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Research Article

Construction of Brauer-Severi Varieties

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Abstract: In this paper, we give an algorithm for computing equations of Brauer-Severi varieties over fields of characteristic 0. As an example, we show the equations of all Brauer-Severi surfaces defined over Q.

***Keywords*:** Brauer-Severi varieties, twists, central simple algebras, Veronese embedding, Hilbert's Theorem 90

MSC: 11G35, 14Q15, 14M99, 14J10, 20G05

1. Introduction

The first who systematically studied Brauer-Severi varieties was Châtelet in his seminal work [1] and under the name of ''variétésde Brauer''. The term ''Severi-Brauer variety'' comes from Beniamino Segre [2], who suggested that Châtelet had omitted previous work by Severi [3], where he studied Brauer-Severi varieties in a more classical geometric context.

In the literature one finds different theoretical constructions of Brauer-Severi varieties: for the classical approach of Châtelet via varieties of left ideals embedded into Grassmannians, which gives a canonical construction, see [1, 4, 5]. When trying to produce explicit equations of Brauer-Severi varieties, this projective embedding is far from being ''optimal'': for instance, Brauer-Severi varieties of dimension 1 are realized not as plane conics, but as curves in $\mathbb{P}^5$ defined by 31 equations, see [4].

Another approach is that of Grothendieck, which is based on general techniques in descent theory. It does not give explicit information on the projective embedding. But when trying to compute it, it yields the same one that in Châtelet idea: see [6].

Even if not canonical, since it is going to depend on the representant of the cocycle class we choose, we will follow the Twisting Theory approach.

Denition 1.1 (Brauer-Severi variety) Let K be a perfect field, $\bar{K}$ its Galois closure and X/K a projective irreducible smooth variety of dimension n. We say that X is a Brauer-Severi variety if there exists an isomorphism $X_{\bar{K}} \simeq_{\bar{K}} \mathbb{P}^n_{\bar{K}}$. Let us denote the set of Brauer-Severi varieties of dimension n defined over K and up to K-isomorphism by BS^n_K.

Denote by $\mathrm{Aut}(\mathbb{P}^n_{\bar{K}})$ the automorphism group of the projective space $\mathbb{P}^n_{\bar{K}}$ over $\bar{K}$, which is isomorphic to the projective general linear group $\mathrm{PGL}_{n+1}(\bar{K})$. Then, we clearly have that

$$\mathrm{BS}_K^n = \mathrm{Twist}_K(\mathbb{P}_K^n) \simeq \mathrm{H}^1(\mathrm{G}_K, \mathrm{Aut}(\mathbb{P}_K^n)) \simeq \mathrm{H}^1(\mathrm{G}_K, \mathrm{PGL}_{n+1}(\bar{K})),$$

where G_K denotes the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. If K is a finite field or the function field of an algebraic curve over an algebraically closed field, then $\mathrm{H}^1(\mathrm{G}_K, \mathrm{PGL}_{n+1}(\bar{K}))$ is trivial (Tsen's Theorem) and there are no non-trivial Brauer-Severi varieties.

The first and only previously known equations of a non-trivial Brauer-Severi variety ($n \geq 2$) were shown in [7]. It is defined over $\mathbb{Q}(\zeta_3)$ where ζ_3 is a third primitive root of unity.

For an application of the explicit construction of a non-trivial Brauer-Severi surface $\mathcal{B}$ see [8]. One could construct a nontrivial cubic surface with a Galois stable set of 6 pairwise skew lines starting from $\mathcal{B}$, see Manin's contruction [9]. This would help to determine whether the condition (a) in the following Theorem of Swinnerton-Dyer is really needed.

Theorem 1.2 (Swinnerton-Dyer [10]) Let S be a smooth cubic surface defined over a number field K. S is birationally trivial if and only if

(a) the cubic S contains a point defined over K and,

(b) the cubic S contains a $\mathrm{Gal}(\mathbb{Q}/K)$-stable set of 2, 3 or 6 pairwise skew lines.

As it was already noticed by Swinnerton-Dyer, a smooth cubic surface containing a stable set of 2 lines contains a rational point, and then it is birationally equivalent to the projective plane. Whether this is also true for a stable set of 3 or 6 lines is still unknown.

Another application of the explicit construction presented in this paper is to the computation of generators of the Picard group of cyclic Brauer-Severi varieties as explained in [11].

In order to compute explicit equations of Brauer-Severi varieties, we need to compute explicite quations of twists of the projective space $\mathbb{P}^n$. As it is shown in [7] for smooth plane curves, in [12] for hyperelliptic curves, and in [13] and [14] for non-hyperelliptic curves, the best idea to compute equations of twists of a variety X/K is to embed its automorphism group $\mathrm{Aut}(X)$ into $\mathrm{GL}_N(K)$ for some $N \in \mathbb{N}$ as a G_K-module, and then apply Hilbert's Theorem 90.

Hence, in order to compute non-trivial Brauer-Severi varieties, we will follow this strategy:

- In Section 2, we will describe the set $\mathrm{Twist}_K(\mathbb{P}_K^n) \simeq \mathrm{H}^1(\mathrm{G}_K, \mathrm{PGL}_{n+1}(\bar{K}))$.
- In Section 3, we will give an embedding of G_K-modules $\mathrm{PGL}_{n+1}(\bar{K}) \subseteq \mathrm{GL}_N(\bar{K})$ for some $N \in \mathbb{N}$ that will allow us to compute explicit equations for Brauer-Severi varieties by explicitly using Hilbert's Theorem 90 [Originally due to Kummer (1855), this result on the vanishing of some first Galois cohomology takes its name from the fact that it is the 90th theorem in David Hilbert's Zahlbericht (1897)].

Finally we present our algorithm to compute equations of Brauer-Severi varieties starting by a cocycle in 4. These algorithmic construction is made explicit in Section 5 for the case of $n = 2$. The output of the algorithm in 4 is smaller than in the construction by Châtelet but still big. This is why in Section 6 we present in Theorem 6.2 nicer equations but of a singular model of Brauer-Severi varieties.

2. Brauer-Severi varieties and central simple algebras

The set of isomorphism classes of central simple algebras of dimension n^2 over K and split over L is denoted by $\mathrm{Az}_n^{L/K}$. The set of isomorphism classes of central simple algebras of dimension n^2 over K is denoted by Az_n^K.

