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Stabilizer Limits of Strongly Stable Triples

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Abstract: Let G be a finite group. We say that (G, H, α) is a strongly stable triple if $H \leq G$, $\alpha \in \mathrm{Irr}(H)$ and $(\alpha^G)_H$ is a multiple of α. In this paper, we study the quasi-primitivity, inductors, and stabilizer limits of strongly stable triples. We show that under certain conditions all stabilizer limits of a strongly stable triple have equal degrees, thus generalizing the corresponding theorem of character triples due to Isaacs.

***Keywords*:** strongly stable triple, quasi-primitivity, inductor, stabilizer limit, degree

MSC: 20C15

1. Introduction

Let G be a finite group, $H \leq G$ and let $\alpha \in \mathrm{Irr}(H)$ be an irreducible complex character of H. If $(\alpha^G)_H$ is a multiple of α, then Shahriari (see [1]) said that α is strongly stable in G. In this situation we say that the triple $\mathscr{T} = (G, H, \alpha)$ is a strongly stable triple, and we call $\alpha(1)$ the degree of $\mathscr{T}$. Note that if $H \lhd G$, then α is G-invariant, and (G, H, α) is a character triple in the sense of [2].

It is natural to consider subtriples and inductors of a strongly stable triple. Let $\mathscr{T} = (G, H, \alpha)$ and $\mathscr{T}' = (G', H', \alpha')$ be two strongly stable triples. If $G = HG'$, $H \cap G' = H'$ and α' is an irreducible constituent of $\alpha_{H'}$, then we call $\mathscr{T}'$ a subtriple of $\mathscr{T}$. If $G' < G$, then we call the subtriple $\mathscr{T}'$ a proper subtriple of $\mathscr{T}$. If $(\alpha')^H = \alpha$, then we call the subtriple $\mathscr{T}'$ an inductor of $\mathscr{T}$. Note that if $\mathscr{T}'$ is an inductor of $\mathscr{T}$, then every inductor of $\mathscr{T}'$ is also an inductor of $\mathscr{T}$.

In order to study the inductive process of characters and some conjectures of M-groups, the concept of multi-Clifford reductions and stabilizer limits of the character triple were studied extensively. For example, Dade (see [3-5]) introduced stabilizer limits and elementary stabilizer limits of an irreducible character and applied it to study some famous conjectures of M-groups. With the help of the concept of inductors and quasi-primitivity of a character triple, Isaacs simplified Dade's results in [6] and proved that under certain conditions all stabilizer limits of a character triple have equal degrees. Shahriari [1] introduced the concept of fully ramified subgroups and proved that a fully ramified character of subgroups is a special strongly stable character. By applying strongly stable characters, Cossey, Isaacs, and Lewis [7] studied when an irreducible character of a Hall π-subgroup can be fully extendible to a π-separable group. In [8], Jin and Chang studied the equivalence of primitive inducing pairs for an irreducible character by using inductors of character triples and strengthened Isaacs' results in [6].

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Suppose that $N \leq H$ is a normal subgroup of G and that $\theta \in \mathrm{Irr}(N)$ is an irreducible constituent of α_N. Let G_θ and H_θ be the stabilizers of θ in G and H respectively, and also let $\alpha_\theta \in \mathrm{Irr}(H_\theta)$ be the Clifford correspondent of α with respect to θ. Then $(G_\theta, H_\theta, \alpha_\theta)$ is a strongly stable triple (see Lemma 2.3 below), and we call it a Clifford reduction of $\mathscr{T} = (G, H, \alpha)$. Also, we see that $G = HG_\theta$ by the Frattini argument, and thus $(G_\theta, H_\theta, \alpha_\theta)$ is always an inductor of $\mathscr{T}$. For any strongly stable triple $\mathscr{T}$, if there exists a finite sequence of subtriples, $\mathscr{T} = \mathscr{T}_0, \mathscr{T}_1, ..., \mathscr{T}_n = \mathscr{T}'$, where $\mathscr{T}_i$ is a Clifford reduction of $\mathscr{T}_{i-1}$ for $i = 1, 2, ..., n$, then we say that $\mathscr{T}'$ is a multi-Clifford reduction of $\mathscr{T}$. If in addition, $\mathscr{T}'$ has no proper Clifford reductions, we say that $\mathscr{T}'$ is a stabilizer limit of $\mathscr{T}$. Clearly, each multi-Clifford reduction of $\mathscr{T}$ is also an inductor of $\mathscr{T}$. Conversely, under certain conditions, an inductor can be a multi-Clifford reduction (see Theorem 3.2 below).

