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Research Article

An Existence Result for a Class of Coupled Polyharmonic Systems

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Abstract: This work deals with the existence of positive continuous solutions for a nonlinear coupled polyharmonic system. Our analysis is based on some potential theory tools, properties of functions in the Kato class $K_{m,n}$ and the Schauder fixed point theorem.

***Keywords*:** positive solutions, Kato class, Green function, nonlinear coupled polyharmonic system, Schauder's fixed point theorem

MSC: 35B40, 35B09, 35J48, 35J58

1. Introduction

Problems of higher-order elliptic equations involving the polyharmonic operator $(-\Delta)^m$ where m is an integer greater than 2, arise in the study of models for stationary surface diffusion flow, thin elastic plates, the Paneitz-Branson equation and the Willmore equation are also known as Helfrich model in membrane biophysics [1]. Accordingly, the consideration of the polyharmonic operator has been investigated several years ago, we refer to [2-7]. Furthermore, many authors have done a lot of work dealing with the existence of positive solutions for nonlinear polyharmonic equations in different domains with various boundary conditions; see [8-14] and the references therein.

Boggio [2] proved that $G_{m,n}$, the Green function of $(-\Delta)^m$ on the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$, under Dirichlet boundary conditions $u = \dfrac{\partial u}{\partial \nu} = \ldots = \dfrac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0$, is given on $B \times B$ by:

$$G_{m,n}(x, y) = k_{m,n} \, |x - y|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(t^2-1)^{m-1}}{t^{n-1}} \mathrm{d}t, \tag{1}$$

where $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative, $k_{m,n} > 0$ and $[x, y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2)$, x, y in B.

From its expression (1), it is obvious that $G_{m,n}$ is positive on B^2. We observe that unlike the elliptic case ($m = 1$), the positivity result of Green's function isn't always true. In fact, many counter-examples [3-7] have shown that the Green's function of $(-\Delta)^m$, $m \geq 2$, does not necessarily keep a constant sign, even when considered with respect to bounded domains.

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In [9], due to the properties of the Green function $G_{m,n}$, the authors introduced the Kato class denoted by $K_{m,n}$ and defined as follows.

Definition 1 ([9])

Let q be a Borel measurable function on B.

The function q is in the Kato class $K_{m,n}$ if the following hypothesis is fulfilled:

$$\lim_{r\to 0}\left(\sup_{x\in B}\int_{B\cap B(x,r)}\left(\frac{\delta(z)}{\delta(x)}\right)^{m} G_{m,n}(x,z)\,|q(z)|\,\mathrm{d}z\right)=0.$$

From here on, $\delta(x) = 1 - |x|$ denotes the Euclidian distance from $x \in B$ to the boundary $\partial B = \{x \in \mathbb{R}^n : |x| = 1\}$.

Remark 1 We note that the Kato class $K_{m,n}$ is a linear space. Besides, if $q \in K_{m,n}$ and p is a Borel measurable function on B such that $|p| \le |q|$ almost everywhere, then $p \in K_{m,n}$.

As a typical example of functions belonging to the class $K_{m,n}$, we quote.

Example 1 ([10])

Put $q(x)=\left(\delta(x)\right)^{-\nu}\left(\log\frac{2}{\delta(x)}\right)^{-\tau}$, $x \in B$. Then, we have

$$q\in K_{m,n} \text{ if and only if } \nu<2m \text{ and } \tau\in\mathbb{R} \text{ or } \nu=2m \text{ and } \tau>1.$$

The investigation of coupled higher-order systems involving the polyharmonic operator $(-\Delta)^m$, $m \ge 2$, has recently appeared in the literature [15-17].

In [15], the authors considered the following system:

$$\begin{cases}(-\Delta)^m u+\lambda a(x)g(v)=0 & \text{in B},\\ (-\Delta)^m v+\mu b(x)h(u)=0 & \text{in B},\\ \lim\limits_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}u(x)=\varphi(\omega), & \\ \lim\limits_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}v(x)=\psi(\omega). & \end{cases} \tag{2}$$

Here λ, μ are parameters in $[0, \infty)$ and φ, ψ are two non-trivial functions in $C(\partial B, [0, \infty))$.

In order to describe the framework of [15], which is a motivation for our work, we need to outline some notations that are also necessary for the rest of the paper. For any function $\varphi \in C(\partial B, [0, \infty))$, we set $H\varphi$ the continuous bounded solution satisfying,

$$\begin{cases}\Delta H\varphi=0 & \text{in B},\\ H\varphi_{/\partial B}=\varphi. & \end{cases}$$

We remark that the map $x\mapsto(1-|x|^2)^{m-1}H\varphi(x)$, $x\in B$, is a continuous bounded solution of the boundary value problem,

$$\begin{cases}(-\Delta)^m u=0 \text{ in B}, \quad \text{(in the distributional sense)}\\ \lim\limits_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}u(x)=\varphi(\omega).\end{cases}$$

In the sequel, we fix φ and ψ two non-trivial functions in $C(\partial B, [0, \infty))$.

