Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

On a Subclass of Close-to-Convex Functions Involving a Zeta–Riemann Fractional Differential Operator

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Received: 22 June 2026; **Revised:** 21 July 2026; **Accepted:** 30 July 2026

Abstract: This study contributes to the advancement of geometric function theory by developing new results through the application of differential subordination techniques to normalized analytic functions associated with a newly defined fractional operator, referred to as the Zeta–Riemann fractional differential operator. By employing this operator, we introduce and investigate a novel subclass of close-to-convex univalent functions in the open unit disk. The proposed class is examined in detail, and several significant analytical properties are established, including inclusion relationships with existing function classes, convolution (Hadamard product) properties, and various structural and geometric characteristics. Furthermore, sufficient conditions are derived to ensure membership in this class, highlighting the role of the operator in shaping the behavior of analytic functions.

***Keywords*:** analytic function, differential subordination, fractional operator, Riemann Zeta function, Hurwitz-Lerch Zeta function

MSC: 30C80, 30C45

1. Introduction

Geometric Function Theory (GFT) is a prominent branch of complex analysis that investigates analytic functions through their geometric behavior. It establishes a deep connection between analytic properties and geometric features such as growth, distortion, rotation, and covering theorems. Among its central themes are the study of univalent (injective) and multivalent functions, both of which have been extensively explored due to their rich theoretical structure and wide applicability. These classes of functions continue to attract significant attention, not only for their intrinsic mathematical elegance but also for their relevance in various applied domains.

The theory of univalent functions, in particular, occupies a foundational position in complex analysis, serving as a cornerstone for developments in both single-variable and multivariable settings. Classical monographs by Duren [1] and Goodman [2] provide comprehensive treatments of this subject, highlighting its fundamental concepts and key results.

Copyright ©2026 Georgia Irina Oros, et al.
DOI: https://doi.org/10.37256/cm.75202610335
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Over time, numerous subclasses of analytic and meromorphic functions have been introduced and investigated, each contributing to a deeper understanding of geometric properties and functional behavior.

An essential tool in GFT is the concept of subordination, which provides a powerful method for comparing analytic functions based on their mapping properties. This notion was first introduced by Littlewood [3] and later formalized and systematically developed by Rogosinski [4], who established several foundational results. Subordination has since become a central technique in the study of analytic function classes, enabling the derivation of inclusion relationships, coefficient bounds, and structural properties.

The scope of subordination theory was significantly broadened with the introduction of differential subordination by Miller and Mocanu [5], which extends classical subordination to include relations involving derivatives of analytic functions. This framework has proven to be highly effective in addressing a wide range of problems in GFT. In addition, Srivastava and Owa [6] applied subordination methods to investigate properties of generalized hypergeometric functions, thereby illustrating its versatility in handling special functions.

There is a growing interest in integrating operator theory, convolution structures, and special functions into the study of analytic and meromorphic function classes. Such approaches have led to the development of new subclasses and the establishment of refined inclusion and characterization results. A considerable body of literature reflects these advancements, as seen in recent studies such as Ali et al. [7] who utilized the structural properties of the q-Sălăgean derivative operator to obtain sharp upper bounds for both the second Hankel determinant and the classical Fekete-Szegő functional for a new class of analytic functions.

Concurrently, the theoretical framework and multidisciplinary applications of fractional calculus have advanced rapidly. Within geometric function theory, these fractional calculus operators are widely used to investigate different forms of differential subordinations. For instance, third-order differential subordinations and superordinations for analytic and p-valent functions were studied using fractional operators in [8], and more recently in [9] and [10].

Let $\mathscr{A}$ denote the class of functions f that are analytic and univalent in the open unit disk $\mathscr{U} = \{\xi \in \mathbb{C}: |\xi| < 1\}$ and satisfy the normalization $f(0) = f'(0) - 1 = 0$. Every function in this class has a power series expansion of the form

$$f(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa \xi^\kappa. \tag{1}$$

For two analytic functions f and g in $\mathscr{U}$, f is subordinate to g, denoted by $f(\xi) \prec g(\xi)$, if there exists an analytic Schwarz function ϖ with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ for $\xi \in \mathscr{U}$ such that $f(\xi) = g(\varpi(\xi))$. When g is univalent in $\mathscr{U}$, this condition is equivalent to $f(0) = g(0)$ and $f(\mathscr{U}) \subseteq g(\mathscr{U})$ (see [5, 11]).

If $f_1, f_2 \in \mathscr{A}$ are given by $f_\iota(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\iota}\xi^\kappa$ for $\iota = 1, 2$, their Hadamard product (or convolution), denoted by $(f_1 * f_2)(\xi)$, is defined by (see [1], p. 246)

$$(f_1 * f_2)(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,1} a_{\kappa,2} \xi^\kappa = (f_2 * f_1)(\xi).$$

A function $f \in \mathscr{A}$ is starlike of order δ if it satisfies

$$\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \delta, \quad \xi \in \mathscr{U},\ 0 \le \delta < 1,$$

and it is convex of order δ if

$$\Re\left[1+\frac{\xi f''(\xi)}{f'(\xi)}\right]>\delta,\quad \xi\in\mathscr{U},\ 0\le\delta<1.$$

The classes $ST(\delta)$ and $CV(\delta)$ were introduced by Robertson [12]. For $\delta=0$, these families reduce to the standard classes of starlike and convex functions, denoted by $ST\equiv ST(0)$ and $CV\equiv CV(0)$, respectively.