In this paper, we only discuss finite groups and complex characters. Our notation and terminology are mostly standard and can be found in [9] and [10] with a few exceptions. That is, we always write G' to denote an arbitrary group, other than the derived subgroup of G.

Now we are ready to state the main theorems of this paper.

Theorem 1.1 Let $\mathscr{T} = (G, H, \alpha)$ be a strongly stable triple, and assume that $N = \mathrm{core}_G(H)$ is nilpotent. Then all stabilizer limits of $\mathscr{T}$ have equal degrees.

It is easy to see that our Theorem 1.1 generalizes a deep theorem due to Isaacs (see Theorem 1.1 of [6]).

A strongly stable triple $\mathscr{T} = (G, H, \alpha)$ is called quasi-primitive if α_N is homogeneous for all $N \leq H$ with $N \lhd G$, that is, α_N is a multiple of some irreducible character of N. Let $\mathscr{T}' = (G', H', \alpha')$ be a strongly stable triple. If $\mathscr{T}'$ is a stabilizer limit of $\mathscr{T}$, then it has no proper Clifford reductions. It is natural to ask when a quasi-primitive inductor is a stabilizer limit. For this, we have the following result, which strengthens Proposition 2.3 of [6].

Theorem 1.2 Let $\mathscr{T} = (G, H, \alpha)$ and $\mathscr{T}' = (G', H', \alpha')$ be strongly stable triples, and let $N = \mathrm{core}_G(H)$. Assume that $\mathscr{T}'$ is an inductor of $\mathscr{T}$. Then the following hold.

(1) If $\mathscr{T}'$ is a stabilizer limit of $\mathscr{T}$, then it is quasi-primitive.

(2) If $\mathscr{T}'$ is quasi-primitive and N is nilpotent, then it is a stabilizer limit of $\mathscr{T}$.

A strongly stable triple $\mathscr{T} = (G, H, \alpha)$ is called primitive if it has no proper inductors. It is clear that if $\mathscr{T}$ is primitive, then we see that it is quasi-primitive. In fact, if $\mathscr{T} = (G, H, \alpha)$ is not quasi-primitive, then there is a normal subgroup N of G contained in H, such that $\alpha_N = e(\gamma_1 + \gamma_2 + \cdots + \gamma_t)$, where γ_i are distinct irreducible characters of N. Then $H_{\gamma_1} < H$. Let $\alpha = \beta^H$, where $\beta \in \mathrm{Irr}(H_{\gamma_1} \mid \gamma_1)$. Then $\mathscr{T}' = (G_{\gamma_1}, H_{\gamma_1}, \beta)$ is a proper inductor of $\mathscr{T}$, contradicting the primitivity of $\mathscr{T}$. Conversely, under certain conditions, a quasi-primitive strongly stable triple can be primitive.

Theorem 1.3 Let $\mathscr{T} = (G, H, \alpha)$ be a strongly stable triple, and assume that $N = \mathrm{core}_G(H)$ is nilpotent. If $\mathscr{T}$ is quasi-primitive, then it is primitive.

This clearly generalizes Corollary 2.4 of [6].

2. Preliminaries

We begin by introducing a few preliminary results, which will be needed for our purpose. The following result is Theorem 2.3 of [7], which we restate here.