Put Φ and Ψ the functions defined on B respectively by:

$$\Phi(x) = (1-|x|^2)^{m-1} H\varphi(x) \text{ and } \Psi(x) = (1-|x|^2)^{m-1} H\psi(x).$$

We refer to $V_{m,n}f$ the m-potential of a Borel measurable function f on B defined by :

$$V_{m,n}f(x) = \int_B G_{m,n}(x,z) f(z)\mathrm{d}z, \ \ x \in B.$$

As usual, let $\mathcal{B}^+(B)$ be the collection of nonnegative Borel measurable functions on B, $\mathbf{L}^1_{Loc}(B)$ refers to the collection of real measurable and locally integratable functions in B. We also denote by $C(\overline{B})$ the collection of continuous functions on $\overline{B}$. The set $C_0(B)$ is the subclass of $C(\overline{B})$ vanishing continuously at ∂B. We remark that $C(\overline{B})$ and $C_0(B)$ endowed with the uniform norm $\|u\|_\infty = \sup_{x\in\overline{B}} |u(x)|$, are Banach spaces. For $(u, v) \in C(\overline{B}) \times C(\overline{B})$ (resp. $C_0(B) \times C_0(B)$), let $\|(u, v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Then clearly $(C(\overline{B}) \times C(\overline{B}), \|(. , .)\|)$ and $(C_0(B) \times C_0(B), \|(. , .)\|)$ are Banach spaces.

We recall that if $f \in \mathbf{L}^1_{Loc}(B)$ and $V_{m,n}f \in \mathbf{L}^1_{Loc}(B)$, then we have (see [13])

$$(-\Delta)^m (V_{m,n}f) = f \text{ in the distributional sense.}$$

To investigate system (2), the authors in [15] assumed the following conditions:

(C_1) The functions $g, h : [0, \infty) \to [0, \infty)$ are continuous and nondecreasing.

(C_2) The maps a, b are in $\mathcal{B}^+(B)$ such that

$$x \mapsto \frac{a(x)}{(\delta(x))^{m-1}} \text{ and } x \mapsto \frac{b(x)}{(\delta(x))^{m-1}}$$

belong to $K_{m,n}$.

(C_3) The constants:

$$\lambda^* := \inf_{x\in B} \frac{\Phi(x)}{V_{m,n}(ag(\Psi))(x)} \text{ and } \mu^* := \inf_{x\in B} \frac{\Psi(x)}{V_{m,n}(bh(\Phi))(x)}$$

are positive.

By the Schauder fixed point method, the authors [15] proved that for $(\lambda, \mu) \in [0, \lambda^*) \times [0, \mu^*)$ system (2) has a positive continuous solution (u, v) which is controlled by (Φ, Ψ), the solution of the homogeneous system associated to (2). In this paper, we consider the following nonlinear coupled polyharmonic system:

$$\begin{cases} (-\Delta)^m u + \lambda g(x, v) = 0 & \text{in B,} \\ (-\Delta)^m v + \mu h(x, u) = 0 & \text{in B,} \\ \lim\limits_{x\to\omega\in\partial B} (1-|x|^2)^{1-m} u(x) = \varphi(\omega), & \\ \lim\limits_{x\to\omega\in\partial B} (1-|x|^2)^{1-m} v(x) = \psi(\omega), & \end{cases} \tag{3}$$

where parameters $\lambda, \mu \in [0, \infty)$ and $\varphi, \psi \in C(\partial B, [0, \infty))$ are non-trivial functions.

Motivated by the paper [15], we aim to investigate the existence of continuous bounded positive solutions of (3) without imposing any special structures on the inhomogeneous terms. Furthermore, we give more general conditions ensuring the existence of solutions. Indeed, as it will be seen, our hypotheses improve and expand those of the previous work [15].

To study (3), we work with the assumptions:

(H_1) The functions $g, h : B \times [0, \infty) \to [0, \infty)$ are nondecreasing and continuous in the second variable.

(H_2) The maps,

$$p := \frac{g(., \Psi)}{\Phi} \text{ and } q := \frac{h(., \Phi)}{\Psi},$$

are in $K_{m,n}$.

Applying a fixed point argument, we achieve our main result as follows.

Theorem 1

Suppose that $(H_1) - (H_2)$ are fulfilled. Then there exist $\lambda^* > 0$ and $\mu^* > 0$ such that for each $(\lambda, \mu) \in [0, \lambda^*) \times [0, \mu^*)$ the system (3) has a positive continuous solution (u, v) satisfying on B,

$$\left(1 - \frac{\lambda}{\lambda^*}\right)\Phi \le u \le \Phi,$$

$$\left(1 - \frac{\mu}{\mu^*}\right)\Psi \le v \le \Psi.$$

We note that for the parameters λ^* and μ^*, it is not a simple existence result.

As in [15] we provide explicit terms of them, see Lemma 1 below.

We point out that this work improves the previous result in [15] since our hypotheses are more general. As it can be seen, the nonlinearities in system (2) are imposed to be separable in their variables while in our system (3) no special structure is required on $g(x, v)$ and $h(x, u)$. Besides, our hypotheses $(H_1) - (H_2)$ imply the conditions $(C_1) - (C_3)$.