We note that the connection between these families is given by the classical Alexander relation:

$$f(\xi)\in CV(\delta)\Leftrightarrow \xi f'(\xi)\in ST(\delta).$$

The class K of close-to-convex functions, introduced by Kaplan [13], consists of all functions $f\in\mathscr{A}$ for which a starlike function $g\in ST$ can be found such that

$$\Re\left(\frac{\xi f'(\xi)}{g(\xi)}\right)>0,\ \xi\in\mathscr{U}.$$

The Zeta–Riemann fractional differential operator incorporates the core properties of the classical Riemann Zeta function into the framework of non-integer calculus. This operator is utilized to model non-local phenomena arising in mathematical physics. It is equally relevant for problems in applied analysis where fractional-order derivatives are required. While fractional calculus extends standard differentiation and integration to arbitrary orders to describe complex dynamical systems, the inclusion of the zeta function introduces structural properties from analytic number theory into the operator framework.

The following fractional differential operator was investigated recently by Tayyah and Atshan [14].

$$\mathfrak{D}^{\alpha}_{(s,k,w)}f(\xi)=\sum_{\kappa=1}^{\infty}\frac{(w+1)^{\frac{s(\alpha-1)+1}{k}}\Gamma\left(\frac{\kappa}{w+1}+1\right)}{k^{\frac{s(\alpha-1)+1}{k}}\Gamma\left(\frac{\kappa}{w+1}-\frac{s(\alpha-1)+1}{k}+1\right)}a_{\kappa}\xi^{(w+1)\left(1-\frac{s(\alpha-1)+1}{k}\right)+\kappa-1},$$

$$\left(s\in\mathbb{N},\ k>0,w\ge0,\ 0\le\frac{s(\alpha-1)+1}{k}<1\right).\tag{2}$$

The classical Hurwitz-Lerch Zeta function is defined in [15] as the series expansion

$$\Phi(\xi,s,\alpha)=\sum_{\kappa=0}^{\infty}\frac{\xi^{\kappa}}{(\kappa+\alpha)^{s}},\tag{3}$$

where $s\in\mathbb{C}$, $\alpha\in\mathbb{C}\backslash\mathbb{Z}^{-}$, and $\Re(s)>1$ when $|\xi|=1$.

For a comprehensive foundation on the Hurwitz-Lerch Zeta function and its generalizations, we refer to the fundamental monograph by Srivastava and Choi [15].

Using the Hadamard product, we consider the Zeta–Riemann fractional differential operator introduced by Shexo et al. [16], defined as:

$$\mathfrak{D}_s^{(\alpha,\beta)} f(\xi) = \left[D^{\alpha}_{(s,1,0)} f(\xi)\right] * \left[\xi^{-(s(\alpha-1)+1)}(\Phi(\xi, s-1, \alpha+\beta) - \alpha^{-s})\right]$$

$$= \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa - s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}} a_\kappa \xi^{\kappa-[s(\alpha-1)+1]}. \tag{4}$$

For the parameter choice $s = 1$, this operator reduces to the fractional differential operator introduced by Srivastava and Owa in [6]:

$$\mathfrak{D}_1^{(\alpha,\beta)} f(\xi) = \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa-\alpha+1)} a_\kappa \xi^{\kappa-\alpha}. \tag{5}$$

Definition 1 [16] By utilizing the operator $\mathfrak{D}_s^{(\alpha,\beta)}$ defined by (4), the normalized Zeta–Riemann fractional differential operator $\mathscr{D}_s^{(\alpha,\beta)}$: $\mathscr{A} \to \mathscr{A}$ (introduced by Shexo et al. [16]) is defined as the convolution of the Hurwitz-Lerch Zeta function and Zeta–Riemann Fractional differential operator as follows:

$$\mathscr{D}_s^{(\alpha,\beta)} f(\xi) = (1+\alpha+\beta)^{s-1}\Gamma(1-s(\alpha-1))\,\xi^{[s(\alpha-1)+1]}\mathfrak{D}_s^{(\alpha,\beta)} f(\xi)$$

$$= \xi + \sum_{\kappa=2}^{\infty} \frac{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}} a_\kappa \xi^{\kappa}. \tag{6}$$

The linearity and boundedness of the operator $\mathscr{D}_s^{(\alpha,\beta)} f(\xi)$ hold in the unit disk $\mathscr{U}$, satisfying

$$\xi\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)' = s(1-\alpha)\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi) - [s(1-\alpha)-1]\mathscr{D}_s^{(\alpha,\beta)} f(\xi) \tag{7}$$

Research into the properties and applications of the Zeta–Riemann fractional differential operator remains highly active. Recent studies offer new insights in pure and computational analysis. By evaluating fundamental criteria such as the stability and existence of solutions, along with numerical approximation methods, researchers can utilize this operator to construct models that capture the empirical behaviors of physical systems. Furthermore, the systematic connection between fractional-order calculus and the analytical properties of the classical Zeta function establishes a foundation for discovering relationships in alternative fields of mathematics, such as chaos theory, fractal geometries, and complex network dynamics.

The main motivation for introducing the Zeta–Riemann fractional differential operator is its ability to combine several familiar fractional operators into a single, unified framework. Standard fractional operators usually rely on simple power-law weights that lack adaptability. In contrast, the kernel of $\mathscr{D}_s^{(\alpha,\beta)}$ is based on the Hurwitz–Lerch Zeta function, which introduces three independent parameters: α, β, and s. From an analytical perspective, these additional parameters provide enhanced flexibility over the mapping behavior and distortion properties of normalized analytic functions. This flexibility allows the operator to uncover new inequalities and inclusion properties that cannot be realized when using traditional, fixed-parameter fractional operators.

This parametric control is explicitly demonstrated by the behavior of the obtained boundary estimates, which dynamically depend on the operational parameters and are expressed in terms of Gaussian hypergeometric functions.