Lemma 2.1 Let $N \leq G$ and $\alpha \in \mathrm{Irr}(H)$, and let $N = \mathrm{core}_G(H)$. Then the following are equivalent.

(1) (G, H, α) is a strongly stable triple.

(2) $\alpha(H - N) = 0$ and α_N is G-invariant.

Let θ be an irreducible constituent of α_N in Lemma 2.1. If α_N is homogeneous, then α is fully ramified over θ. In this situation, we also say that θ is fully ramified in H, and α is fully ramified over N. Moreover, the relevant definitions and properties of normal fully ramified subgroups can be found in Section 4 of [11].

We need the following elementary result of normal fully ramified subgroups, and the proof can be found in Lemma 2.2 of [12].

Lemma 2.2 Let $N \lhd G$ and $G' \leq G$, and assume that $G = G'N$. Write $N' = N \cap G'$. Let $\chi' \in \mathrm{Irr}(G')$ and $(\chi')^G = \chi \in \mathrm{Irr}(G)$. If χ' is fully ramified over $\theta' \in \mathrm{Irr}(N')$, then χ is fully ramified over $(\theta')^N \in \mathrm{Irr}(N)$.

Finally, we present the following result, which easily follows from Lemma 3.1 of [7].

Lemma 2.3 Let $N \leq H \leq G$ with $N \lhd G$ and let $\alpha \in \mathrm{Irr}(H)$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of α_N, and let G_θ and H_θ be the stabilizers of θ in G and H respectively. Let $\alpha_\theta \in \mathrm{Irr}(H_\theta)$ be the Clifford correspondent of α with respect to θ. If (G, H, α) is a strongly stable triple, then $G = HG_\theta$ and $(G_\theta, H_\theta, \alpha_\theta)$ is a strongly stable triple.

3. Main results

In this section, we will prove all theorems in the introduction. First, we recall the following definition of fully ramified subgroups in [1]. Let $H \le G$. If there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = e\chi$ and $\chi_H = e\alpha$, where $\chi \in \mathrm{Irr}(G)$ and e is an integer, then we say that H is fully ramified in G. In this situation we also say that α is fully ramified in G, χ is fully ramified over H, or χ is fully ramified over α.

Next, we recall a bit of notation. For any $H \le G$, we write $H^G = \langle H^g \mid g \in G \rangle$ to denote the normal closure of H in G. It is clear that H^G is the unique smallest normal subgroup of G that contains H.

We will also need the following technical result.

Lemma 3.1 Let $\mathscr{T} = (G, H, \alpha)$ and $\mathscr{T}' = (G', H', \alpha')$ be strongly stable triples. Write $N = \mathrm{core}_G(H)$, and let $G'' = NG'$ and $H'' = NH'$. Assume that $\mathscr{T}'$ is an inductor of $\mathscr{T}$ and that $\alpha'_{N'}$ is a multiple of $\theta' \in \mathrm{Irr}(N')$, where $N' = \mathrm{core}_{G'}(H')$. Let $\alpha'' = (\alpha')^{H''}$ and $\theta'' = (\theta')^N$. Then the following hold.

(1) $\alpha'' \in \mathrm{Irr}(H'')$ is fully ramified over $\theta'' \in \mathrm{Irr}(N)$.

(2) (G'', N, θ'') is a character triple.

(3) (G'', H'', α'') is a strongly stable triple and it is a Clifford reduction of $\mathscr{T}$.

Proof. We will prove (1), (2) and (3) simultaneously. Since $\mathscr{T}'$ is a strongly stable triple, it follows by Lemma 2.1 that $\alpha'(H' - N') = 0$ and $\alpha'_{N'}$ is G'-invariant. Also, since $\alpha'_{N'}$ is a multiple of $\theta' \in \mathrm{Irr}(N')$, we see that θ' is G'-invariant, and thus (G', N', θ') is a character triple. See Figure 1 below. We carry out the proof by considering two cases.