First, (H_1) implies clearly (C_1).

Secondly, if $(H_1) - (H_2)$ are satisfied and $g(0)$, $h(0) > 0$, then (C_2) is fulfilled.

In fact, in case of the system (2) assumption (H_2) can be formulated as:

$$p := \frac{ag(\Psi)}{\Phi} \text{ and } q := \frac{bh(\Phi)}{\Psi}$$

are in $K_{m,n}$.

We claim that the functions

$$x \mapsto \frac{a(x)}{(\delta(x))^{m-1}} \text{ and } x \mapsto \frac{b(x)}{(\delta(x))^{m-1}}$$

belong to $K_{m,n}$.

Let $x \in B$,

$$\frac{a(x)}{(\delta(x))^{m-1}} = \frac{p(x)\Phi(x)}{(\delta(x))^{m-1} g(\Psi(x))}.$$

Since g is nondecreasing, then we have for $x \in B$,

$$0 \le \frac{a(x)}{(\delta(x))^{m-1}} \le \frac{2^{m-1}\| H\varphi\|_\infty}{g(0)} p(x). \tag{4}$$

Similarly, we obtain for $x \in B$,

$$0 \le \frac{b(x)}{(\delta(x))^{m-1}} \le \frac{2^{m-1}\| H\psi\|_\infty}{h(0)} q(x). \tag{5}$$

The assertions (4) and (5) imply that (C_2) is satisfied.

Moreover, we remark that due to Lemma 1 stated below, our hypotheses imply condition (C_3).

For the rest of the paper, the letter C will denote a generic positive constant which may vary from line to line.

The plan of the article is arranged as follows. In Section 2, we state some preparing results concerning the Green function and the Kato class $K_{m,n}$. Section 3 is dedicated to the proof of our Theorem 1. Some examples illustrating our main result are presented in Section 4.

2. Preliminaries

In this paragraph, we state a key result on the Green function $G_{m,n}$. Then, we give some properties of the functions belonging to the polyharmonic Kato class $K_{m,n}$ and a careful analysis about continuity is performed.

Proposition 1 ([9])

Let $r > 0$. If $x, y \in B$ satisfies $|x - y| \geq r$, then there is $C > 0$ such that

$$G_{m,n}(x,y) \leq C\frac{\left(\delta(x)\delta(y)\right)^m}{r^n}.$$

In the following proposition, we provide some useful properties of functions in $K_{m,n}$, which are taken from [9, 11].

Proposition 2

Let $q \in K_{m,n}$. Then the following holds:

(i) The constant $\alpha_q := \sup_{x,y\in B}\int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)}|q(z)|\,\mathrm{d}z$ is finite.

(ii) The function $x \mapsto \left(\delta(x)\right)^{2m-1} q(x)$ belongs to $L^1(B)$.

(iii) For any nonnegative harmonic function H on B, we have for $x \in B$

$$\int_B G_{m,n}(x,z)\left(1-|z|^2\right)^{m-1} H(z)|q(z)|\,\mathrm{d}z \leq \alpha_q \left(1-|x|^2\right)^{m-1} H(x).$$

(iv) Let $\beta \in \{m-1, m\}$. For each $x_0 \in \overline{B}$, we have

$$\lim_{r\to 0}\left(\sup_{x\in B}\int_{B\cap B(x_0,r)}\left(\frac{\delta(z)}{\delta(x)}\right)^{\beta} G_{m,n}(x,z)|q(z)|\,\mathrm{d}z\right) = 0.$$

(v) The function $x \mapsto \int_B \left(\frac{\delta(z)}{\delta(x)}\right)^{m-1} G_{m,n}(x,z)|q(z)|\,\mathrm{d}z$ is in $C_0(B)$.

Proposition 3

Suppose that hypotheses $(H_1) - (H_2)$ are satisfied. Let

$$S = \{(u,v) \in C_0(B)\times C_0(B) \text{ such that } 0 \leq u \leq \Phi \text{ and } 0 \leq v \leq \Psi\}$$

and

$$\Gamma = \{(V_{m,n}(g(.,v)), V_{m,n}(h(.,u))) : (u,v) \in S\}.$$

Then Γ is relatively compact in $C(\overline{B}) \times C(\overline{B})$. In particular, Γ is relatively compact in $C_0(B) \times C_0(B)$.

Proof.