Theorem 2.1 (Serre, chap. X, §5, Prop. 8, [15]). Let L/K be a finite Galois extension of fields, $G = \mathrm{Gal}(L/K)$ its Galois group, and n be a natural number. Then there is a natural bijection of pointed sets

$$a_n^{L/K}: \quad \mathrm{Az}_n^{L/K} \quad \xrightarrow{\simeq} \quad \mathrm{H}^1(G, \mathrm{PGL}_n(L)).$$

Notice that previous Theorem implies

$$\mathrm{Az}_n^K = \cup_{L/K} \mathrm{Az}_n^{L/K} \simeq \mathrm{H}^1(\mathrm{G}_K, \mathrm{PGL}_n(\bar{K})) \simeq \mathrm{BS}_K^{n-1}.$$

It is well-known that for $n = 2$, 3 all the algebrasin Az_n^K are cyclic algebras [16]. We show the equivalent definition for cyclic simple central algebras given in [17].

Proposition 2.2 There is a bijection between the set of isomorphism classes of cyclic algebras of degree n over K and the set of equivalence classes of pairs (χ, a) where $\chi : \mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$ is a group isomorphism with L a cyclic Galois extension of degree n of K and $a \in K^*$. The equivalent relation is $(\chi, a) \sim (\chi', a')$ if and only if $\chi = \chi'$ and $a'a^{-1} \in \mathrm{Nm}_{L/K}(L^*)$.

Moreover, given a pair (χ, a), the corresponding algebra is given (by Theorem 2.1) by the cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{PGL}_n(L))$ that maps

$$\chi^{-1}(1) \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ a & 0 & 0 & \dots & 0 \end{pmatrix}.$$

3. The key embedding

The following lemma will be the key point for constructing equations dening Brauer-Severi varieties via Hilbert's Theorem 90. The n-Veronese embedding, $V_n : \mathbb{P}^n \to \mathbb{P}^m$ with $m = \binom{2n+1}{n} - 1$ induces an embedding $\mathrm{PGL}_{n+1}(\bar{K}) \to \mathrm{PGL}_{m+1}(\bar{K})$ of $\mathrm{Gal}(\bar{K}/K)$-modules. See for instance Theorem 5.2.2 in [17] for a slightly variation of it. We go a little bit further:

Lemma 3.1 For every $n \in \mathbb{N}$, there exists an embedding of $\mathrm{Gal}(\bar{K}/K)$-modules $\iota_n : \mathrm{PGL}_{n+1}(\bar{K}) \to \mathrm{GL}_{m+1}(\bar{K})$ where $m = \binom{2n+1}{n} - 1$.

Example 3.2 The case $n = 1$ is Proposition 3.5 in [12] and it was used to compute twists of hyperelliptic curves. In this case, the Veronese embedding has degree $n + 1 = 2$ and it is given by

$$V_1 : \mathbb{P}^1 \to \mathbb{P}^2 : (x : y) \mapsto (x^2 : xy : y^2).$$

It induces the embeddin

$$\iota_1 : \mathrm{PGL}_2(\bar{K}) \to \mathrm{GL}_3(\bar{K}) : [A] = \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] \mapsto \frac{1}{\det(A)} \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix}.$$

Remark 3.3 The case $n = 2$ is implicitly used in [7] for computing the only previous known equations for a non-trivial Brauer-Severi variety of dimension greater than 1.

Proof. (Lemma 3.1) Let us consider the Veronese embedding of dimension n and degree $n+1 : V_n : \mathbb{P}^n \to \mathbb{P}^m$ with $m = \binom{2n+1}{n} - 1$. We name the coordinates as follows $V_n : \mathbb{P}^n \to \mathbb{P}^m : (x_0 : \dots : x_n) \mapsto (\omega_0 : \dots : \omega_m)$, where the ω_k are equal to the products $\omega_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} = \prod_i x_i^{\alpha_i}$ with $\sum_i \alpha_i = n+1$ numbered in lexicographical order.

The embedding V_n induces another embedding on automorphism groups $[\iota_n] : \mathrm{PGL}_{n+1}(\bar{K}) \to \mathrm{PGL}_{m+1}(\bar{K})$. We will see that indeed we can lift it

$$\iota_n : \mathrm{PGL}_{n+1}(\bar{K}) \to \mathrm{GL}_{m+1}(\bar{K}).$$

Let be $[A] = [a_{ij}] \in \mathrm{PGL}_{n+1}(\bar{K})$, then $[\iota_n]([A]) = [(L^k)_{k=0 \dots m}]$ is the matrix whose rows are L^k, where again named in lexicographical order, the coordinates of L^k are given by the formula

$$\prod_i (\sum_j a_{ij} x_j)^{\alpha_i} = \sum_{(\beta_0 \dots \beta_n)} L^k_{\beta_0 \dots \beta_n} x_0^{\beta_0} \dots x_n^{\beta_n}.$$

Hence, the matrix $[\iota_n]([A])$ is a matrix whose entries are polynomials of degree n in the entries of A. We can now fix a lift of $[\iota_n]([A])$ to $\mathrm{GL}_m(\bar{K})$ by doing $[\iota_n]([A]) = \frac{1}{\det(A)} (L^k)_{k=0 \dots m}$. This is an embedding of $\mathrm{Gal}(\bar{K}/K)$-modules.

Remark 3.4 The anticanonical sheaf in $\mathbb{P}^n$ is equal to $\mathcal{O}(n+1)$ [18] and it gives the Veronese embedding of degree $n+1$ of $\mathbb{P}^n$ into $\mathbb{P}^m$ with $m = \binom{2n+1}{n} - 1$. So, previous embeddings can be seen as the natural action of the automorphism group of $\mathbb{P}^n$ on the vector space of global section of the anticanonical sheaf $\mathcal{O}(n+1)$.

Proposition 3.5 The equations of the image of the Veronese embeddin $V_n : \mathbb{P}^n \to \mathbb{P}^m : (x_0 : \dots : x_n) \mapsto (\omega_0 : \dots : \omega_m)$, where the ω_k are equal to the products $\omega_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} = \prod_i x_i^{\alpha_i}$ with $\sum_i \alpha_i = n+1$ in lexicographical order, are

$$\omega_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} \omega_{x_n^{n+1}}^{n-\alpha_n} = \omega_{x_0 x_n^n}^{\alpha_0} \dots \omega_{x_{n-1} x_n^n}^{\alpha_{n-1}},$$

together with the equations given by permuting the indices by $\{\sigma, \sigma^2, \dots, \sigma^n\}$ where $\sigma : \{0, 1, \dots, n\} \mapsto \{1, 2, \dots, n, 0\}$.