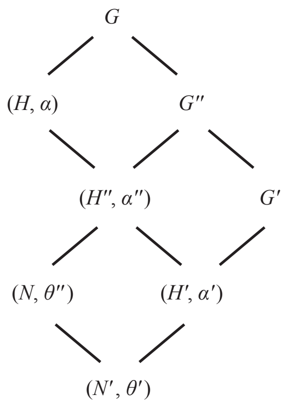

Figure 1. Clifford reduction

Case 1 $G'' = NG' < G$. Then $H'' = NH' < H$. It is clear that $H \cap G'' = H''$ and $HG'' = G$. We claim that $N = \mathrm{core}_{G''}(H'')$. To see this, write $N'' = \mathrm{core}_{G''}(H'')$, and since $N \le H''$ and $N \lhd G''$, we have $N \le N''$. Also, since $N'' \lhd G''$ and $N'' \le H'' < H$, we deduce that $(N'')^G = (N'')^{G''H} = (N'')^H \le H$. Since $(N'')^G \lhd G$, we see that $(N'')^G \le N$, and thus $N'' = N$, as claimed.

Since $H''G' = G''$ and $H'' \cap G' = H'$, it is clear that $N' = N \cap G'$. Since $((\alpha')^{H''})^H = \alpha$ is irreducible, it follows that $\alpha'' = (\alpha')^{H''} \in \mathrm{Irr}(H'')$ and α'' lies under α. Since θ' is G'-invariant and $\alpha'(H' - N') = 0$, we see that α' is fully ramified over θ'. Let $\theta'' = (\theta')^N$. By Lemma 2.2, α'' is fully ramified over $\theta'' \in \mathrm{Irr}(N)$. Note that G' stabilizes θ', so it also stabilizes θ'', and thus θ'' is G''-invariant. Now we see that (G'', N, θ'') is a character triple. Since α'' is fully ramified over θ'', we see that $\alpha''(H'' - N) = 0$ and α''_N is G''-invariant. By Lemma 2.1, (G'', H'', α'') is a strongly stable triple.

Now we claim that (G'', H'', α'') is a Clifford reduction of $\mathscr{T}$. Let $G_{\theta''}$ and $H_{\theta''}$ be the stabilizers of θ'' in G and H respectively. Since θ'' is G''-invariant, we have $G'' \le G_{\theta''}$, and thus $H'' \le H_{\theta''}$. Let $\gamma = (\alpha'')^{H_{\theta''}}$. Since $\gamma^H = \alpha$ is irreducible,

γ is irreducible. Also, since γ lies over θ'', we see that $\gamma \in \mathrm{Irr}(H_{\theta''})$ is the Clifford correspondent of α with respect to θ''. By Lemma 2.3, $G = HG_{\theta''}$ and $(G_{\theta''}, H_{\theta''}, \gamma)$ is a strongly stable triple. Moreover, note that $H \cap G_{\theta''} = H_{\theta''}$ and $G = HG_{\theta''}$, so $N = \mathrm{core}_{G_{\theta''}}(H_{\theta''})$. By Lemma 2.1, we have $\gamma(H_{\theta''} - N) = 0$, and thus γ is fully ramified over θ''. Since α'' is fully ramified over θ'', it is clear that γ is fully ramified over α'', and thus $\gamma_{H''}$ is a multiple of α''. But $(\alpha'')^{H_{\theta''}} = \gamma$ is irreducible, and we have $H'' = H_{\theta''}$. Note that $HG_{\theta''} = G = HG''$, so $G'' = G_{\theta''}$, and thus (G'', H'', α'') is a Clifford reduction of $\mathscr{T}$, as claimed.

Case 2 $G'' = NG' = G$. Then $H'' = NH' = H$. Since α' is fully ramified over θ' and (G', N', θ') is a character triple, it follows by the proof of the first case that $(\alpha')^{H} = \alpha$ is fully ramified over $(\theta')^{N} = \theta'' \in \mathrm{Irr}(N)$, and thus (G, N, θ'') is a character triple. Also, by the proof in the previous paragraph, we see that $G_{\theta''} = G'' = G$, and thus θ'' is G-invariant. The proof is now complete.