Let $(u, v) \in S$ then by hypothesis (H_1), we get:

$$0 \le V_{m,n}(g(.,v)) \le V_{m,n}(g(.,\Psi)) = V_{m,n}(p\Phi). \tag{6}$$

Applying Proposition 2 (*iii*) with $H = H\varphi$, we reach:

$$V_{m,n}(p\Phi) \le \alpha_p \Phi \le \alpha_p \left\| H\varphi \right\|_\infty . \tag{7}$$

Using (6) and (7), we obtain that:

$$0 \le V_{m,n}(g(.,v)) \le \alpha_p \left\| H\varphi \right\|_\infty .$$

Similarly, we have

$$0 \le V_{m,n}(h(.,u)) \le \alpha_q \left\| H\psi \right\|_\infty .$$

Thus, the family Γ is uniformly bounded.
Now, we shall prove that Γ is equicontinuous on B.
Let $\varepsilon > 0$ and $x_0 \in B$. From Proposition 2 (*iv*), there is $r > 0$ such that,

$$0 \le \sup_{\xi \in B} \int_{B \cap B(x_0, 2r)} \left(\frac{\delta(z)}{\delta(\xi)} \right)^{m-1} G_{m,n}(\xi, z) p(z) \mathrm{d}z \le \frac{\varepsilon}{2^m \ \left\| H\varphi \right\|_\infty} . \tag{8}$$

Let $x, y \in B \cap B(x_0, r)$, then for any $(u, v) \in S$, we get:

$$\begin{aligned}
\left| V_{m,n}(g(.,v))(x) - V_{m,n}(g(.,v))(y) \right| &= \left| \int_B G_{m,n}(x,z) p(z) \Phi(z) \mathrm{d}z - \int_B G_{m,n}(y,z) p(z) \Phi(z) \mathrm{d}z \right| \\
&\le \int_B \left| G_{m,n}(x,z) - G_{m,n}(y,z) \right| p(z) \Phi(z) \mathrm{d}z \\
&\le \int_B \left| G_{m,n}(x,z) - G_{m,n}(y,z) \right| p(z) (1 - |z|^2)^{m-1} H\varphi(z) \mathrm{d}z \\
&\le 2^{m-1} \ \left\| H\varphi \right\|_\infty \int_B \left| G_{m,n}(x,z) - G_{m,n}(y,z) \right| (\delta(z))^{m-1} p(z) \mathrm{d}z \\
&\le 2^{m-1} \left\| H\varphi \right\|_\infty \int_{B \cap B(x_0, 2r)} \left| G_{m,n}(x,z) - G_{m,n}(y,z) \right| (\delta(z))^{m-1} p(z) dz \\
&+ 2^{m-1} \ \left\| H\varphi \right\|_\infty \int_{B \cap B^c(x_0, 2r)} \left| G_{m,n}(x,z) - G_{m,n}(y,z) \right| (\delta(z))^{m-1} p(z) \mathrm{d}z \\
&:= I_1 + I_2 .
\end{aligned}$$

Using the facts that, $1 \le \dfrac{1}{(\delta(x))^{m-1}}$, $1 \le \dfrac{1}{(\delta(y))^{m-1}}$ and (8), we get that:

$$I_1 \le 2^{m-1} \ \left\| H\varphi \right\|_\infty \int_{B \cap B(x_0, 2r)} \left(\frac{\delta(z)}{\delta(x)} \right)^{m-1} G_{m,n}(x,z) p(z) \mathrm{d}z$$

$$+ 2^{m-1} \|H\varphi\|_\infty \int_{B\cap B(x_0,2r)} \left(\frac{\delta(z)}{\delta(y)}\right)^{m-1} G_{m,n}(y,z)p(z)\mathrm{d}z$$

$$\leq 2^m \|H\varphi\|_\infty \sup_{\xi\in B} \int_{B\cap B(x_0,2r)} \left(\frac{\delta(z)}{\delta(\xi)}\right)^{m-1} G_{m,n}(\xi,z)p(z)\mathrm{d}z$$

$$\leq \varepsilon.$$

On the other hand, if $|z - x_0| \geq 2r$, then $|z - y| \geq r$ and $|z - x| \geq r$. Hence, by applying Proposition 1, we obtain that

$$\begin{aligned}
|G_{m,n}(x,z) - G_{m,n}(y,z)| \left(\delta(z)\right)^{m-1} p(z) &\leq \left(G_{m,n}(x,z) + G_{m,n}(y,z)\right)\left(\delta(z)\right)^{m-1} p(z) \\
&\leq \frac{C}{r^n}\Big(\left(\delta(x)\delta(z)\right)^m + \left(\delta(y)\delta(z)\right)^m\Big)\left(\delta(z)\right)^{m-1} p(z) \\
&\leq \frac{C}{r^n}\Big(\left(\delta(x)\right)^m + \left(\delta(y)\right)^m\Big)\left(\delta(z)\right)^{2m-1} p(z) \\
&\leq \frac{2C}{r^n}\left(\delta(z)\right)^{2m-1} p(z).
\end{aligned}$$

Since for $z \in B \cap B^c(x_0, 2r)$, $x \mapsto G_{m,n}(x, z)$ is continuous on $B \cap B(x_0, r)$, we deduce from Proposition 2 (*ii*) and Lebesgue's dominated convergence theorem

$$I_2 \to 0 \ \text{ as } \ |x - y| \to 0.$$

Hence, $\{V_{m,n}(g(., v)), (u, v) \in S\}$ is equicontinuous on B.

Similarly, $\{V_{m,n}(h(., u)), (u, v) \in S\}$ is equicontinuous on B.

Thus, the family Γ is equicontinuous on B.