When these parameters are chosen as $\alpha = \beta = 0$ and $s = 1$, the operator reduces to the classical derivative, and the parametric bounds reduce to the well-known sharp constants. The corollaries of the general theorems and the corresponding examples presented throughout this work directly demonstrate this versatility of the results, as the generalized subordination criteria and functional constraints reduce to the classical baseline properties of standard close-to-convex function classes. These special cases confirm that our multi-parameter approach provides a natural extension of the classical close-to-convex and bounded turning families.

A sequence $\{c_\kappa\}_{\kappa=1}^{\infty}$ of real numbers is called a convex null sequence if $c_\kappa \to 0$ as $\kappa \to \infty$, and it satisfies the second-order difference condition $\Delta^2 c_\kappa = c_\kappa - 2c_{\kappa+1} + c_{\kappa+2} \geq 0$ for all $\kappa \geq 1$. Geometrically, this analytic property ensures that the sequence components form a convex configuration relative to their convergence limit.

In the framework of geometric function theory, convex hull sequences and convex null sequences serve as powerful analytical tools. Rather than being confined to computational geometry boundaries, their significance in complex analysis stems from their ability to establish positivity conditions for trigonometric sums and to facilitate the application of convolution techniques. These sequences provide a rigorous foundation for constructing specialized subclasses of analytic and univalent functions, determining sharp coefficient estimates, and deriving various inclusion and subordination properties. Consequently, they have been extensively employed in the structural investigation of starlike, convex, and close-to-convex functions.

From a broader perspective, the theory of convex null sequences is deeply rooted in classical results concerning positive real parts of analytic functions and sequence transformations, while convex hull properties arise naturally in the analysis of coefficient inequalities and extremal problems. Their applications extend beyond geometric function theory to approximation theory, operator theory, and other branches of mathematical analysis. Consequently, these concepts serve as valuable background material, providing the theoretical motivation for many modern investigations involving differential operators, convolution structures, and generalized classes of analytic functions.

The central goal of the present work is to introduce and evaluate a novel family of univalent functions of bounded turning of order δ defined in the open unit disk through the action of this Zeta–Riemann fractional differential operator. To fully characterize this family, we establish distinct inclusion criteria, analyze its behavior under Hadamard convolution products, and derive sufficient conditions for functions to belong to this class.

2. Preliminary results

We define the following class of functions using the operator defined in (6).

Definition 2 A function $f \in \mathscr{A}$ is said to be in the class of functions with bounded turning (which forms a subclass of close-to-convex functions) $\Omega_s^{(\alpha,\beta)}(\delta)$, if and only if

$$\Re\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)' > \delta, \quad \xi \in \mathscr{U}.$$

The geometric condition $\Re\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)' > \delta$ shares a close relationship with Kaplan's classical formulation of close-to-convexity [13]. Specifically, a function belongs to the standard close-to-convex class if it satisfies the condition

$$\Re\left\{\frac{\xi\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'}{g(\xi)}\right\} > 0,$$

for some starlike function $g \in \mathscr{S}^*$. Choosing the identity function $g(\xi) = \xi$, which is starlike, reduces this condition directly to the bounded turning inequality used here when $\delta = 0$. Since every function of bounded turning of order δ is structurally close-to-convex, the class $\Omega_s^{(\alpha,\beta)}(\delta)$ forms a specialized subclass of the broader close-to-convex family.

When $\alpha = \beta = 0$ and $s = 1$, we obtain the class $K(\delta)$ as follows:

$$\Re\left(f'(\xi)\right) > \delta, \quad \xi \in \mathscr{U}.$$

This class of functions was introduced by MacGregor [17] and further investigated in the fundamental monographs by Pommerenke [18] and Ruscheweyh [19].

The next lemmas are used to demonstrate our results:

Lemma 1 [20] If $p(\xi)$ is analytic in $\mathscr{U}$ with $p(0) = 1$ and $\Re\{p(\xi)\} > 1/2$, $\xi \in \mathscr{U}$, then for any analytic function F, in $\mathscr{U}$, the function $p * F$ takes its values in the convex hull of $F(\mathscr{U})$.

A sequence of non-negative numbers $\{c_n\}$ is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$ and

$$c_0 - c_1 \geq c_1 - c_2 \geq \ldots \geq c_k - c_{k+1} \geq \ldots \geq 0.$$

Lemma 2 [21] Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function

$$p(\xi) = \frac{c_0}{2} + \sum_{\kappa=1}^{\infty} c_\kappa \xi^\kappa, \quad \xi \in \mathscr{U}.$$

is analytic and $\Re\{p(\xi)\} > 0$.

Remark 1 When a convex null sequence $\{c_k\}_{k=0}^{\infty}$ fulfills the condition $c_0 = 1$, Lemma 1 implies that $\Re\{p(\xi)\} > 1/2$. Consequently, application of this sequence to an arbitrary analytic function generates a mapping of the open unit disk onto the convex hull of the original image domain, as established by Lemma 2.

Remark 2 Given that the convex hull represents the minimal convex set encompassing a given domain, the property $q(\mathscr{U}) = \overline{\text{co}}(q(\mathscr{U}))$ holds whenever $q(\xi)$ is a convex mapping. Under these conditions, Lemmas 1 and 2 ensure that any convex null sequence $\{c_k\}_{k=0}^{\infty}$ with $c_0 = 1$ operates as a subordinating factor sequence for $q(\xi)$ (cf. [22]).

We utilize the following result established by Hallenbeck and Ruscheweyh.