Now we prove the following result, which is inspired by Proposition 2.3 in [6].

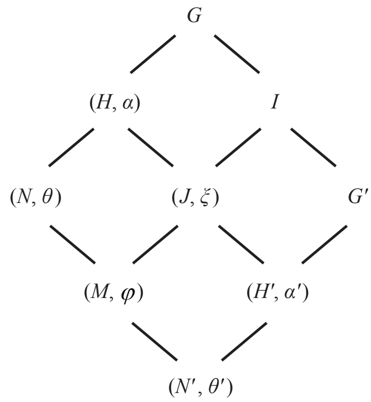

Figure 2. Multi-Clifford reduction

Theorem 3.2 Let $\mathscr{T} = (G, H, \alpha)$ and $\mathscr{T}' = (G', H', \alpha')$ be strongly stable triples, and assume that $N = \mathrm{core}_G(H)$ is nilpotent. If $\mathscr{T}'$ is an inductor of $\mathscr{T}$ and $\alpha'_{N'}$ is homogeneous, where $N' = \mathrm{core}_{G'}(H')$, then $\mathscr{T}'$ is a multi-Clifford reduction of $\mathscr{T}$.

Proof. We proceed by induction on $|G : G'|$. If $G' = G$, then there is nothing to prove, so we may assume that $G' < G$. Since $HG' = G$ and $H \cap G' = H'$, it is clear that $N' = N \cap G'$. Since $\mathscr{T}'$ is a strongly stable triple, it follows by Lemma 2.1 that $\alpha'(H' - N') = 0$ and $\alpha'_{N'}$ is G'-invariant. Let $\theta' \in \mathrm{Irr}(N')$ be an irreducible constituent of $(\alpha')_{N'}$. Since $\alpha'_{N'}$ is homogeneous, we see that θ' is G'-invariant, and thus (G', N', θ') is a character triple.

Suppose first that $NG' < G$. Let $G'' = NG'$ and $H'' = NH'$. It is clear that $H \cap G'' = H''$ and $HG'' = G$. Since $G'' < G$, we have $H'' < H$. Let $\alpha'' = (\alpha')^{H''}$. By Lemma 3.1 (3), (G'', H'', α'') is a strongly stable triple and it is a Clifford reduction of $\mathscr{T}$. Since $|G'' : G'| < |G : G'|$, we can apply the inductive hypothesis to deduce that $\mathscr{T}'$ is a multi-Clifford reduction of (G'', H'', α''). Also, we know that (G'', H'', α'') is a Clifford reduction of $\mathscr{T}$. By definition, we conclude that $\mathscr{T}'$ is a multi-Clifford reduction of $\mathscr{T}$. The theorem is proved in the case where $NG' < G$.

Now suppose that $NG' = G$, so that $NH' = H$. Let $\theta = (\theta')^{N}$. By Lemma 3.1 (2), we see that (G, N, θ) is a character triple. Let $M = (N')^{N}$. Since N is nilpotent, we have $M < N$. Also, since G' normalizes N' and $NG' = G$, we deduce that $M \lhd G$. Let $I = MG'$ and $J = MH'$. Then $I \le G$ and $J \le H$. Let $\varphi = (\theta')^{M}$. Since $\varphi^{H} = \alpha \in \mathrm{Irr}(H)$, we see that $\varphi \in \mathrm{Irr}(M)$. Note that G' stabilizes θ', so it also stabilizes φ, and thus φ is I-invariant. Since $\varphi^{N} = \theta \in \mathrm{Irr}(N)$ and $M \lhd N$, we have $N_{\varphi} = M$. By the Frattini argument, we deduce that $G = NG_{\varphi}$. Since $N \cap G_{\varphi} = M = N \cap I$ and $NI = G$, we have $|I| = |G_{\varphi}|$. Also, note that $I \le G_{\varphi}$, so $I = G_{\varphi}$. Certainly, we have $J = H_{\varphi}$. Thus we conclude that (I, M, φ) is a character triple. See Figure 2 above.