Proof. Let us call V the variety in $\mathbb{P}^m$ defined by this set of equations that we call $\mathcal{F}$. If $\omega_m = \omega_{x_n^{n+1}} \neq 0$, we make $\omega_m = x_n = 1$, then $x_i = \omega_{x_i x_n^n}$ and $\omega_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} = \omega_{x_0 x_n^n}^{\alpha_0} \dots \omega_{x_{n-1} x_n^n}^{\alpha_n}$. The map $V_n : \mathbb{P}^n \setminus \{x_n = 0\} \to \mathrm{V} \setminus \{\omega_m = 0\}$ is clearly a bijection. If $\omega_m = 0$, then $x_n = 0$ and at least for another i we have $\omega_{x_i^{n+1}} \neq 0$ and we repeat the previous argument with the transformed equations. This proves that $V_n(\mathbb{P}^n) = V$ and that V_n is a bijection. We finally check that V is a non-singular variety: Let $\mathcal{F}$ be the set of equations in the statement of the proposition, we need to check that the matrix $(\partial f_i / \partial x_j)_{f_j \in \mathcal{F}, 0 \le i \le n}$ has rank at least $m - n$. If $\omega_m = \omega_{x_n^{n+1}} \neq 0$, the set of $m - n$ columns $\partial / \partial x_i (\omega_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} \omega_{x_n^{n+1}}^{n-\alpha_n} - \omega_{x_0 x_n^n}^{\alpha_0} \dots \omega_{x_{n-1} x_n^n}^{\alpha_{n-1}})_{i=0,\dots,n}$ with $\alpha_n < n$ has maximal rank $m - n$. If $\omega_m = \omega_{x_n^{n+1}} = 0$, we take an i with $\omega_{x_i^{n+1}} \neq 0$ and we repeat the previous argument with the permutated equations.

Example 3.6 For $n = 1$, we get the equation of the conic $\omega_0 \omega_2 = \omega_1^2$.

For $n = 2$, we get the equations:

$$V_2(\mathbb{P}^2) : \begin{cases} \omega_0 \omega_9^2 = \omega_5^3, & \omega_0 \omega_6^2 = \omega_3^3, & \omega_3 \omega_0^2 = \omega_0 \omega_1^2 \\ \omega_1 \omega_9^2 = \omega_5^2 \omega_8, & \omega_1 \omega_6^2 = \omega_3^2 \omega_6, & \omega_4 \omega_0^2 = \omega_0 \omega_1 \omega_2 \\ \omega_2 \omega_9^2 = \omega_5^2 \omega_9, & \omega_2 \omega_6^2 = \omega_3^2 \omega_7, & \omega_5 \omega_0^2 = \omega \omega_2^2 \\ \omega_3 \omega_9^2 = \omega_5 \omega_8^2, & \omega_4 \omega_6^2 = \omega_3 \omega_6 \omega_7, & \omega_6 \omega_0^2 = \omega_1^3 \\ \omega_4 \omega_9^2 = \omega_5 \omega_8 \omega_9, & \omega_5 \omega_6^2 = \omega_3 \omega_7^2, & \omega_7 \omega_0^2 = \omega_1^2 \omega_2 \\ \omega_6 \omega_9^2 = \omega_8^3, & \omega_8 \omega_6^2 = \omega_6 \omega_7^2, & \omega_8 \omega_0^2 = \omega_1 \omega_2^2 \\ \omega_7 \omega_9^2 = \omega_8^2 \omega_9, & \omega_9 \omega_6^2 = \omega_7^3, & \omega_9 \omega_0^2 = \omega_2^3 \end{cases} \subseteq \mathbb{P}^9$$

4. The algorithm

Given a cocycl $\bar{\xi}$ in $\mathrm{H}^1(G_k, \mathrm{PGL}_{n+1}(\bar{K}))$, it defines a Brauer-Severi variety as in Theorem 2.1. This algorithm gives equations dening the Brauer-Severi variety.

(i) Transform the cocycl $\bar{\xi}$ into a cocycle ξ in $\mathrm{H}^1(G_k, \mathrm{GL}_{m+1}(\bar{K}))$ with Lemma 3.1.

(ii) Use an explicit version of Hilbert's Theorem 90 to get $\phi \in \mathrm{GL}_{m+1}(\bar{K})$ such that $\xi_\sigma = \phi \circ {}^\sigma \phi^{-1}$ for all $\sigma \in G_K$. This can be done by taking a sufficiently general matrix in $\mathrm{GL}_{m+1}(K)$ and applying the recipe in [15], or by searching for a basis of fixed vectors of a special action of $\mathrm{GL}_{m+1}(K)$ on $\bar{K}^{m+1}$, see [13, 14].

(iii) Get the equations of the Brauer-Severi variety in $\mathbb{P}^m$ by using the computed ϕ in (ii) and the equations of $V_n(\mathbb{P}^n) \subseteq \mathbb{P}^m$ in Proposition 3.5.

5. The case $n = 2$ made explicit

In this section, we look closer at the output of the previously presented algorithm for the case $n = 2$. In particular, a nice parametrization of degree 3 cyclic extensions is given in Subsection 5.1. This allows to explicitly give the first known equations of a non-trivial Brauer-Severi variety of dimension greater than 1 defined over $\mathbb{Q}$ in Subsection 5.2.

The Veronese embedding of dimension 2 and degree 3 looks like

$$V_2 : \mathbb{P}^2 \to \mathbb{P}^9 : (x : y : z) \mapsto (x^3 : x^2y : x^2z : xy^2 : xyz : xz^2 : y^3 : y^2z : yz^2 : z^3).$$

This embedding induces another one

$$\iota_2 : \mathrm{PGL}_3(\bar{K}) \to \mathrm{GL}_{10}(\bar{K}) : [A] = \left[\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right] \mapsto \frac{1}{\det(A)} \cdot M(A)$$

where the matrix $M(A)$ is given by

$$\begin{pmatrix}
a^3 & 3a^2b & 3a^2c & 3ab^2 & 6abc & 3ac^2 & b^3 & 3b^2c & 3bc^2 & c^3 \\
a^2d & a^2e+2abd & a^2f+2acd & 2abe+b^2d & 2abf+2ace+2bcd & 2acf+c^2d & b^2e & b^2f+2bce & 2bcf+c^2e & c^2f \\
a^2g & a^2h+2abg & a^2i+2acg & 2abh+b^2g & 2abi+2ach+2bcg & 2aci+c^2g & b^2h & b^2i+2bch & 2bci+c^2h & c^2i \\
ad^2 & 2ade+bd^2 & 2adf+cd^2 & ae^2+2bde & 2aef+2bdf+2cde & af^2+2cdf & be^2 & 2bef+ce^2 & bf^2+2cef & cf^2 \\
adg & adh+aeg+bdg & adi+afg+cdg & aeh+bdh+beg & aei+afh+bdi+bfg+cdh+ceg & afi+cdi+cfg & beh & bei+bfh+ceh & bfi+cei+cfh & cfi \\
ag^2 & 2agh+bg^2 & 2agi+cg^2 & ah^2+2bgh & 2ahi+2bgi+2cgh & ai^2+2cgi & bh^2 & 2bhi+ch^2 & bi^2+2chi & ci^2 \\
d^3 & 3d^2e & 3d^2f & 3de^2 & 6def & 3df^2 & e^3 & 3e^2f & 3ef^2 & f^3 \\
d^2g & d^2h+2deg & d^2i+2dfg & 2deh+e^2g & 2dei+2dfh+2efg & 2dfi+f^2g & e^2h & e^2i+2efh & 2efi+f^2h & f^2i \\
dg^2 & 2dgh+eg^2 & 2dgi+fg^2 & dh^2+2egh & 2dhi+2egi+2fgh & di^2+2fgi & eh^2 & 2ehi+fh^2 & ei^2+2fhi & fi^2 \\
g^3 & 3g^2h & 3g^2i & 3gh^2 & 6ghi & 3gi^2 & h^3 & 3h^2i & 3hi^2 & i^3
\end{pmatrix}$$