Lemma 3 (Hallenbeck and Ruscheweyh, see [5], Th. 3.1.b, p.71). Let $\mathfrak{h}$ be a convex with $\mathfrak{h}(0) = 1$ and let $\gamma \neq 0$ with $\Re(\gamma) \geq 0$. suppose that $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ is analytic in $\mathscr{U}$ and satisfying

$$\mathfrak{p}(\xi) + \frac{\xi \mathfrak{p}'(\xi)}{\gamma} \prec \mathfrak{h}(\xi), \qquad \xi \in \mathscr{U},$$

then

$$\mathfrak{p}(\xi) \prec q(\xi) \prec \mathfrak{h}(\xi), \qquad \xi \in \mathscr{U},$$

where

$$q(\xi)=\frac{\gamma}{\xi^{\gamma}}\int_0^{\xi}t^{\gamma-1}\mathfrak{h}(t)dt,\qquad \xi\in\mathscr{U}.$$

The function $q(\xi)$ is convex and is the best dominant.

Lemma 4 [23] Let $q(\xi)$ be a convex function in $\mathscr{U}$ and let

$$h(\xi)=q(\xi)+\beta\xi q^{'}(\xi),$$

where $\beta>0$. If $p(\xi)$ is analytic and satisfies

$$p(\xi)+\beta\xi\mathfrak{p}^{'}(\xi)\prec h(\xi)\quad(\xi\in\mathscr{U}),$$

then

$$p(\xi)\prec q(\xi),$$

and $q(\xi)$ is convex and is the best dominant.

The Gaussian hypergeometric function, defined for real or complex parameters a, b, and c with $c\notin\mathbb{Z}_0^-$, is expressed by the series

$${}_2F_1(a,b;c;\xi)=1+\frac{ab}{c}\frac{\xi}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{\xi^2}{2!}+\cdots.$$

This series is absolutely convergent within the open unit disk $\mathscr{U}$ and defines an analytic function therein (cf. [24, Chapter 14]; see also [5]).

Lemma 5 [24] For a, b and $c\,(c\notin\mathbb{Z}_0^-)$, real or complex parameters

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\xi t)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;\xi)\quad(\Re(c)>\Re(b)>0);\tag{8}$$

$${}_2F_1(a,b;c;\xi)\;={}_2F_1(b,a;c;\xi);\tag{9}$$

$${}_2F_1(a,b;c;\xi)\;=(1-\xi)^{-a}\,{}_2F_1(a,c-b;c;\frac{\xi}{\xi-1});\tag{10}$$

$${}_2F_1(1,1;2;\frac{a\xi}{a\xi+1})\;=\frac{(1+a\xi)\ln(1+a\xi)}{a\xi};\tag{11}$$

$$ {}_2F_1(1,\,1;\,3;\,\frac{a\xi}{a\xi+1}) = \frac{2(1+a\xi)}{a\xi}\left(1-\frac{\ln(1+a\xi)}{a\xi}\right). \tag{12} $$

Lemma 6 [25] (see also Noor et al. [26]) Let $p(\xi)$ be an analytic function in $\mathscr{U}$ with $p(0)=1$.
1. If $0<\delta\leq\frac{1}{2}$ and $\Re\left\{1+\frac{\xi p'(\xi)}{p(\xi)}\right\}>\delta-1$, then $\Re\{p(\xi)\}>\delta$ $(\xi\in\mathscr{U})$.
2. If $\frac{1}{2}<\delta<1$ and $\Re\left\{1+\frac{\xi p'(\xi)}{p(\xi)}\right\}>\frac{\delta}{2}-1$, then $\Re\{p(\xi)\}>\delta$ $(\xi\in\mathscr{U})$.

Remark 3 It is worth noting that while the results in reference [25] are originally presented within the framework of meromorphic functions, and subsequently implemented in peer-reviewed proofs by Noor et al. [26], their underlying proofs rely strictly on the local analyticity of the auxiliary function $p(\xi)$ satisfying $p(0)=1$. Therefore, Lemma 6 remains entirely valid for the normalized analytic operators investigated throughout this work.

3. Main results

Unless stated otherwise, we assume throughout this work that $s\geq 1$, $\alpha,\,\beta\in\mathbb{R}^+$, and $0\leq\delta<1$.

Our initial subordination theorem, established by using the operator given in (6), provides the sharpest result as the function obtained constitutes the best dominant.

Theorem 1 If a function f belongs to the class $\Omega_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}(\delta)$, then $f\in\Omega_s^{(\alpha,\,\beta)}(\delta)$. Moreover, we have

$$ \left(\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)\right)' \prec q(\xi) \prec \frac{1+(1-2\delta)\xi}{1-\xi} \quad (\xi\in\mathscr{U}), \tag{13} $$

where

$$ q(\xi)=\frac{s(1-\alpha)}{\xi^{s(1-\alpha)}}\int_0^{\xi}\frac{1+(1-2\delta)t}{1-t}t^{(s(1-\alpha)-1)}dt, $$

and $q(\xi)$ constitutes the best dominant. Furthermore, the following sharp bound holds:

$$ \Re\left(\mathscr{D}_s^{(\alpha,\,\beta)}f(\xi)\right)' > \eta \quad (\xi\in\mathscr{U},\ \ \eta\in\mathbb{R}), \tag{14} $$

where η is defined by

$$ \eta=-(1-2\delta)+(1-\delta)\,{}_2F_1(1,\,1,\,s(1-\alpha)+1,\,\frac{1}{2}), $$

and the estimate given in (14) is sharp.