Then, we claim that $V_{m,n}(g(., v)) \to 0$ and $V_{m,n}(h(., u)) \to 0$ as $x \to \omega \in \partial B$ uniformly in $(u, v) \in S$.

Consider $\varepsilon > 0$ and $\omega \in \partial B$. Proposition 2 (*iv*) gives that there is $r > 0$ such that,

$$0 \leq \sup_{\xi\in B} \int_{B\cap B(\omega,2r)} \left(\frac{\delta(z)}{\delta(\xi)}\right)^{m-1} G_{m,n}(\xi,z)p(z)\mathrm{d}z \leq \frac{\varepsilon}{2^{m-1}\ \|H\varphi\|_\infty}. \tag{9}$$

Let $x \in B \cap B(\omega, r)$. Then for any $(u, v) \in S$, we get from (6) and (9),

$$\begin{aligned}
V_{m,n}\left(g(.,v)\right)(x) &\leq \int_B G_{m,n}(x,z)p(z)\Phi(z)\mathrm{d}z \\
&\leq \int_B G_{m,n}(x,z)p(z)(1-|z|^2)^{m-1} H\varphi(z)\mathrm{d}z \\
&\leq 2^{m-1}\ \|H\varphi\|_\infty \int_B G_{m,n}(x,z)\left(\delta(z)\right)^{m-1} p(z)\mathrm{d}z \\
&\leq 2^{m-1} \|H\varphi\|_\infty \int_{B\cap B(\omega,2r)} G_{m,n}(x,z)\left(\frac{\delta(z)}{\delta(\xi)}\right)^{m-1} p(z)\mathrm{d}z
\end{aligned}$$

$$+2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\omega,2r)} G_{m,n}(x,z)\left(\delta(z)\right)^{m-1} p(z)\mathrm{d}z$$

$$\leq \varepsilon + 2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\omega,2r)} G_{m,n}(x,z)\left(\delta(z)\right)^{m-1} p(z)\mathrm{d}z.$$

For $y \in B \cap B^c(\omega, 2r)$ we have $|x - y| \geq r$. So, Propositions 1 and 2 (*ii*) imply that

$$2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\omega,2r)} G_{m,n}(x,z)\left(\delta(z)\right)^{m-1} p(z)\mathrm{d}z \leq C\left(\int_B \left(\delta(z)\right)^{2m-1} p(z)\mathrm{d}z\right)\left(\delta(x)\right)^m \to 0$$

as $x \longrightarrow \omega$.

By the same arguments, we have $V_{m,n}(h(., u)) \longrightarrow 0$ as $x \longrightarrow \omega \in \partial B$ uniformly in $(u, v) \in S$.

So, the Arzela-Ascoli Theorem implies that the set Γ is relatively compact in $C(\overline{B}) \times C(\overline{B})$. Then, since $\Gamma \subset C_0(B) \times C_0(B)$ which is a Banach space included in $C(\overline{B}) \times C(\overline{B})$, we conclude that Γ is relatively compact in $C_0(B) \times C_0(B)$.

3. Proof of Theorem 1

Before getting started with the proof of our Theorem 1, we give the following preliminary result.

Lemma 1

If g, h satisfies (H_2), then

$$\lambda^* := \inf_{x\in B} \frac{\Phi(x)}{V_{m,n}\left(g(.,\Psi)\right)(x)} > 0 \text{ and } \mu^* := \inf_{x\in B} \frac{\Psi(x)}{V_{m,n}\left(h(.,\Phi)\right)(x)} > 0.$$

Proof.

According to (H_2) the maps $p = \dfrac{g(.,\Psi)}{\Phi}$ and $q = \dfrac{h(.,\Phi)}{\Psi}$ belong to $K_{m,n}$. As in the proof of Proposition 3, we obtain that on B:

$$V_{m,n}\left(g(.,\Psi)\right) \leq \alpha_p \Phi$$

and

$$V_{m,n}\left(h(.,\Phi)\right) \leq \alpha_q \Psi.$$

These estimates imply that for each $x \in B$:

$$\frac{1}{\alpha_p} \leq \frac{\Phi(x)}{V_{m,n}\left(g(.,\Psi)\right)(x)} \text{ and } \frac{1}{\alpha_q} \leq \frac{\Psi(x)}{V_{m,n}\left(h(.,\Phi)\right)(x)},$$

which gives that

$$\lambda^* \geq \frac{1}{\alpha_p} > 0 \text{ and } \mu^* \geq \frac{1}{\alpha_q} > 0.$$

Now, we are prepared to show Theorem 1.

Proof of Theorem 1 We consider the non-empty convex closed set Λ given by:

$$\Lambda = \left\{(u,v) \in C_0(B) \times C_0(B) \text{ such that } \left(1 - \frac{\lambda}{\lambda^*}\right)\Phi \le u \le \Phi \text{ and } \left(1 - \frac{\mu}{\mu^*}\right)\Psi \le v \le \Psi\right\}.$$

We define the operator T on Λ by:

$$T(u,v) = (y,z),$$

where

$$y = \Phi - \lambda V_{m,n}\left(g(.,v)\right) \text{ and } z = \Psi - \mu V_{m,n}\left(h(.,u)\right).$$

We attempt to show that T admits a fixed point in Λ.