This matrix has been computed with Magma [19], we show below the code that is easily generalizable to other $n \in \mathbb{N}$.

```
K <x, y, z, a, b, c, d, e, f, g, h, i> : = PolynomialRing(Rationals(), 12);
X : = ax + by + cz; Y : = dx + ey + fz; Z: = gx + hy + iz;
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X^3; X^2Y; X^2Z; XY^2; XYZ; XZ^2; Y^3; Y^2Z; YZ^2; Z^3;

By Theorem 2.1, the fact that all the elements in Az_3^K are cyclic and Proposition 2.2, we need to study the image by ι_2 of the matrix

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 0 & 0 \end{pmatrix}.$$

So, in this case: $b = 1, f = 1, g = \alpha, a = c = d = e = f = i = 0$, and

$$\iota_2(A_\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\alpha & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 5.1 Let L/K be a degree 3 Galois extension of number fields. Write $L = K(l_1, l_2, l_3)$ and $\mathrm{Gal}(L/K) = \langle\sigma\rangle$ with $\sigma(l_1) = l_2$ and $\sigma(l_2) = l_3$. Let $\alpha \in K$ and let us define a cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(L/K), \mathrm{GL}_3(L))$ by its value $\xi_\sigma = \iota_2(A_\alpha)$ at a generator of $\mathrm{Gal}(L/K)$ and extended by the cocycle condition. Then $\xi_\tau = \phi \circ {}^\tau\phi^{-1}$ for all $\tau \in \mathrm{Gal}(L/K)$ with

$$\phi = \begin{pmatrix} l_1 & 0 & 0 & 0 & 0 & 0 & l_2 & 0 & 0 & l_3 \\ 0 & l_1 & 0 & 0 & 0 & l_2 & 0 & l_3 & 0 & 0 \\ 0 & 0 & l_1 & l_2 & 0 & 0 & 0 & 0 & l_3 & 0 \\ 0 & 0 & l_3 & l_1 & 0 & 0 & 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_2\alpha & 0 & 0 & 0 & l_3\alpha & 0 & l_1\alpha & 0 & 0 \\ l_3\alpha & 0 & 0 & 0 & 0 & 0 & l_1\alpha & 0 & 0 & l_2\alpha \\ 0 & l_3\alpha & 0 & 0 & 0 & l_1\alpha & 0 & l_2\alpha & 0 & 0 \\ 0 & 0 & l_2\alpha & l_3\alpha & 0 & 0 & 0 & 0 & l_1\alpha & 0 \\ l_2\alpha^2 & 0 & 0 & 0 & 0 & 0 & l_3\alpha^2 & 0 & 0 & l_1\alpha^2 \end{pmatrix}$$

Proof. It is easily checked that the matrix ϕ satisfies the equation $\xi_\tau = \phi \circ {}^\tau\phi^{-1}$.

Theorem 5.2 The set of isomorphism classes of Brauer-Severi surfaces defined over $\mathbb{Q}$ is in bijection with the set of equivalence classes of pairs (χ, α) where L is a Galois extension of degree 3, $\chi : \mathrm{Gal}(L/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z}$ is an isomorphism and $\alpha \in \mathbb{Q}^*$. Two pairs (χ, α) and (χ', α') are equivalent if and only if $\chi = \chi'$ and $\alpha'\alpha^{-1} \in \mathrm{Nm}_{L/K}(L^*)$. Given (χ, α) with $L = \mathbb{Q}(l_1, l_2, l_3)$ where the l_i are conjugate numbers, the corresponding Brauer-Severi variety $\mathcal{B}$ is given by the intersection $\cap_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})} {}^\sigma X$, where X/L is the variety in $\mathbb{P}^9$ defined by the set of equations:

$$\alpha(l_1\omega_0 + l_2\omega_6 + l_3\omega_9)(l_2\omega_0 + l_3\omega_6 + l_1\omega_9)^2 = (l_2\omega_1 + l_3\omega_5 + l_1\omega_7)^3$$

$$\alpha^3(l_1\omega_1+l_2\omega_5+l_3\omega_7)(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_1+l_3\omega_5+l_1\omega_7)^2(l_3\omega_1+l_1\omega_5+l_2\omega_7)$$

$$\alpha^2(l_1\omega_2+l_2\omega_3+l_3\omega_8)(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_1+l_3\omega_5+l_1\omega_7)^2(l_2\omega_0+l_3\omega_6+l_1\omega_9)$$

$$\alpha(l_3\omega_2+l_1\omega_3+l_2\omega_8)(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_1+l_3\omega_5+l_1\omega_7)(l_2\omega_2+l_3\omega_3+l_1\omega_8)^2$$

$$\omega_4(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_1+l_3\omega_5+l_1\omega_7)(l_2\omega_2+l_3\omega_3+l_1\omega_8)(l_2\omega_0+l_3\omega_6+l_1\omega_9)$$

$$\alpha^2(l_3\omega_0+l_1\omega_6+l_2\omega_9)(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_2+l_3\omega_3+l_1\omega_8)^3$$

$$\alpha(l_3\omega_1+l_1\omega_5+l_2\omega_7)(l_2\omega_0+l_3\omega_6+l_1\omega_9)^2=(l_2\omega_2+l_3\omega_3+l_1\omega_8)^2(l_2\omega_0+l_3\omega_6+l_1\omega_9).$$

Proof. By Theorem 2.1 we know that all the Brauer-Severi surfaces are parametrized by $\mathrm{Az}_3^{\mathbb{Q}}$. We also know that all the elements in $\mathrm{Az}_3^{\mathbb{Q}}$ are cyclic algebras [16]. We use the description in Proposition 2.2 for cyclic algebras.