Let $\xi = (\alpha')^{J}$. Then $\xi \in \mathrm{Irr}(J)$ and lies over φ. We see that $\xi \in \mathrm{Irr}(J)$ is the Clifford correspondent of α with respect to φ, so it follows by Lemma 2.3 that (I, J, ξ) is a strongly stable triple. Certainly, (I, J, ξ) is a Clifford reduction of $\mathscr{T}$.

Note that $\mathscr{T}'$ is an inductor of (I, J, ξ) and $M = N \cap I = \mathrm{core}_I(J)$ is nilpotent. Since $|I : G'| < |G : G'|$, we can apply the inductive hypothesis to deduce that $\mathcal{T}'$ is a multi-Clifford reduction of the strongly stable triple (I, J, ξ). Also, we know that (I, J, ξ) is a Clifford reduction of $\mathcal{T}$. By definition, we conclude that $\mathcal{T}'$ is a multi-Clifford reduction of $\mathcal{T}$. The proof is now complete.

Now we prove Theorem 1.3, which is an immediate consequence of Theorem 3.2.

Theorem 3.3 Let $\mathcal{T} = (G, H, \alpha)$ be a strongly stable triple, and assume that $N = \mathrm{core}_G(H)$ is nilpotent. If $\mathcal{T}$ is quasi-primitive, then it is primitive.

Proof. Since $\mathcal{T}$ is quasi-primitive, we see that α_N is homogeneous. Moreover, $\mathcal{T}$ has no proper Clifford reductions. By Theorem 3.2, $\mathcal{T}$ has no proper inductors, so we conclude that $\mathcal{T}$ is primitive, as required.

The following is Theorem 1.2 in the introduction.

Theorem 3.4 Let $\mathcal{T} = (G, H, \alpha)$ and $\mathcal{T}' = (G', H', \alpha')$ be strongly stable triples, and let $N = \mathrm{core}_G(H)$. Assume that $\mathcal{T}'$ is an inductor of $\mathcal{T}$. Then the following hold.

(1) If $\mathcal{T}'$ is a stabilizer limit of $\mathcal{T}$, then it is quasi-primitive.

(2) If $\mathcal{T}'$ is quasi-primitive and N is nilpotent, then it is a stabilizer limit of $\mathcal{T}$.

Proof. Since $\mathcal{T}' = (G', H', \alpha')$ is a stabilizer limit of $\mathcal{T}$, it follows by definition that $\mathcal{T}'$ is a multi-Clifford reduction of $\mathcal{T}$, and thus $\mathcal{T}'$ is an inductor of $\mathcal{T}$. Also, since $\mathcal{T}'$ has no proper Clifford reductions, $\alpha'_{N'}$ is homogeneous for all $N' \leq H'$ with $N' \lhd G'$. Thus we conclude that $\mathcal{T}'$ is quasi-primitive. This proves (1).

To prove (2), write $N' = \mathrm{core}_{G'}(H')$. Since $H \cap G' = H'$ and $HG' = G$, we have $N' = N \cap G'$. Also, since $\mathcal{T}'$ is quasi-primitive, $\alpha'_{N'}$ is homogeneous. Since N is nilpotent, it follows by Theorem 3.2 that $\mathcal{T}'$ is a multi-Clifford induction of $\mathcal{T}$.