From Proposition 3, we have that $\{(V_{m,n}(g(.,v)), V_{m,n}(h(.,u))), (u,v) \in \Lambda\}$ is relatively compact in $C_0(B) \times C_0(B)$. Since Φ and Ψ are in $C_0(B)$, we deduce that $T\Lambda$ is also relatively compact in $C_0(B) \times C_0(B)$.

Next, we intend to show that T is a compact operator from Λ into itself.

Let $(u,v) \in \Lambda$. Then by hypothesis (H_1), the maps g, h are nondecreasing in their second variables. So, we obtain:

$$V_{m,n}\left(g(.,v)\right) \le V_{m,n}\left(g(.,\Psi)\right),$$

$$V_{m,n}\left(h(.,u)\right) \le V_{m,n}\left(h(.,\Phi)\right).$$

Then, since $\lambda \in [0, \lambda^*)$ and $\mu \in [0, \mu^*)$, we obtain that:

$$\Phi - \lambda V_{m,n}\left(g(.,\Psi)\right) \le \Phi - \lambda V_{m,n}\left(g(.,v)\right) \le \Phi,$$

$$\Psi - \mu V_{m,n}\left(h(.,\Phi)\right) \le \Psi - \mu V_{m,n}\left(h(.,u)\right) \le \Psi.$$

This implies, by using Lemma 1, that:

$$\Phi\left(1 - \frac{\lambda}{\lambda^*}\right) \le y \le \Phi \quad \text{and} \quad \Psi\left(1 - \frac{\mu}{\mu^*}\right) \le z \le \Psi.$$

Taking into account that $T\Lambda \subset C_0(B) \times C_0(B)$, we conclude that T is a self-map on Λ.

Now, we establish that $T: \Lambda \to \Lambda$ is continuous in norm $\|(.\,,.)\|$.

Consider $((u_k, v_k))_{k \in \mathbb{N}}$ a sequence in Λ converging to $(u,v) \in \Lambda$.

Put

$$(y_k, z_k) = T(u_k, v_k),\ k \in \mathbb{N} \quad \text{and} \quad (y,z) = T(u,v).$$

We get for $k \in \mathbb{N}$, $x \in B$,

$$\begin{aligned} |y_k(x) - y(x)| &= |-\lambda V_{m,n}\left(g(.,v_k)\right)(x) + \lambda V_{m,n}\left(g(.,v)\right)(x)| \\ &\le \lambda \int_B G_{m,n}(x,\xi)\,|g\left(\xi, v_k(\xi)\right) - g\left(\xi, v(\xi)\right)|\,\mathrm{d}\xi. \end{aligned}$$

On the other hand, since the function g is nondecreasing in the second variable, we get for $k \in \mathbb{N}$ and $(x, \xi) \in B^2$,

$$\begin{aligned} G_{m,n}(x,\xi)\,|\,g(\xi, v_k(\xi)) - g(\xi, v(\xi))\,| &\le 2G_{m,n}(x,\xi)g(\xi, \Psi(\xi)) \\ &\le 2G_{m,n}(x,\xi)p(\xi)\Phi(\xi). \end{aligned}$$

The continuity of g in the second variable and the fact that $V_{m,n}(p\Phi) \le \alpha_p\Phi \le \alpha_p\|H\varphi\|_\infty < +\infty$, imply due to the Lebesgue's theorem that for $x \in B$,

$$|\,y_k(x) - y(x)\,| \to 0 \;\; as \;\; k \to \infty.$$

With the same arguments as before, we get that for each $x \in B$, $|z_k(x) - z(x)| \to 0$ as $k \to +\infty$.
The relative compactness of $T\Lambda$ in $C(\overline{B}) \times C(\overline{B})$ yields the uniform convergence, that is

$$\|(y_k - y, z_k - z)\| \to 0 \;\; as \;\; k \to +\infty.$$

Thereby we have shown that T is a compact operator from Λ into itself.
Therefore, Schauder's fixed point theorem implies the existence of a point $(u, v) \in \Lambda$ satisfying:

$$u = \Phi - \lambda V_{m,n}\left(g(.,v)\right), \tag{10}$$

and

$$v = \Psi - \mu V_{m,n}\left(h(.,u)\right). \tag{11}$$

The pair (u, v) is clearly positive continuous satisfying:

$$\left(1 - \frac{\lambda}{\lambda^*}\right)\Phi \le u \le \Phi \;\; \text{and} \;\; \left(1 - \frac{\mu}{\mu^*}\right)\Psi \le v \le \Psi.$$