In order to compute the equations, we plug the equation of the isomorphism ϕ in Lemma 5.1 into the equations of $V_2(\mathbb{P}^2)\subseteq\mathbb{P}^9$ given in Corollary 3.6. These equations are, a priori, not defined over $\mathbb{Q}$, even if the ideal generating $\mathcal{B}$ is. Notice that after plugging ϕ in the equations in the second and the third columns in Corollary 3.6, we get the conjugate equations to the ones appearing in the first column and shown here. Hence, the intersection $\cap_{\sigma\in\mathrm{Gal}(L/\mathbb{Q})}{}^{\sigma}X$ gives the equations for $\mathcal{B}/\mathbb{Q}$.

5.1 *Degree 3 cyclic extensions of* $\mathbb{Q}$

In order to show equations defined over $\mathbb{Q}$ for the Brauer-Severi surfaces shown in Theorem 5.2, we need to work with a good basis for the degree 3 cyclic extensions L of $\mathbb{Q}$. Ideally, we would like to find a basis l_1, l_2, l_3 of $L/\mathbb{Q}$ such that we could easily write each equation in Theorem 5.2 as $l_i^2f_1+l_if_2+f_3=0$ with $f_i\in\mathbb{Q}[\omega_0,...,\omega_9]$. Then, since the matrix

$$\begin{pmatrix} l_1 & l_2 & l_3\\ l_2 & l_3 & l_1\\ l_3 & l_1 & l_2\end{pmatrix}$$

is invertible, the equations of the Brauer-Severi surface would be given by $f_1=f_2=f_3=0$.

In this subsection we will find an element t_1 in L such that l_1, l_2, l_3 with $l_1=t_1$ is a basis in which we can easily write the equations in Theorem 5.2 as $l_i^2f_1+l_if_2+f_3=0$ for some $f_i\in\mathbb{Q}[\omega_0,...,\omega_9]$.

Proposition 5.3 Let $L/\mathbb{Q}$ be a cyclic degree 3 extension given by the decomposition field of the polynomial $P(t)=t^3+At^2+Bt+C$ with $A,B,C\in\mathbb{Z}$. Let t_1 be a fixed root of $P(t)=0$. Then there exists $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ such that $t_2=\frac{\alpha t_1+\beta}{\gamma t_1+\delta}$ and $t_3=\frac{\alpha t_2+\beta}{\gamma t_2+\delta}$ are the other two roots of $P(t)=0$.

Proof. Since $L/\mathbb{Q}$ is Galois and of degree 3, there exists $a,b,c\in\mathbb{Q}$ such that $t_2=at_1^2+bt_1+c$. We want to look for $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ such that $(at_1^2+bt_1+c)(\alpha t_1+\beta)=(\gamma t_1+\delta)$. We can take α equal to the product of the numerator and the denominator of a and we make $\beta=(\alpha aA-b\alpha)/a$, $\gamma=\beta b+c\alpha-\alpha aB$ and $\delta=c\beta-\alpha aC$. Now, it is easy to check that $t_3=\frac{\alpha t_2+\beta}{\gamma t_2+\delta}$ is the third root and that $t_1\neq t_2\neq t_3\neq t_1$.

Lemma 5.4 With the notation above, $M:=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathcal{M}_3(\mathbb{Z})$ has order 3, $\delta=-(\alpha+1)$ and $\beta\gamma=-(\alpha^2+\alpha+1)$. Moreover, if we have $A=0$, then $B=3\frac{\beta}{\gamma}$ and $C=\frac{\beta(2\alpha+1)}{\gamma^2}$.

Proof. It is enough with checking that $t_1=\frac{\alpha t_3+\beta}{\gamma t_3+\delta}$, so $M^3=1$, which implies $\alpha\delta-\beta\gamma=1$, $\alpha+\delta=-1$, $\alpha^2+\beta\gamma=\delta$ and $\delta^2+\beta\gamma=\alpha$. For the last statement, we just write $0=-A=t_1+t_2+t_3$.

5.2 *A particular example*

Let us take $A = 0$, $B = -3$ and $C = 1$, that is, $\alpha = \gamma = -1$, $\beta = 1$ and $\delta = 0$, and the cocycle given by

$$\sigma \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix},$$

where $\sigma(t_i) = t_{i+1}$. Notice that 2 is not a norm in the decomposition field $\mathbb{Q}_P(t)$ since 2 is inert in $\mathbb{Q}_P(t)$. We prove it by checking that 2 does not divide the discriminant $\Delta_P = 81$ and that the polynomial $P(t) = t^3 - 3t + 1$ is irreducible in $\mathbb{F}_2$.

The equations in Theorem 5.2 for this particular example look like:

$$(-2\omega_0^3 + 6\omega_0^2\omega_6 - 6\omega_0\omega_6^2 - 3\omega_1^3 + 6\omega_1^2\omega_5 + 3\omega_1^2\omega_7 - 6\omega_1\omega_5^2 - 3\omega_1\omega_7^2 + 3\omega_5^3 - 3\omega_5^2\omega_7 + 3\omega_5\omega_7^2 +$$

$$4\omega_6^3 - 6\omega_6^2\omega_9 + 6\omega_6\omega_9^2 - 2\omega_9^3)\boldsymbol{t^2} + (2\omega_0^3 - 6\omega_0^2\omega_9 + 6\omega_0\omega_9^2 + 3\omega_1^2\omega_5 - 3\omega_1^2\omega_7 -$$

$$3\omega_1\omega_5^2 + 3\omega_1\omega_7^2 + 3\omega_5^3 - 6\omega_5^2\omega_7 + 6\omega_5\omega_7^2 + 2\omega_6^3 - 6\omega_6^2\omega_9 + 6\omega_6\omega_9^2 - 3\omega_7^3 - 4\omega_9^3)\mathbf{t}$$