We have to show that $\mathcal{T}'$ has no proper Clifford reductions. Suppose that $\mathcal{T}'' = (G'', H'', \alpha'')$ is a proper Clifford reduction of $\mathcal{T}'$. Then there exist a normal subgroup $M' \leq H'$ of G' and an irreducible constituent θ' of $(\alpha')_{M'}$ such that $G_{\theta'} = G'' < G'$, and thus $(\alpha')_{M'}$ is inhomogeneous. This contradicts the fact that $\mathcal{T}'$ is quasi-primitive, and so we conclude that $\mathcal{T}'$ is a stabilizer limit of $\mathcal{T}$, as required.

To prove Theorem 1.1, we need the following result. See Theorem 3.1 of [6] for a proof.

Lemma 3.5 Let (G, N, θ), (G', N', θ') and (G'', N'', θ'') be character triples. Assume that (G', N', θ') and (G'', N'', θ'') are quasi-primitive inductors of (G, N, θ). If N is nilpotent, then $\theta'(1) = \theta''(1)$.

Now, we can prove Theorem 1.1 in the introduction, which we restate here.

Theorem 3.6 Let $\mathcal{T} = (G, H, \alpha)$ be a strongly stable triple, and assume that $N = \mathrm{core}_G(H)$ is nilpotent. Then all stabilizer limits of $\mathcal{T}$ have equal degrees.

Proof. Let $\mathcal{T}_1 = (G_1, H_1, \alpha_1)$ and $\mathcal{T}_2 = (G_2, H_2, \alpha_2)$ be stabilizer limits of $\mathcal{T}$. We have to prove that $\alpha_1(1) = \alpha_2(1)$. By Theorem 3.4, $\mathcal{T}_1$ and $\mathcal{T}_2$ are quasi-primitive inductors of $\mathcal{T}$. Write $N_i = \mathrm{core}_{G_i}(H_i)$ for $i = 1, 2$. Note that $N_i = N \cap G_i$ since $H \cap G_i = H_i$ and $HG_i = G$. Let $\theta_i \in \mathrm{Irr}(N_i)$ be an irreducible constituent of $(\alpha_i)_{N_i}$. Since $\mathcal{T}_i$ is quasi-primitive, we see that $(\alpha_i)_{N_i}$ is a multiple of θ_i. By Lemma 2.1, we know that θ_i is G_i-invariant, and thus (G_i, N_i, θ_i) is a character triple. Moreover, it is clear that the character triple (G_i, N_i, θ_i) is quasi-primitive. See Figure 3 below.

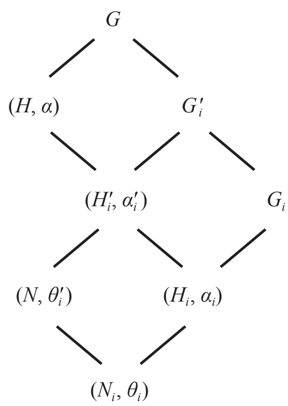

Figure 3. Stabilizer limits

Suppose first that $NG_i = G$ for $i = 1, 2$. Let $\theta = (\theta_i)^N$. By Lemma 3.1, α is fully ramified over $\theta \in \mathrm{Irr}(N)$ and (G, N, θ) is a character triple, so we have

$$\frac{\alpha(1)}{\theta(1)} = \sqrt{|H:N|} = \sqrt{|H_i:N_i|} = \frac{\alpha_i(1)}{\theta_i(1)}.$$

Moreover, note that the character triple (G_i, N_i, θ_i) is a quasi-primitive inductor of (G, N, θ). By Lemma 3.5, we have $\theta_1(1) = \theta_2(1)$, and so $\alpha_1(1) = \alpha_2(1)$. The theorem is proved in the case where $NG_i = G$.