As the rest of the proof, we need to prove that (u, v) is a solution of system (3).
We have $V_{m,n}(g(., v)) \in C_0(B)$ which implies $V_{m,n}(g(., v)) \in \mathbf{L}^1_{\mathbf{loc}}(B)$. On the other hand,

$$0 \le g(.,v) \le 2^{m-1}\|H\varphi\|_\infty \left(\delta(.)\right)^{m-1} p.$$

Since we have $x \to \left(\delta(x)\right)^{2m-1} p(x) \in L^1(B)$, we get $g(., v) \in \mathbf{L}^1_{\mathbf{loc}}(B)$.
Hence we have

$$(-\Delta)^m V_{m,n}\left(g(.,v)\right) = g(.,v) \;\;\text{ in } B \;\; \text{(in the distributional sense).}$$

In the same way

$$(-\Delta)^m V_{m,n}\left(h(.,u)\right) = h(.,u) \;\;\text{ in } B \;\; \text{(in the distributional sense).}$$

Now, applying the operator $(-\Delta)^m$ in (10) and (11) we obtain that:

$$(-\Delta)^m u = -\lambda g(.,v),$$

$$(-\Delta)^m v = -\mu h(.,u).$$

Finally we have

$$\lim_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}u(x) = \varphi(\omega) - \lambda \lim_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}V_{m,n}\left(g(.,v)\right)(x).$$

Since for $x \in B$, we have

$$0 \le (1-|x|^2)^{1-m}V_{m,n}\left(g(.,v)\right)(x) \le 2^{m-1}\left\|H\varphi\right\|_\infty \int_B \left(\frac{\delta(z)}{\delta(x)}\right)^{m-1} G_{m,n}(x,z)p(z)\mathrm{d}z,$$

we deduce by Proposition 2 (v) that

$$\lim_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}V_{m,n}\left(g(.,v)\right)(x) = 0.$$

Hence

$$\lim_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}u(x) = \varphi(\omega).$$

Similarly,

$$\lim_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}v(x) = \psi(\omega).$$

This completes the proof.

4. Examples

In this section, we present two examples for the illustration of our Theorem 1.

Example 2 Let φ, ψ be two continuous positive functions on ∂B, λ, μ be nonnegative constants and $\alpha, \beta > 1$. We consider the functions a and b defined on B by:

$$a(x) = \frac{1}{\left(\delta(x)\right)^\gamma}, \text{ with } \gamma < 2m$$

and

$$b(x) = \frac{1}{\left(\delta(x)\right)^{2m}\left(\log\left(\frac{2}{\delta(x)}\right)\right)^\nu}, \text{ with } \nu \in \mathbb{R}.$$

We consider the system,

$$
\begin{cases}
(-\Delta)^m u + \lambda a(x) v^{\alpha} = 0, & \text{in B,} \\
(-\Delta)^m v + \mu b(x) u^{\beta} = 0, & \text{in B,} \\
\lim_{x \to \omega \in \partial B} (1-|x|^2)^{1-m} u(x) = \varphi(\omega), \\
\lim_{x \to \omega \in \partial B} (1-|x|^2)^{1-m} v(x) = \psi(\omega).
\end{cases}
\tag{12}
$$

Here $g(x, t) = a(x)t^{\alpha}$ and $h(x, t) = b(x)t^{\beta}$, $(x, t) \in B \times [0, \infty)$.

It is clear that (H_1) is satisfied.

On the one hand, for $x \in B$,

$$
\frac{g\left(x, \Psi(x)\right)}{\Phi(x)} = \frac{(1-|x|^2)^{(m-1)(\alpha-1)}\left(H\psi(x)\right)^{\alpha}}{\left(\delta(x)\right)^{\gamma} H\varphi(x)}.
$$

Since the functions φ and ψ are positive and continuous on the compact ∂B, the functions $H\varphi$ and $H\psi$ are bounded and bounded away from zero.

Hence, there is $C > 0$ such for $x \in B$,

$$
\frac{g\left(x, \Psi(x)\right)}{\Phi(x)} \leq \frac{C}{\left(\delta(x)\right)^{\gamma}}. \tag{13}
$$

By Example 1 and (13) we obtain that

$$
p = \frac{g(.,\Psi)}{\Phi} \in K_{m,n}.
$$

On the other hand for $x \in B$,

$$
\begin{aligned}
0 \leq \frac{h\left(x, \Phi(x)\right)}{\Psi(x)} &= \frac{\left(1+|x|\right)^{(m-1)(\beta-1)}\left(H\varphi(x)\right)^{\beta}}{\left(\delta(x)\right)^{2m-(m-1)(\beta-1)}\left(\log\left(\frac{2}{\delta(x)}\right)\right)^{\nu} H\psi(x)} \\
&\leq \frac{C}{\left(\delta(x)\right)^{2m-(m-1)(\beta-1)}\left(\log\left(\frac{2}{\delta(x)}\right)\right)^{\nu}}.
\end{aligned}
\tag{14}
$$

Since $2m - (m-1)(\beta-1) < 2m$, we deduce from Example 1 and (14) that

$$
q = \frac{h(.,\Phi)}{\Psi} \in K_{m,n}.
$$

Hence, (H_2) is fulfilled.

