,q)$. Since $s_t \in \varphi^{-1}(\Im)$, then we obtain

$$s_t \in \mathscr{G}I_{\vartheta}(\varphi^{-1}(C_{\zeta}(\Im,p,q)),p,q) = \Re \text{ (say).}$$

Hence, $\Re \in I^S$ is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq C_\zeta(\Im, p, q)$.

($\Leftarrow$) Let $s_t \in P_t(S)$ and $\Im \in I^Z$, with $\zeta(\Im) \geq p$, $\zeta^*(\Im) \leq q$, and $s_t \in \varphi^{-1}(\Im)$, there is $\Re \in I^S$ that is (p,q)-$F\mathscr{G}$-open containing s_t, with $\varphi(\Re) \leq C_\zeta(\Im, p, q)$. Thus, $s_t \in \Re \leq \varphi^{-1}(C_\zeta(\Im, p, q))$, and $s_t \in \mathscr{G}I_\vartheta(\varphi^{-1}(C_\zeta(\Im, p, q)), p, q)$. Then, $\varphi^{-1}(\Im) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(C_\zeta(\Im, p, q)), p, q)$. Therefore, φ is $DFW\mathscr{G}$-continuous. □

Theorem 13 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a fuzzy function, $\Im \in I^Z$, $p \in I_\circ$, and $q \in I_1$. It follows that the following statements hold equivalently:

(1) φ is $DFW\mathscr{G}$-continuous.

(2) $\varphi^{-1}(\Im) \geq \mathscr{G}C_\vartheta(\varphi^{-1}(I_\zeta(\Im, p, q)), p, q)$ if $\zeta(\Im^c) \geq p$ and $\zeta^*(\Im^c) \leq q$.

(3) $\mathscr{G}I_\vartheta(\varphi^{-1}(C_\zeta(\Im, p, q)), p, q) \geq \varphi^{-1}(I_\zeta(\Im, p, q))$.

(4) $\mathscr{G}C_\vartheta(\varphi^{-1}(I_\zeta(\Im, p, q)), p, q) \leq \varphi^{-1}(C_\zeta(\Im, p, q))$.

Proof. (1) $\Leftrightarrow$ (2) The proof can be readily established using Proposition 3 and Definition 16.

(2) $\Rightarrow$ (3) Let $\Im \in I^Z$. Hence, by (2),

$$\mathscr{G}C_\vartheta(\varphi^{-1}(I_\zeta(C_\zeta(\Im^c, p, q), p, q)), p, q) \leq \varphi^{-1}(C_\zeta(\Im^c, p, q)).$$

Thus, $\varphi^{-1}(I_\zeta(\Im, p, q)) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(C_\zeta(\Im, p, q)), p, q)$.

(3) $\Leftrightarrow$ (4) This can be readily established using Propositions 1 and 3.

(4) $\Rightarrow$ (1) Let $\Im \in I^Z$, with $\zeta(\Im) \geq p$ and $\zeta^*(\Im) \leq q$. Hence, by (4),

$$\mathscr{G}C_\vartheta(\varphi^{-1}(I_\zeta(\Im^c, p, q)), p, q) \leq \varphi^{-1}(C_\zeta(\Im^c, p, q)) = \varphi^{-1}(\Im^c).$$

Thus, $\varphi^{-1}(\Im) \leq \mathscr{G}I_\vartheta(\varphi^{-1}(C_\zeta(\Im, p, q)), p, q)$, so φ is $DFW\mathscr{G}$-continuous. □

Lemma 7 Each $DFA\mathscr{G}$-continuous function is $DFW\mathscr{G}$-continuous.

Proof. This can be readily established using Definitions 15 and 16. □

Remark 14 The converse of Lemma 7 fails, as demonstrated in Example 16.