$$+2\omega_0^3 - 18\omega_0^2\omega_6 + 12\omega_0^2\omega_9 + 24\omega_0\omega_6^2 - 12\omega_0\omega_6\omega_9 - 6\omega_0\omega_9^2 + 7\omega_1^3 - 18\omega_1^2\omega_5 -$$

$$3\omega_1^2\omega_7 + 15\omega_1\omega_5^2 + 6\omega_1\omega_5\omega_7 - 5\omega_5^3 - 3\omega_5\omega_7^2 - 10\omega_6^3 + 6\omega_6^2\omega_9 + \omega_7^3 + 2\omega_9^3 = 0$$

$$(-2\omega_0^2\omega_1 + 6\omega_0^2\omega_5 - 4\omega_0^2\omega_7 + 4\omega_0\omega_1\omega_9 - 8\omega_0\omega_5\omega_6 - 4\omega_0\omega_5\omega_9 + 8\omega_0\omega_6\omega_7 -$$

$$3\omega_1^2\omega_2 + 2\omega_1^2\omega_3 + \omega_1^2\omega_8 + 4\omega_1\omega_2\omega_5 + 2\omega_1\omega_2\omega_7 - 4\omega_1\omega_3\omega_5 + 2\omega_1\omega_6^2 -$$

$$4\omega_1\omega_6\omega_9 - 2\omega_1\omega_7\omega_8 - 2\omega_2\omega_5^2 - \omega_2\omega_7^2 + 3\omega_3\omega_5^2 - 2\omega_3\omega_5\omega_7 + \omega_3\omega_7^2 - \omega_5^2\omega_8 +$$

$$4\omega_5\omega_6^2 + 2\omega_5\omega_7\omega_8 + 2\omega_5\omega_9^2 - 6\omega_6^2\omega_7 + 4\omega_6\omega_7\omega_9 - 2\omega_7\omega_9^2)\boldsymbol{t^2}$$

$$+(2\omega_0^2\omega_1 - 2\omega_0^2\omega_7 - 4\omega_0\omega_1\omega_9 - 4\omega_0\omega_5\omega_6 + 4\omega_0\omega_5\omega_9 + 4\omega_0\omega_6\omega_7 + \omega_1^2\omega_3 -$$

$$\omega_1^2\omega_8 + 2\omega_1\omega_2\omega_5 - 2\omega_1\omega_2\omega_7 - 2\omega_1\omega_3\omega_5 + 4\omega_1\omega_6^2 - 8\omega_1\omega_6\omega_9 + 2\omega_1\omega_7\omega_8 +$$

$$6\omega_1\omega_9^2 - \omega_2\omega_5^2 + \omega_2\omega_7^2 + 3\omega_3\omega_5^2 - 4\omega_3\omega_5\omega_7 + 2\omega_3\omega_7^2 - 2\omega_5^2\omega_8 + 2\omega_5\omega_6^2 + 4\omega_5\omega_7\omega_8 -$$

$$2\omega_5\omega_9^2 - 6\omega_6^2\omega_7 + 8\omega_6\omega_7\omega_9 - 3\omega_7^2\omega_8 - 4\omega_7\omega_9^2)\mathbf{t}$$

$$+2\omega_0^2\omega_1 - 14\omega_0^2\omega_5 + 12\omega_0^2\omega_7 - 4\omega_0\omega_1\omega_6 + 24\omega_0\omega_5\omega_6 + 4\omega_0\omega_5\omega_9 - 20\omega_0\omega_6\omega_7 - 4\omega_0\omega_7\omega_9 +$$

$$7\omega_1^2\omega_2 - 6\omega_1^2\omega_3 - \omega_1^2\omega_8 - 12\omega_1\omega_2\omega_5 - 2\omega_1\omega_2\omega_7 + 10\omega_1\omega_3\omega_5 + 2\omega_1\omega_3\omega_7 + 2\omega_1\omega_5\omega_8 +$$

$$4\omega_1\omega_6\omega_9 - 2\omega_1\omega_9^2 + 5\omega_2\omega_5^2 + 2\omega_2\omega_5\omega_7 - 5\omega_3\omega_5^2 - \omega_3\omega_7^2 - 10\omega_5\omega_6^2 - 4\omega_5\omega_6\omega_9 -$$

$$2\omega_5\omega_7\omega_8 + 10\omega_6^2\omega_7 + \omega_7^2\omega_8 + 2\omega_7\omega_9^2 = 0$$

$$(-\omega_0^2\omega_2 + 3\omega_0^2\omega_3 - 2\omega_0^2\omega_8 - 3\omega_0\omega_1^2 + 4\omega_0\omega_1\omega_5 + 2\omega_0\omega_1\omega_7 + 2\omega_0\omega_2\omega_9 - 4\omega_0\omega_3\omega_6 -$$

$$2\omega_0\omega_3\omega_9 - 2\omega_0\omega_5^2 + 4\omega_0\omega_6\omega_8 - \omega_0\omega_7^2 + 2\omega_1^2\omega_6 + \omega_1^2\omega_9 - 4\omega_1\omega_5\omega_6 - 2\omega_1\omega_7\omega_9 +$$

$$\omega_2\omega_6^2 - 2\omega_2\omega_6\omega_9 + 2\omega_3\omega_6^2 + \omega_3\omega_9^2 + 3\omega_5^2\omega_6 - \omega_5^2\omega_9 - 2\omega_5\omega_6\omega_7 + 2\omega_5\omega_7\omega_9 -$$

$$3\omega_6^2\omega_8 + \omega_6\omega_7^2 + 2\omega_6\omega_8\omega_9 - \omega_8\omega_9^2)\boldsymbol{t^2}$$

$$+(\omega_0^2\omega_2 - \omega_0^2\omega_8 + 2\omega_0\omega_1\omega_5 - 2\omega_0\omega_1\omega_7 - 2\omega_0\omega_2\omega_9 - 2\omega_0\omega_3\omega_6 + 2\omega_0\omega_3\omega_9 -$$

$$\omega_0\omega_5^2 + 2\omega_0\omega_6\omega_8 + \omega_0\omega_7^2 + \omega_1^2\omega_6 - \omega_1^2\omega_9 - 2\omega_1\omega_5\omega_6 + 2\omega_1\omega_7\omega_9 + 2\omega_2\omega_6^2 -$$

$$4\omega_2\omega_6\omega_9 + 3\omega_2\omega_9^2 + \omega_3\omega_6^2 - \omega_3\omega_9^2 + 3\omega_5^2\omega_6 - 2\omega_5^2\omega_9 - 4\omega_5\omega_6\omega_7 +$$

$$4\omega_5\omega_7\omega_9 - 3\omega_6^2\omega_8 + 2\omega_6\omega_7^2 + 4\omega_6\omega_8\omega_9 - 3\omega_7^2\omega_9 - 2\omega_8\omega_9^2)\boldsymbol{t}$$

$$+\omega_0^2\omega_2 - 7\omega_0^2\omega_3 + 6\omega_0^2\omega_8 + 7\omega_0\omega_1^2 - 12\omega_0\omega_1\omega_5 - 2\omega_0\omega_1\omega_7 - 2\omega_0\omega_2\omega_6 +$$

$$12\omega_0\omega_3\omega_6 + 2\omega_0\omega_3\omega_9 + 5\omega_0\omega_5^2 + 2\omega_0\omega_5\omega_7 - 10\omega_0\omega_6\omega_8 - 2\omega_0\omega_8\omega_9 -$$