Next, suppose that $NG_i < G$ for $i = 1, 2$, so that $NH_i < H$. Let $G_i' = NG_i$, $H_i' = NH_i$ and $N_i' = \mathrm{core}_{G_i'}(H_i')$. Note that $H \cap G_i' = H_i'$ and $HG_i' = G$, so we have $N_i' = N$. Let $\theta_i' = (\theta_i)^N$ and $\alpha_i' = (\alpha_i)^{H_i'}$. By Lemma 3.1, $\alpha_i' \in \mathrm{Irr}(H_i')$ is fully ramified over $\theta_i' \in \mathrm{Irr}(N)$ and (G_i', N, θ_i') is a character triple, so we have

$$\left[(\alpha_i')_N, \theta_i'\right] = \frac{\alpha_i'(1)}{\theta_i'(1)} = \sqrt{|H_i':N|} = \sqrt{|H_i:N_i|} = \frac{\alpha_i(1)}{\theta_i(1)}.$$

Clearly, the character triple (G_i, N_i, θ_i) is a quasi-primitive inductor of (G_i', N, θ_i').

By Lemma 3.1 (3), we see that (G_i', H_i', α_i') is a Clifford reduction of $\mathscr{T}$, where $G_i' = G_{\theta_i'}$, $H_i' = H_{\theta_i'}$ and $\alpha_i' \in \mathrm{Irr}(H_i')$ is the Clifford correspondent of α with respect to θ_i'. Note that θ_1' and θ_2' are two irreducible constituents of α_N, so it follows by Clifford's theorem that θ_1' and θ_2' are H-conjugate, and thus $[\alpha_N, \theta_1'] = [\alpha_N, \theta_2']$. By the Clifford correspondence, we see that $[\alpha_N, \theta_i'] = [(\alpha_i')_N, \theta_i']$, so it follows that

$$\frac{\alpha_1(1)}{\theta_1(1)} = \frac{\alpha_2(1)}{\theta_2(1)}.$$

Now we claim that $\theta_1(1) = \theta_2(1)$. Since $(\theta_1')^h = \theta_2'$ for some $h \in H$, we have $(G_1')^h = G_2'$. Since the character triple (G_1, N_1, θ_1) is a quasi-primitive inductor of (G_1', N, θ_1'), we see that the character triple $(G_1, N_1, \theta_1)^h = ((G_1)^h, (N_1)^h, (\theta_1)^h)$ is a quasi-primitive inductor of $(G_1', N, \theta_1')^h = (G_2', N, \theta_2')$. Also, since (G_2, N_2, θ_2) is also a quasi-primitive inductor of (G_2', N, θ_2'), it follows by Lemma 3.5 that $\theta_2(1) = (\theta_1)^h(1) = \theta_1(1)$, as claimed. Now we have $\alpha_1(1) = \alpha_2(1)$, and the theorem is proved in the case where $NG_i < G$.

We are left the case that $NG_1 = G$ and $NG_2 < G$ or $NG_1 < G$ and $NG_2 = G$. Without loss of generality, we can assume that $NG_1 = G$ and that $NG_2 < G$. Let $NG_i = G_i'$, $NH_i = H_i'$ and $N_i' = \mathrm{core}_{G_i'}(H_i')$ for $i = 1, 2$. Then $G_2' < G_1' = G$ and $H_2' < H_1' = H$. Let $\theta_1' = (\theta_1)^N$. Similar to the proof of the first case, we see that α is fully ramified over $\theta_1' \in \mathrm{Irr}(N)$. Let $\theta_2' = (\theta_2)^N$ and $\alpha_2' = (\alpha_2)^{H_2'}$. Similar to the proof of the second case, we see that $N = N_2'$ and that $\alpha_2' \in \mathrm{Irr}(H_2')$ is fully ramified over $\theta_2' \in \mathrm{Irr}(N)$. Now, both θ_1' and θ_2' lie under α and since θ_1' is the unique irreducible constituent of α_N, we deduce that $\theta_2' = \theta_1'$.

Since α is fully ramified over θ_1' and α_2' is fully ramified over θ_1', it is clear that α is fully ramified over α_2', and thus $\alpha_{H_2'}$ is a multiple of α_2'. But $(\alpha_2')^H = \alpha$ is irreducible, and we have $H_2' = H$. This contradicts the fact that $H_2' < H$, and so this case is impossible. The proof is now complete.

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