Example 16 Let $\mathscr{S} = \{s_1, s_2, s_3\}$ and define $\partial, \Re \in I^S$ as follows: $\partial = \{\frac{s_1}{0.5}, \frac{s_2}{0.6}, \frac{s_3}{0.7}\}$, $\Re = \{\frac{s_1}{0.5}, \frac{s_2}{0.4}, \frac{s_3}{0.3}\}$. Define $\vartheta, \vartheta^*, \zeta, \zeta^* : I^S \longrightarrow I$ as follows:

$$\vartheta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nabla = \partial, \\ 0, & \text{otherwise}, \end{cases} \qquad \vartheta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \partial, \\ 1, & \text{otherwise}, \end{cases}$$

$$\zeta(\nabla) = \begin{cases} 1, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \Re, \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta^*(\nabla) = \begin{cases} 0, & \text{if} \quad \nabla \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nabla = \Re, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity function $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{S}, \zeta, \zeta^*)$ is $DFW\mathscr{G}$-continuous, but it is not $DFA\mathscr{G}$-continuous.

Remark 15 Based on the definitions outlined earlier, the following diagram is established.

DF$\mathscr{G}$-continuity $\longrightarrow$ **DFA$\mathscr{G}$-continuity** $\longrightarrow$ **DFW$\mathscr{G}$-continuity**

Definition 17 Let $s_t \in P_t(S)$ and $\Re, \Im \in I^S$. An $DFTS$ $(\mathscr{S}, \vartheta, \vartheta^*)$ is called:

(1) (p,q)-$F\mathscr{G}$-regular space if $s_t \overline{\Lambda}\, \Re$ for any (p,q)-$F\mathscr{G}$-closed set $\Re$, there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$ such that $s_t \in \partial_1$, $\Re \leq \partial_2$, and $\partial_1 \overline{\Lambda}\, \partial_2$.

(2) (p,q)-$F\mathscr{G}$-normal space if $\Re \overline{\Lambda}\, \Im$ for any (p,q)-$F\mathscr{G}$-closed set $\Re$ and $\Im$, there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$ such that $\Re \leq \partial_1$, $\Im \leq \partial_2$, and $\partial_1 \overline{\Lambda}\, \partial_2$.

Theorem 14 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$ and $s_t \in P_t(S)$, and let $\Re, \Im \in I^S$. It follows that the following statements hold equivalently:

(1) $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular.

(2) If $s_t \in \Re$ for any (p,q)-$F\mathscr{G}$-open set $\Re$, there is $\Im$ with $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$ such that

$$s_t \in \Im \leq C_\vartheta(\Im, p, q) \leq \Re.$$

(3) If $s_t \overline{\Lambda}\, \Re$ for any (p,q)-$F\mathscr{G}$-closed set $\Re$, there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$ such that $s_t \in \partial_1$, $\Re \leq \partial_2$, and $C_\vartheta(\partial_1, p, q) \overline{\Lambda}\, C_\vartheta(\partial_2, p, q)$.

Proof. (1) $\Rightarrow$ (2) Let $s_t \in \Re$ for any (p,q)-$F\mathscr{G}$-open set $\Re$, then $s_t \overline{\Lambda}\, \Re^c$. By (1), there are $\Im, \hbar \in I^S$ with $\vartheta(\Im) \geq p$, $\vartheta^*(\Im) \leq q$, $\vartheta(\hbar) \geq p$, and $\vartheta^*(\hbar) \leq q$ such that $s_t \in \Im$, $\Re^c \leq \hbar$, and $\Im \overline{\Lambda}\, \hbar$. Therefore, $s_t \in \Im \leq C_\vartheta(\Im, p, q) \leq \Re$.

(2) $\Rightarrow$ (3) Let $s_t \overline{\Lambda}\, \Re$ for any (p,q)-$F\mathscr{G}$-closed set $\Re$, then $s_t \in \Re^c$. By (2), there is $\Im$ with $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$ such that $s_t \in \Im \leq C_\vartheta(\Im, p, q) \leq \Re^c$. Since $\vartheta(\Im) \geq p$ and $\vartheta^*(\Im) \leq q$, then $\Im$ is (p,q)-$F\mathscr{G}$-open and $s_t \in \Im$. Again, by (2), there is ∂_1 with $\vartheta(\partial_1) \geq p$ and $\vartheta^*(\partial_1) \leq q$ such that $s_t \in \partial_1 \leq C_\vartheta(\partial_1, p, q) \leq \Im \leq C_\vartheta(\Im, p, q) \leq \Re^c$. Hence, $\Re \leq (C_\vartheta(\Im, p, q))^c = I_\vartheta(\Im^c, p, q) \leq \Im^c$. Set $\partial_2 = I_\vartheta(\Im^c, p, q)$, and thus, $\vartheta(\partial_2) \geq p$ and $\vartheta^*(\partial_2) \leq q$. Then, $C_\vartheta(\partial_2, p, q) \leq \Im^c \leq (C_\vartheta(\partial_1, p, q))^c$. Therefore, $C_\vartheta(\partial_1, p, q) \overline{\Lambda}\, C_\vartheta(\partial_2, p, q)$.