$$6\omega_1^2\omega_6 - \omega_1^2\omega_9 + 10\omega_1\omega_5\omega_6 + 2\omega_1\omega_5\omega_9 + 2\omega_1\omega_6\omega_7 + 2\omega_2\omega_6\omega_9 -$$

$$\omega_2\omega_9^2 - 5\omega_3\omega_6^2 - 2\omega_3\omega_6\omega_9 - 5\omega_5^2\omega_6 - 2\omega_5\omega_7\omega_9 + 5\omega_6^2\omega_8 - \omega_6\omega_7^2 + \omega_7^2\omega_9 + \omega_8\omega_9^2 = 0$$

$$(-4\omega_0^2\omega_2 - 2\omega_0^2\omega_3 + 6\omega_0^2\omega_8 + 8\omega_0\omega_2\omega_6 + 4\omega_0\omega_3\omega_9 - 8\omega_0\omega_6\omega_8 - 4\omega_0\omega_8\omega_9 - 3\omega_1\omega_2^2 +$$

$$4\omega_1\omega_2\omega_3 + 2\omega_1\omega_2\omega_8 - 2\omega_1\omega_3^2 - \omega_1\omega_8^2 + 2\omega_2^2\omega_5 + \omega_2^2\omega_7 - 4\omega_2\omega_3\omega_5 - 6\omega_2\omega_6^2 +$$

$$4\omega_2\omega_6\omega_9 - 2\omega_2\omega_7\omega_8 - 2\omega_2\omega_9^2 + 3\omega_3^2\omega_5 - \omega_3^2\omega_7 - 2\omega_3\omega_5\omega_8 + 2\omega_3\omega_6^2 -$$

$$4\omega_3\omega_6\omega_9 + 2\omega_3\omega_7\omega_8 + \omega_5\omega_8^2 + 4\omega_6^2\omega_8 + 2\omega_8\omega_9^2)\boldsymbol{t^2}$$

$$-(2\omega_0^2\omega_2 + 2\omega_0^2\omega_3 + 4\omega_0\omega_2\omega_6 - 4\omega_0\omega_3\omega_9 - 4\omega_0\omega_6\omega_8 + 4\omega_0\omega_8\omega_9 + 2\omega_1\omega_2\omega_3 -$$

$$2\omega_1\omega_2\omega_8 - \omega_1\omega_3^2 + \omega_1\omega_8^2 + \omega_2^2\omega_5 - \omega_2^2\omega_7 - 2\omega_2\omega_3\omega_5 - 6\omega_2\omega_6^2 + 8\omega_2\omega_6\omega_9 +$$

$$2\omega_2\omega_7\omega_8 - 4\omega_2\omega_9^2 + 3\omega_3^2\omega_5 - 2\omega_3^2\omega_7 - 4\omega_3\omega_5\omega_8 + 4\omega_3\omega_6^2 -$$

$$8\omega_3\omega_6\omega_9 + 4\omega_3\omega_7\omega_8 + 6\omega_3\omega_9^2 + 2\omega_5\omega_8^2 + 2\omega_6^2\omega_8 - 3\omega_7\omega_8^2 - 2\omega_8\omega_9^2)\boldsymbol{t}$$

$$+12\omega_0^2\omega_2+2\omega_0^2\omega_3-14\omega_0^2\omega_8-20\omega_0\omega_2\omega_6-4\omega_0\omega_2\omega_9-4\omega_0\omega_3\omega_6+24\omega_0\omega_6\omega_8+$$

$$4\omega_0\omega_8\omega_9+7\omega_1\omega_2^2-12\omega_1\omega_2\omega_3-2\omega_1\omega_2\omega_8+5\omega_1\omega_3^2+2\omega_1\omega_3\omega_8-$$

$$6\omega_2^2\omega_5-\omega_2^2\omega_7+10\omega_2\omega_3\omega_5+2\omega_2\omega_3\omega_7+2\omega_2\omega_5\omega_8+10\omega_2\omega_6^2+2\omega_2\omega_9^2-$$

$$5\omega_3^2\omega_5+4\omega_3\omega_6\omega_9-2\omega_3\omega_7\omega_8-2\omega_3\omega_9^2-\omega_5\omega_8^2-$$

$$10\omega_6^2\omega_8-4\omega_6\omega_8\omega_9+\omega_7\omega_8^2=0$$

$$(-\omega_0^2\omega_4-3\omega_0\omega_1\omega_2+2\omega_0\omega_1\omega_3+\omega_0\omega_1\omega_8+2\omega_0\omega_2\omega_5+$$

$$\omega_0\omega_2\omega_7-2\omega_0\omega_3\omega_5+2\omega_0\omega_4\omega_6-\omega_0\omega_7\omega_8+2\omega_1\omega_2\omega_6+$$

$$\omega_1\omega_2\omega_9-2\omega_1\omega_3\omega_6-\omega_1\omega_8\omega_9-2\omega_2\omega_5\omega_6-\omega_2\omega_7\omega_9+3\omega_3\omega_5\omega_6-$$

$$\omega_3\omega_5\omega_9-\omega_3\omega_6\omega_7+\omega_3\omega_7\omega_9-2\omega_4\omega_6\omega_9+\omega_4\omega_9^2-\omega_5\omega_6\omega_8+\omega_5\omega_8\omega_9+\omega_6\omega_7\omega_8)\boldsymbol{t^2}$$

$$-(\omega_0^2\omega_4+\omega_0\omega_1\omega_3-\omega_0\omega_1\omega_8+\omega_0\omega_2\omega_5-\omega_0\omega_2\omega_7-\omega_0\omega_3\omega_5+2\omega_0\omega_4\omega_9+$$

$$\omega_0\omega_7\omega_8+\omega_1\omega_2\omega_6-\omega_1\omega_2\omega_9-\omega_1\omega_3\omega_6+\omega_1\omega_8\omega_9-\omega_2\omega_5\omega_6+$$

$$\omega_2\omega_7\omega_9+3\omega_3\omega_5\omega_6-2\omega_3\omega_5\omega_9-2\omega_3\omega_6\omega_7+2\omega_3\omega_7\omega_9+\omega_4\omega_6^2-$$

$$2\omega_4\omega_6\omega_9-2\omega_5\omega_6\omega_8+2\omega_5\omega_8\omega_9+2\omega_6\omega_7\omega_8-3\omega_7\omega_8\omega_9)\boldsymbol{t}$$

$$+4\omega_0^2\omega_4+7\omega_0\omega_1\omega_2-6\omega_0\omega_1\omega_3-\omega_0\omega_1\omega_8-6\omega_0\omega_2\omega_5-\omega_0\omega_2\omega_7+$$

$$5\omega_0\omega_3\omega_5+\omega_0\omega_3\omega_7-6\omega_0\omega_4\omega_6-2\omega_0\omega_4\omega_9+\omega_0\omega_5\omega_8-6\omega_1\omega_2\omega_6-$$

$$\omega_1\omega_2\omega_9+5\omega_1\omega_3\omega_6+\omega_1\omega_3\omega_9+\omega_1\omega_6\omega_8+5\omega_2\omega_5\omega_6+\omega_2\omega_5\omega_9+$$