Proof. Let $f\in\Omega_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}(\delta)$, then from Definition 2, we have

$$ \Re\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\,\beta-\frac{1}{s})}f(\xi)\right)' > \delta. $$

Set

$$p(\xi) = \left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'. \tag{15}$$

Then $p(\xi)$ is analytic in $\mathscr{U}$ with $p(0) = 1$. Differentiation of (15) with the use of (7), yields

$$p(\xi) + \xi p'(\xi) = s(1-\alpha)\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi)\right)' - [s(1-\alpha)-1]p(\xi),$$

which can be written as

$$p(\xi) + \frac{\xi p'(\xi)}{s(1-\alpha)} = \left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi)\right)' \prec h(\xi) = \frac{1+(1-2\delta)\xi}{1-\xi}.$$

where $\gamma = s(1-\alpha)$. Using Lemma 3, we have

$$p(\xi) \prec q(\xi) \prec h(\xi), \quad \xi \in \mathscr{U}.$$

Applying Lemma 3, we thus obtain

$$p(\xi) \prec q(\xi) = \frac{s(1-\alpha)}{\xi^{s(1-\alpha)}} \int_0^{\xi} \frac{1+(1-2\delta)t}{1-t} t^{(s(1-\alpha)-1)} dt,$$

hence by taking in (8) $b = s(1-\alpha),\ a = 1$ and $c = b+1,$ from Lemma 5 and using finally (10), it follows that

$$q(\xi) = -(1-2\delta) + (2-2\delta)(1-\xi)^{-1}\ {}_2F_1(1,\ 1,\ s(1-\alpha)+1,\ \frac{\xi}{\xi-1}).$$

This establishes the differential subordination given in (13).
Furthermore, the infimum of the real part of $q(\xi)$ within the unit disk is given by

$$\inf\{\Re\, q(\xi) : |\xi| < 1\} = q(-1). \tag{16}$$

Indeed, we have

$$\Re\left(\frac{1+(1-2\delta)\xi}{1-\xi}\right) \geq \left(\frac{1-(1-2\delta)\sigma}{1-\sigma}\right) \quad (|\xi| < \sigma < 1).$$

Setting

$$g(\rho, \xi) = \left(\frac{1+(1-2\delta)\rho\xi}{1-\rho\xi} \right) \quad (0 \le \rho \le 1;\ \xi \in \mathscr{U})$$

and

$$d\upsilon(\rho) = \rho(1-\alpha)\rho^{s(1-\alpha)-1} d\rho$$

which is a positive measure on the closed interval [0, 1], we get that

$$q(\xi) = \int_0^1 g(\rho, \xi) dv(\rho),$$

so that

$$\Re\, q(\xi) \ge \int_0^1 \left(\frac{1+(1-2\delta)\rho\sigma}{1-\rho\sigma} \right) dv(\rho) = q(-\sigma) \quad (|\xi| < \sigma < 1).$$

By letting $\sigma \to 1^-$ in this inequality, we obtain the required result given in (16). The bound in (14) is sharp because $q(\xi)$ constitutes the best dominant for the differential subordination (13). □

Putting $\alpha = \beta = 0$ and $s = 1$, in Theorem 1, we get the following result:

Corollary 1 Let $f \in \Omega_1^{(1,-1)}(\delta)$ then $f \in \Omega_1^{(0,0)}(\delta)$. Further

$$f'(\xi) \prec q(\xi) \prec \frac{1+(1-2\delta)\xi}{1-\xi}, \quad \xi \in \mathscr{U},$$

where

$$q(\xi) = \frac{1}{\xi} \int_0^\xi \frac{1+(1-2\delta)t}{1-t} dt,$$

$$= -(1-2\delta) + (2-2\delta)(1-\xi)^{-1}\, {}_2F_1(1, 1, 2, \frac{\xi}{\xi-1})$$

and $q(\xi)$ is the best dominant. Furthermore, we have

$$\Re\left(f'(\xi)\right) > \eta, \quad \eta \in \mathbb{R},$$

where η is given by

$$\eta = -(1-2\delta) + (1-\delta)\ {}_2F_1(1, 1, 2, \frac{1}{2}).$$

For $\delta = 0$ in Corollary 1, we obtain the following example:

Example 1 Let $f \in \Omega_1^{(1,-1)}(0)$ then $f \in \Omega_1^{(0,0)}(0)$. Further

$$f'(\xi) \prec q(\xi) \prec \frac{1+\xi}{1-\xi}, \quad \xi \in \mathscr{U},$$

where

$$q(\xi) = -1 + 2(1-\xi)^{-1}\ {}_2F_1(1, 1, 2, \frac{\xi}{\xi-1}),$$

and $q(\xi)$ is the best dominant. Furthermore, we have

$$\Re\left(f'(\xi)\right) > \eta, \quad \xi \in \mathscr{U},$$

where η is given by

$$\eta = -1 + 2\ln 2.$$

The following theorem establishes a new differential subordination involving the normalized operator $\mathscr{D}_s^{(\alpha,\beta)} f(\xi)$ given in (6), for which the function $q(\xi)$ constitutes the best dominant.

Theorem 2 Let $q(\xi)$ be a convex function with $q(0) = 1$ and let $h(\xi)$ be a function given by

$$h(\xi) = q(\xi) + \frac{\xi q'(\xi)}{s(1-\alpha)}, \quad \xi \in \mathscr{U}.$$

If $f \in \Omega_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}(\delta)$ satisfies

$$\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi)\right)' \prec h(\xi), \quad \xi \in \mathscr{U},$$

then

$$\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)' \prec q(\xi), \quad \xi \in \mathscr{U},$$

and $q(\xi)$ is the best dominant.