(3) $\Rightarrow$ (1) This can be readily established using Definition 17. □

Theorem 15 Let $(\mathscr{S}, \vartheta, \vartheta^*)$ be an $DFTS$ and $\Re, \Im \in I^S$. It follows that the following statements hold equivalently:

(1) $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-normal.

(2) If $\Im \leq \Re$ for any (p,q)-$F\mathscr{G}$-closed set $\Im$ and (p,q)-$F\mathscr{G}$-open set $\Re$, there is $\partial \in I^S$ with $\vartheta(\partial) \geq p$ and $\vartheta^*(\partial) \leq q$ such that $\Im \leq \partial \leq C_\vartheta(\partial, p, q) \leq \Re$.

(3) If $\Re \overline{\Lambda}\, \Im$ for any (p,q)-$F\mathscr{G}$-closed set $\Re$ and $\Im$, there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$ such that $\Re \leq \partial_1$, $\Im \leq \partial_2$, and $C_\vartheta(\partial_1, p, q) \overline{\Lambda}\, C_\vartheta(\partial_2, p, q)$.

Proof. The proof follows by applying the same method used in Theorem 14. □

Theorem 16 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a bijective DF-open and $DF\mathscr{G}$-irresolute function. If $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal), thus $(\mathscr{Z}, \zeta, \zeta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal).

Proof. If $z_t \overline{\Lambda}\, \Im$ for any (p,q)-$F\mathscr{G}$-closed set $\Im \in I^Z$ and $DF\mathscr{G}$-irresolute function φ, thus $\varphi^{-1}(\Im)$ is (p,q)-$F\mathscr{G}$-closed. Moreover, set $z_t = \varphi(s_t)$, and then $s_t \overline{\Lambda}\, \varphi^{-1}(\Im)$. Since $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular, then there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$, such that $s_t \in \partial_1$, $\varphi^{-1}(\Im) \leq \partial_2$, and $\partial_1 \overline{\Lambda}\, \partial_2$. Since φ is bijective DF-open, then $z_t \in \varphi(\partial_1)$, $\Im = \varphi(\varphi^{-1}(\Im)) \leq \varphi(\partial_2)$, and $\varphi(\partial_1) \overline{\Lambda}\, \varphi(\partial_2)$. Therefore, $(\mathscr{Z}, \zeta, \zeta^*)$ is (p,q)-$F\mathscr{G}$-regular. The other case also follows similar lines. □

Theorem 17 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be an injective DF-continuous and $DF\mathscr{G}$-irresolute closed function. If $(\mathscr{Z}, \zeta, \zeta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal), thus $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal).

Proof. Let $s_t \overline{\Lambda}\, \Re$ for any (p,q)-$F\mathscr{G}$-closed set $\Re \in I^S$ and injective $DF\mathscr{G}$-irresolute closed function φ, and then $\varphi(\Re)$ is (p,q)-$F\mathscr{G}$-closed and $\varphi(s_t) \overline{\Lambda}\, \varphi(\Re)$. Since $(\mathscr{Z}, \zeta, \zeta^*)$ is (p,q)-$F\mathscr{G}$-regular, there are $\partial_i \in I^S$ with $\vartheta(\partial_i) \geq p$ and $\vartheta^*(\partial_i) \leq q$ for $i \in \{1,2\}$, such that $\varphi(s_t) \in \partial_1$, $\varphi(\Re) \leq \partial_2$, and $\partial_1 \overline{\Lambda}\, \partial_2$. Since φ is DF-continuous function, thus $s_t \in \varphi^{-1}(\partial_1)$ and $\Re \leq \varphi^{-1}(\partial_2)$ with $\vartheta(\varphi^{-1}(\partial_i)) \geq p$ and $\vartheta^*(\varphi^{-1}(\partial_i)) \leq q$ for $i \in \{1,2\}$, such that $\varphi^{-1}(\partial_1) \overline{\Lambda}\, \varphi^{-1}(\partial_2)$. Therefore, $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular. The other case also follows similar lines. □