Then by Theorem 1, there exists λ^*, $\mu^* > 0$ such that for each $(\lambda, \mu) \in [0, \lambda^*) \times [0, \mu^*)$, system (12) has a positive continuous solution (u, v) satisfying,

$$
(1-\frac{\lambda}{\lambda^*})\Phi \leq u \leq \Phi,
$$

$$(1-\frac{\mu}{\mu^*})\Psi \le v \le \Psi.$$

Example 3 Let φ, ψ be two continuous positive functions on ∂B, $\alpha > 0$, $\beta \ge 1$ and $\gamma > 1$. Consider the system

$$\begin{cases} (-\Delta)^m u + \lambda\left(v(x) - \sin\left((\delta(x))^\alpha v(x)\right)\right) = 0, & \text{in } B, \\ (-\Delta)^m v + \mu \ln\left(1 + \dfrac{u^\beta(x)}{(\delta(x))^{2m+(\beta-1)(m-1)}\left(\log\dfrac{2}{\delta(x)}\right)^\gamma}\right) = 0, & \text{in } B, \\ \lim\limits_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}u(x) = \varphi(\omega), \\ \lim\limits_{x\to\omega\in\partial B}(1-|x|^2)^{1-m}v(x) = \psi(\omega), \end{cases} \tag{15}$$

where

$$g(x,t) = t - \sin\left((\delta(x))^\alpha t\right) \text{ and } h(x,t) = \ln\left(1 + \frac{t^\beta}{(\delta(x))^{2m+(\beta-1)(m-1)}\left(\log\frac{2}{\delta(x)}\right)^\gamma}\right).$$

We note that, since for $(x, t) \in B \times [0, \infty)$, $|\sin((\delta(x))^\alpha t)| \le (\delta(x))^\alpha t \le t$, then g is a nonnegative function. It is clear that g is continuous in the second variable.

Moreover, we remark that for $(x, t) \in B \times [0, \infty)$,

$$\frac{\partial g}{\partial t}(x,t) = 1 - (\delta(x))^\alpha \cos\left((\delta(x))^\alpha t\right).$$

Using the fact that, on $B \times [0, \infty)$, $|(\delta(x))^\alpha \cos((\delta(x))^\alpha t)| \le 1$, we obtain that g is nondecreasing in the second variable.

On the other hand, it is obvious to see that the function h is a nonnegative function defined on $B \times [0, \infty)$, which is continuous and nondecreasing in the second variable.

Now, let's verify that the hypothesis (H_2) is fulfilled.

Let $x \in B$,

$$\begin{aligned} 0 \le p(x) &= \frac{\Psi(x) - \sin\left((\delta(x))^\alpha \Psi(x)\right)}{\Phi(x)} \\ &\le \frac{H\psi(x)}{H\varphi(x)} + \left|\frac{\sin\left((\delta(x))^\alpha (1-|x|^2)^{m-1} H\psi(x)\right)}{(1-|x|^2)^{m-1} H\varphi(x)}\right| \\ &\le \frac{H\psi(x)}{H\varphi(x)}\left(1 + (\delta(x))^\alpha\right) \\ &\le C\left(1 + (\delta(x))^\alpha\right). \end{aligned}$$

By Example 1 and since $\alpha > 0$, we deduce that the map $x \mapsto 1 + (\delta(x))^{\alpha}$ belong to $K_{m,n}$. This implies that $p \in K_{m,n}$. Besides, for $x \in B$,

$$0 \le q(x) = \frac{\ln\left(1 + \dfrac{\Phi^{\beta}(x)}{\left(\delta(x)\right)^{2m+(\beta-1)(m-1)}\left(\log\dfrac{2}{\delta(x)}\right)^{\gamma}}\right)}{\Psi(x)}$$

$$\le \frac{\Phi^{\beta}(x)}{\left(\delta(x)\right)^{2m+(\beta-1)(m-1)}\left(\log\dfrac{2}{\delta(x)}\right)^{\gamma}\Psi(x)}$$

$$\le \frac{C}{\left(\delta(x)\right)^{2m}\left(\log\dfrac{2}{\delta(x)}\right)^{\gamma}}.$$

Taking into account that $\gamma > 1$, we conclude from Example 1, that $q \in K_{m,n}$.

Hence, Theorem 1 implies the existence of $\lambda^*, \mu^* > 0$ such that for each $(\lambda, \mu) \in [0, \lambda^*) \times [0, \mu^*)$, system (15) has a positive continuous solution (u, v) satisfying,

$$\left(1 - \frac{\lambda}{\lambda^*}\right)\Phi \le u \le \Phi,$$

$$\left(1 - \frac{\mu}{\mu^*}\right)\Psi \le v \le \Psi.$$

5. Conclusion

We have improved and expanded the result proved in [15]. We emphasize that notable features of this work are that the nonlinearities are not required to have any special structure and include a large class of functions. Our examples illustrate these facts. For instance, taking $\psi = m + 1$ in Example 2 we have found an example where $x \mapsto \dfrac{a(x)}{(\delta(x))^{m-1}}$ is not in $K_{m,n}$ (thus (C_2) is not fulfilled), yet Theorem 1 remains applicable. Furthermore, we may consider nonlinearities which are not separable in their variables, as demonstrated in Example 3.

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