Proof. Let the function $p(\xi)$ be defined by

$$p(\xi) = \left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'. \tag{17}$$

It follows that $p(\xi)$ is analytic in $\mathscr{U}$ and satisfies $p(0) = 1$. Differentiating (17) and applying the identity (7), we obtain

$$p(\xi) + \frac{\xi p'(\xi)}{s(1-\alpha)} = \left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi)\right)' \prec h(\xi) = q(\xi) + \xi q'(\xi).$$

By applying Lemma 4 with the choice $\beta = s(1-\alpha)$, we conclude that

$$p(\xi) \prec q(\xi), \quad \xi \in \mathscr{U},$$

which is equivalent to

$$\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)' \prec q(\xi), \quad \xi \in \mathscr{U},$$

where $q(\xi)$ is the best dominant. □

Theorem 3 Let $f \in \mathscr{A}$, and $0 < \delta \leq \frac{1}{2}$. Assume that for each $r \in (0, 1)$, the function $f(\xi)$ satisfies:

$$\min_{|\xi|\leq r} \Re\left\{\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'\right\} = \min_{|\xi|\leq r}\left|\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'\right|,$$

and

$$s(1-\alpha)\Re\left\{\frac{\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})} f(\xi)\right)'}{\left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)'} - 1\right\} > \delta - 1, \quad \xi \in \mathscr{U}.$$

Then, we have $f \in \Omega_s^{(\alpha,\beta)}(\delta)$

Proof. Let

$$p_1(\xi) = \left(\mathscr{D}_s^{(\alpha,\beta)} f(\xi)\right)',$$

then $p_1(\xi)$ is analytic in $\mathscr{U}$ with $p_1(0) = 1$. From (7) it follows that

$$\xi p_1(\xi) = s(1-\alpha)\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}f(\xi) - [s(1-\alpha)-1]\mathscr{D}_s^{(\alpha,\beta)}f(\xi)$$

which on differentiation, yields

$$1+\frac{\xi p_1'(\xi)}{p_1(\xi)} = s(1-\alpha)\frac{\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}f(\xi)\right)'}{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'} - [s(1-\alpha)-1]$$

$$\frac{\xi p_1'(\xi)}{p_1(\xi)} = \left\{ s(1-\alpha)\frac{\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}f(\xi)\right)'}{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'} - [s(1-\alpha)-1] \right\} - 1$$

From the hypothesis and Lemma 6, we obtain

$$\Re\{p_1(\xi)\} > \delta, \quad \xi \in \mathscr{U}.$$

This concludes the proof. □

Setting $\alpha = \beta = 0$ and $s = 1$ in Theorem 3, we obtain the following corollary:

Corollary 2 Let $f \in \mathscr{A}$ and $0 < \delta \leq \frac{1}{2}$. Assume that for each $r \in (0, 1)$, the function $f(\xi)$ satisfies

$$\min_{|\xi|\leq r} \Re\left\{f'(\xi)\right\} = \min_{|\xi|\leq r}\left|f'(\xi)\right|,$$

and

$$\Re\left\{\frac{\left(\xi f'(\xi)\right)'}{f'(\xi)} - 1\right\} > \delta - 1, \quad \xi \in \mathscr{U}.$$

Then, we have $f \in K(\delta)$.

The following theorem provides the counterpart to Theorem 3 for the complementary range of the parameter δ.

Theorem 4 Let $f \in \mathscr{A}$ and $\frac{1}{2} < \delta < 1$. Assume that for each $r \in (0, 1)$, the function $f(\xi)$ satisfies

$$\min_{|\xi|\leq r} \Re\left\{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'\right\} = \min_{|\xi|\leq r}\left|\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'\right|,$$

and

$$s(1-\alpha)\Re\left\{\frac{\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}f(\xi)\right)'}{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'}-1\right\}>\frac{\delta}{2}-1,\quad \xi\in\mathscr{U}.$$

Then, $f\in\Omega_s^{(\alpha,\beta)}(\delta)$.

Proof. Let the function $p_1(\xi)$ be defined by

$$p_1(\xi)=\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'.$$

It follows that $p_1(\xi)$ is analytic in $\mathscr{U}$ and fulfills $p_1(0)=1$. Following the same steps as in the proof of Theorem 3, we obtain

$$\frac{\xi p_1'(\xi)}{p_1(\xi)}=\left\{s(1-\alpha)\frac{\left(\mathscr{D}_s^{(\alpha+\frac{1}{s},\beta-\frac{1}{s})}f(\xi)\right)'}{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'}-[s(1-\alpha)-1]\right\}-1.$$

From the hypothesis and Lemma 6, we conclude that

$$\Re\{p_1(\xi)\}>\delta,\quad \xi\in\mathscr{U}.$$

This concludes the proof of Theorem 4. □

Setting $\alpha=\beta=0$ and $s=1$ in Theorem 4, we obtain the following corollary.

Corollary 3 Let $f\in\mathscr{A}$, and $\frac{1}{2}<\delta<1$. Suppose that for arbitrary r $(0<r<1)$, $f(\xi)$ satisfies the conditions

$$\min_{|\xi|\le r}\Re\left\{f'(\xi)\right\}=\min_{|\xi|\le r}\left|f'(\xi)\right|,$$

and

$$\Re\left\{\frac{\left(\xi f'(\xi)\right)'}{f'(\xi)}-1\right\}>\frac{\delta}{2}-1,\quad \xi\in\mathscr{U}.$$

Then, we have $f\in K(\delta)$.

It should be noted that the subsequent results, particularly Theorems 5, 6, and 7, are established for the specific parameter choices $\alpha=\beta=0$ and $s=1$. This restriction is mathematically motivated by the structural behavior of the Hadamard convolution techniques employed in the proofs. For these particular values, the operator reduces to a form

that allows us to directly utilize the underlying properties of the standard normalized classes, thereby ensuring that the coefficient estimates and geometric bounds can be explicitly obtained.