Theorem 18 Let $\varphi : (\mathscr{S}, \vartheta, \vartheta^*) \longrightarrow (\mathscr{Z}, \zeta, \zeta^*)$ be a surjective DF-closed, DF-open, and $DF\mathscr{G}$-irresolute function. If $(\mathscr{S}, \vartheta, \vartheta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal), thus $(\mathscr{Z}, \zeta, \zeta^*)$ is (p,q)-$F\mathscr{G}$-regular (resp. (p,q)-$F\mathscr{G}$-normal).

Proof. The proof follows by applying the same method used in Theorem 16. □

6. Conclusions and future work

In this work, new classes of fuzzy open sets, called (p,q)-$F\mathcal{G}$-open (resp. (p,q)-$F\mathcal{G}P$-open, (p,q)-$F\mathcal{G}S$-open, (p,q)-$F\mathcal{G}\alpha$-open, and (p,q)-$F\mathcal{G}\gamma$-open) sets, have been introduced in $DFTSs$ based on fuzzy topologies in view of Šostak [16]. A detailed examination of the relationships among these classes of open sets was carried out, supported by illustrative examples. However, the notions of $DF\mathcal{G}$-interior (resp. $DF\mathcal{G}$-closure) operators, have been proposed and examined. Thereafter, we displayed and studied new fuzzy functions via (p,q)-$F\mathcal{G}$-open sets, called $DF\mathcal{G}$-continuous (resp. $DF\mathcal{G}$-open and $DF\mathcal{G}$-closed) functions within the context of $DFTSs$ $(\mathcal{S}, \vartheta, \vartheta^*)$ and $(\mathcal{Z}, \zeta, \zeta^*)$. We also explored the concepts of $DF\mathcal{G}P$-continuous (resp. $DF\mathcal{G}S$-continuous, $DF\mathcal{G}\alpha$-continuous, and $DF\mathcal{G}\gamma$-continuous) functions, which represent weaker variants of $DF\mathcal{G}$-continuous functions. As an application, we demonstrate how these new forms of continuity can serve to generalize, extend, and unify a variety of previously known results in $DFTSs$. Furthermore, we explored and discussed new fuzzy functions via (p,q)-$F\mathcal{G}$-open sets and (p,q)-$F\mathcal{G}$-closed sets, referred to as $DF\mathcal{G}$-irresolute (resp. $DF\mathcal{G}$-irresolute open, $DF\mathcal{G}$-irresolute closed, and $DF\mathcal{G}$-irresolute homeomorphism) functions within the context of $DFTSs$. The interrelationships among the function classes established in this work are examined systematically and clarified through suitable illustrative examples. In particular, $DFA\mathcal{G}$-continuity and $DFW\mathcal{G}$-continuity are introduced as weaker forms of $DF\mathcal{G}$-continuity, and their respective properties are investigated in the setting of $DFTSs$. In addition, new fuzzy separation axioms are formulated through the notions of (p,q)-$F\mathcal{G}$-normal and (p,q)-$F\mathcal{G}$-regular spaces, which are defined in terms of (p,q)-$F\mathcal{G}$-closed sets. The obtained results enrich the existing framework of double fuzzy topology and provide a basis for investigating further generalizations and related structures. The following directions are suggested for future research:

- Introducing and investigating double fuzzy upper $\mathcal{G}$-continuity and double fuzzy lower $\mathcal{G}$-continuity.
- Extending the notions developed in this study to fuzzy soft minimal spaces [32].
- Exploring possible applications of the proposed notions in the framework of primal approximation spaces [33].
- Developing analogous formulations based on lattice-theoretic structures and rough set theory [34].

Author contributions

Conceptualization, W.K.A. and I.M.T.; Methodology, W.K.A. and I.M.T.; Formal analysis, W.K.A. and I.M.T.; Investigation, W.K.A. and I.M.T.; Resources, I.M.T.; Writing—original draft, I.M.T.; Writing—review & editing, W.K.A. and I.M.T.; Supervision, I.M.T. All authors have read and agreed to the published version of the manuscript.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Conflict of interest

The authors declare no competing financial interest.

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