The next theorem evaluates the convolution condition of the function $f(\xi)$ as follows:

Theorem 5 If $f \in \Omega_1^{(0,0)}(\delta) = K(\delta)$, then

$$\frac{1}{\xi}\left\{f(\xi) * m\frac{\xi}{(1-\xi)^2}\right\} - 1 \neq 0,$$

where

$$m = \frac{(1-e^{i\theta})(1+(1-2\delta)e^{-i\theta})}{1+(1-2\delta)^2+2(1-2\delta)\cos\theta}, \quad (\theta \in [0, 2\pi), \quad \xi \in \mathscr{U}). \tag{18}$$

Proof. If $f \in \Omega_1^{(0,0)}(\delta) = K(\delta)$, the defining condition yields

$$\Re\left(f'(\xi)\right) > \delta$$

which is equivalent to the differential subordination

$$f'(\xi) \prec \frac{1+(1-2\delta)\xi}{1-\xi}.$$

By the definition of subordination, there exists a Schwarz function $\omega(\xi)$ analytic in $\mathscr{U}$, with $\omega(0) = 0$ and $|\omega(\xi)| < 1$, $\xi \in \mathscr{U}$, such that

$$f'(\xi) = \frac{1+(1-2\delta)\omega(\xi)}{1-\omega(\xi)}, \quad \xi \in \mathscr{U}.$$

This relationship can be rewritten in the boundary form

$$f'(\xi)\frac{(1-e^{i\theta})}{1+(1-2\delta)e^{i\theta}} - 1 \neq 0, \quad (\theta \in [0, 2\pi), \quad \xi \in \mathscr{U}). \tag{19}$$

Using the foundational convolution identity

$$\xi f'(\xi) = f(\xi) * \frac{\xi}{(1-\xi)^2},$$

Eq. (19) becomes

$$\frac{1}{\xi}\left[f(\xi)*m\frac{\xi}{(1-\xi)^2}\right]-1\neq 0$$

where m is given by (18). This establishes the required convolution condition and the proof is complete. □

The next theorem prove a convex set for the class $\Omega_s^{(\alpha,\beta)}(\delta)$ as follows:

Theorem 6 The class $\Omega_s^{(\alpha,\beta)}(\delta)$, is a convex set.

Proof. Let the functions

$$f_1(\xi)=\xi+\sum_{\kappa=2}^{\infty}a_{\kappa_1}\xi^{\kappa},$$

and

$$f_2(\xi)=\xi+\sum_{\kappa=2}^{\infty}a_{\kappa_2}\xi^{\kappa},$$

be in the class $\Omega_s^{(\alpha,\beta)}(\delta)$. For $t\in(0,1)$, it suffices to show that the function

$$h(\xi)=(1-t)f_1(\xi)+tf_2(\xi),$$

is in the class $\Omega_s^{(\alpha,\beta)}(\delta)$. Since

$$h(\xi)=\xi+\sum_{\kappa=2}^{\infty}\left[(1-t)a_{\kappa_1}+ta_{\kappa_2}\right]\xi^{\kappa},$$

then

$$\left(\mathscr{D}_s^{(\alpha,\beta)}h(\xi)\right)'=1+\sum_{\kappa=2}^{\infty}\kappa(1-t)\frac{\Gamma(1-s(\alpha-1))(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))(\kappa+\alpha+\beta)^{s-1}}a_{\kappa_1}\xi^{\kappa-1}$$

$$+\sum_{\kappa=2}^{\infty}\kappa t\frac{\Gamma(1-s(\alpha-1))(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))(\kappa+\alpha+\beta)^{s-1}}a_{\kappa_2}\xi^{\kappa-1},$$

from which we can write

$$\Re\left(\mathscr{D}_s^{(\alpha,\beta)}h(\xi)\right)'=(1-t)\Re\left\{1+\sum_{\kappa=2}^{\infty}\kappa\frac{\Gamma(1-s(\alpha-1))(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))(\kappa+\alpha+\beta)^{s-1}}a_{\kappa_1}\xi^{\kappa-1}\right\}$$

$$+t\Re\left\{1+\sum_{\kappa=2}^{\infty}\kappa\frac{\Gamma(1-s(\alpha-1))(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))(\kappa+\alpha+\beta)^{s-1}}a_{\kappa_2}\xi^{\kappa-1}\right\}. \tag{20}$$

Since $f_1(\xi)$ and $f_2(\xi)$ belongs to $\Omega_s^{(\alpha,\beta)}(\delta)$, this implies that

$$\Re\left\{1+\sum_{\kappa=2}^{\infty}\kappa\frac{\Gamma(1-s(\alpha-1))(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))(\kappa+\alpha+\beta)^{s-1}}a_{\kappa_i}\xi^{\kappa-1}\right\}>\delta \quad (i=1,2). \tag{21}$$

From (20) and (21), we have

$$\Re\left(\mathscr{D}_s^{(\alpha,\beta)}h(\xi)\right)'>\delta.$$

This completes the proof. □

Theorem 7 The class $f(\xi)\in\Omega_s^{(\alpha,\beta)}(\delta)$ and $g(\xi)\in\mathscr{A}$ such that $\Re(\frac{g(\xi)}{\xi})>\frac{1}{2}$. Then

$$(f*g)(\xi)\in\Omega_s^{(\alpha,\beta)}(\delta).$$

Proof. Let

$$h(\xi)=(f*g)(\xi).$$

Using convolution properties, we have

$$\left(\mathscr{D}_s^{(\alpha,\beta)}h(\xi)\right)'=\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'*\frac{g(\xi)}{\xi},\quad \xi\in\mathscr{U}. \tag{22}$$

Since $f(\xi)\in\Omega_s^{(\alpha,\beta)}(\delta)$ and $\Re(\frac{g(\xi)}{\xi})>\frac{1}{2},$ then it follows from Lemma 1

$$(f*g)(\xi)\in\Omega_s^{(\alpha,\beta)}(\delta),\quad \xi\in\mathscr{U}.$$

This completes the proof. □

Theorem 8 For $1\leq\mu\leq 2$ and let $f(\xi)$ and $g(\xi)$ belongs to the class $\Omega_s^{(\alpha,\beta)}(\delta),$ with

$$f(\xi)=\xi+\sum_{\kappa=2}^{\infty}a_\kappa\xi^\kappa,$$

and

$$g(\xi) = \xi + \sum_{\kappa=2}^{\infty} b_\kappa \xi^\kappa.$$

Then

$$(f * g)(\xi) \in \Omega_s^{(\alpha,\beta)}(\delta),$$

where

$$\delta \leq \rho = \frac{4\delta - \mu(2\delta+1) - 1}{2(1-\mu)}$$

Proof. Since $g(\xi) \in \Omega_s^{(\alpha,\beta)}(\delta)$, we have

$$\Re\left\{1 + \sum_{\kappa=2}^{\infty} \kappa \frac{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}} b_\kappa \xi^{\kappa-1}\right\} > \delta. \tag{23}$$

For $1 \leq \mu \leq 2$, let $c_0 = 1$ and

$$c_\kappa = \frac{\mu-1}{\kappa} \frac{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}}{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!}$$

Then $\{c_\kappa\}_{\kappa=0}^{\infty}$ is a convex null sequence. Therefore by Lemma 2, we have

$$\Re\left\{1 + \sum_{\kappa=2}^{\infty} \frac{\mu-1}{\kappa} \frac{\Gamma(\kappa-s(\alpha-1))\,(\kappa+\alpha+\beta)^{s-1}}{\Gamma(1-s(\alpha-1))\,(1+\alpha+\beta)^{s-1}\kappa!} b_\kappa \xi^{\kappa-1}\right\} > \frac{1}{2}, \quad \xi \in \mathscr{U}. \tag{24}$$

Now taking the convolution of (23) and (24) and applying Lemma 1, to have

$$\Re\left\{1 + \sum_{\kappa=2}^{\infty} (1-\mu) b_\kappa \xi^{\kappa-1}\right\} > \delta, \quad \xi \in \mathscr{U}.$$

Or

$$\Re\left\{\frac{g(\xi)}{\xi}\right\} = \Re\left\{1 + \sum_{\kappa=2}^{\infty} b_\kappa \xi^{\kappa-1}\right\} > \frac{\delta-\mu}{(1-\mu)}.$$

Thus

$$\Re\left\{\frac{g(\xi)}{\xi}-\frac{2\delta-\mu-1}{2(1-\mu)}\right\}>\frac{1}{2}.$$

Since $f(\xi)\in\Omega_s^{(\alpha,\beta)}(\delta)$, applying Lemma 1, we obtain

$$\Re\left\{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'*\left(\frac{g(\xi)}{\xi}-\frac{2\delta-\mu-1}{2(1-\mu)}\right)\right\}>\delta,$$

or we can write

$$\Re\left\{\left(\mathscr{D}_s^{(\alpha,\beta)}f(\xi)\right)'*\frac{g(\xi)}{\xi}\right\}>\frac{4\delta-\mu(2\delta+1)-1}{2(1-\mu)}.$$

Hence the result follows from (22). □

4. Conclusions

In this paper, we have introduced and investigated a new subclass of close-to-convex univalent functions in the open unit disk defined via the Zeta–Riemann fractional differential operator $\mathscr{D}_s^{(\alpha,\beta)}f(\xi)$. By employing the framework of differential subordination, several meaningful results have been established, including inclusion relationships, convolution (Hadamard product) properties, and additional structural characteristics of the proposed class. These findings demonstrate the effectiveness of combining fractional operators with subordination techniques to generate richer and more flexible families of analytic functions within geometric function theory.

The introduction of the Zeta–Riemann fractional differential operator provides a promising analytical tool that enhances the study of univalent function classes and broadens the scope of fractional calculus in complex analysis. The results obtained in this work not only extend existing theories but also contribute to a deeper understanding of how fractional operators influence geometric properties such as univalence and close-to-convexity.

There are several directions in which this research can be further developed. One natural extension is to investigate alternative subclasses of analytic and bi-univalent functions, such as bi-Bazilević-type or pseudo-starlike functions, by applying our normalized Zeta–Riemann fractional differential operator to the structural frameworks established in [27]. In addition, the application of higher-order differential subordinations and superordination techniques may yield sharper results and more generalized frameworks. A future research direction involves extending the current results to other types of differential subordinations, such as fuzzy third-order differential subordinations and superordinations following the approach in [28], or strong differential subordination and sandwich properties as investigated in [29]. Additionally, the current framework can be expanded to incorporate generalized q-calculus structures, such as starlike functions of negative order defined by q-fractional derivatives [30] or new subclasses of (λ, q)-fractional differential operators associated with generating functions [31].

A promising direction for future research involves examining coefficient estimates, growth and distortion theorems, and radius problems associated with the newly defined class, following the approach discussed in [32] for the Tremblay fractional derivative operator. Similarly, unified sufficient conditions and inclusion properties based on coefficient criteria can be established for this class, as investigated for the Rabotnov fractional operator in [33]. It would also be of interest to investigate the interaction of the Zeta–Riemann operator with other well-known fractional and integral operators, leading to unified operator theories.

Declaration of generative AI and AI-assisted technologies

The authors confirm that no generative AI or AI-assisted technologies were used in the research, data analysis, or preparation of this manuscript.

Author contributions

All authors contributed equally to the conception, development, and preparation of this manuscript. All authors reviewed the article and gave their approval for the final version.

Funding

The APC of this paper was funded by University of Oradea, Romania.

Data availability

All data generated or analyzed during this study are included in this published article.

Conflict of interest

The authors declare no competing financial interest.

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