$$\omega_2\omega_6\omega_7-5\omega_3\omega_5\omega_6-\omega_3\omega_7\omega_9+2\omega_4\omega_6^2+$$

$$2\omega_4\omega_6\omega_9-\omega_5\omega_8\omega_9-\omega_6\omega_7\omega_8+\omega_7\omega_8\omega_9=0$$

$$(-8\omega_0^3+12\omega_0^2\omega_6+12\omega_0^2\omega_9-12\omega_0\omega_6^2-12\omega_0\omega_9^2-3\omega_2^3+$$

$$6\omega_2^2\omega_3+3\omega_2^2\omega_8-6\omega_2\omega_3^2-3\omega_2\omega_8^2+3\omega_3^3-$$

$$3\omega_3^2\omega_8+3\omega_3\omega_8^2+4\omega_6^3+4\omega_9^3)\boldsymbol{t^2}$$

$$+(-4\omega_0^3+12\omega_0^2\omega_6-12\omega_0\omega_6^2+3\omega_2^2\omega_3-3\omega_2^2\omega_8-3\omega_2\omega_3^2+3\omega_2\omega_8^2+$$

$$3\omega_3^3-6\omega_3^2\omega_8+6\omega_3\omega_8^2+8\omega_6^3-12\omega_6^2\omega_9+12\omega_6\omega_9^2-3\omega_8^3-4\omega_9^3)\boldsymbol{t}$$

$$+24\omega_0^3-36\omega_0^2\omega_6-36\omega_0^2\omega_9+12\omega_0\omega_6^2+48\omega_0\omega_6\omega_9+$$

$$12\omega_0\omega_9^2+7\omega_2^3-18\omega_2^2\omega_3-3\omega_2^2\omega_8+15\omega_2\omega_3^2+6\omega_2\omega_3\omega_8-$$

$$5\omega_3^3-3\omega_3\omega_8^2-12\omega_6^2\omega_9-12\omega_6\omega_9^2+\omega_8^3=0$$

$$(-4\omega_0^2\omega_1-2\omega_0^2\omega_5+6\omega_0^2\omega_7+8\omega_0\omega_1\omega_6-3\omega_0\omega_2^2+4\omega_0\omega_2\omega_3+$$

$$2\omega_0\omega_2\omega_8-2\omega_0\omega_3^2+4\omega_0\omega_5\omega_9-8\omega_0\omega_6\omega_7-4\omega_0\omega_7\omega_9-\omega_0\omega_8^2-6\omega_1\omega_6^2+$$

$$4\omega_1\omega_6\omega_9-2\omega_1\omega_9^2+2\omega_2^2\omega_6+\omega_2^2\omega_9-4\omega_2\omega_3\omega_6-2\omega_2\omega_8\omega_9+3\omega_3^2\omega_6-$$

$$\omega_3^2\omega_9-2\omega_3\omega_6\omega_8+2\omega_3\omega_8\omega_9+2\omega_5\omega_6^2-4\omega_5\omega_6\omega_9+4\omega_6^2\omega_7+\omega_6\omega_8^2+2\omega_7\omega_9^2)\boldsymbol{t}^2$$

$$+(-2\omega_0^2\omega_1+2\omega_0^2\omega_5+4\omega_0\omega_1\omega_6+2\omega_0\omega_2\omega_3-2\omega_0\omega_2\omega_8-\omega_0\omega_3^2-4\omega_0\omega_5\omega_9-$$

$$4\omega_0\omega_6\omega_7+4\omega_0\omega_7\omega_9+\omega_0\omega_8^2-6\omega_1\omega_6^2+8\omega_1\omega_6\omega_9-4\omega_1\omega_9^2+$$

$$\omega_2^2\omega_6-\omega_2^2\omega_9-2\omega_2\omega_3\omega_6+2\omega_2\omega_8\omega_9+3\omega_3^2\omega_6-2\omega_3^2\omega_9-$$

$$4\omega_3\omega_6\omega_8+4\omega_3\omega_8\omega_9+4\omega_5\omega_6^2-8\omega_5\omega_6\omega_9+6\omega_5\omega_9^2+2\omega_6^2\omega_7+$$

$$2\omega_6\omega_8^2-2\omega_7\omega_9^2-3\omega_8^2\omega_9)\boldsymbol{t}$$

$$+12\omega_0^2\omega_1+2\omega_0^2\omega_5-14\omega_0^2\omega_7-20\omega_0\omega_1\omega_6-4\omega_0\omega_1\omega_9+$$

$$7\omega_0\omega_2^2-12\omega_0\omega_2\omega_3-2\omega_0\omega_2\omega_8+5\omega_0\omega_3^2+2\omega_0\omega_3\omega_8-4\omega_0\omega_5\omega_6+$$

$$24\omega_0\omega_6\omega_7+4\omega_0\omega_7\omega_9+10\omega_1\omega_6^2+2\omega_1\omega_9^2-6\omega_2^2\omega_6-\omega_2^2\omega_9+$$

$$10\omega_2\omega_3\omega_6+2\omega_2\omega_3\omega_9+2\omega_2\omega_6\omega_8-5\omega_3^2\omega_6-2\omega_3\omega_8\omega_9+$$

$$4\omega_5\omega_6\omega_9-2\omega_5\omega_9^2-10\omega_6^2\omega_7-4\omega_6\omega_7\omega_9-\omega_6\omega_8^2+\omega_8^2\omega_9=0.$$

Now the intersection $\cap_{\sigma\in \mathrm{Gal}(L/\mathbb{Q})}{}^{\sigma}X$ is just generated by the equations $f_1=f_2=f_3=0$ where the previous equations are written as $f_1\boldsymbol{t}^2=f_2\boldsymbol{t}=f_3=0$ with $f_i\in\mathbb{Q}[\omega_0,\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8,\omega_9]$.

6. Singular equations

As we have already seen, equations for smooth models of Brauer-Severi varieties are quite impractical since they have many variables and terms. However, if we work with singular models we can show "nicer" models, meaning, having less variables and shorter equations.

Lemma 6.1 Let K be a perfect field, and let be $B_n : X_1X_2 \ldots X_{n+1} = X_0^{n+1} \subseteq \mathbb{P}_K^{n+1}$. Then B_n is birationally equivalent to $\mathbb{P}^n$ over $\mathbb{Q}$.

Proof. Let us consider de map $\psi : \mathbb{P}^n \to \mathbb{P}^{n+1} : (x_0 : x_1 : \ldots : x_n) \mapsto (x_0 : x_1 : \ldots : x_n : \frac{x_0^{n+1}}{x_1 \ldots x_n})$. It gives a birational map between $\mathbb{P}^n$ and $\psi(\mathbb{P}^n)$. Moreover, $\psi(\mathbb{P}^n) \simeq B_n$.

Theorem 6.2 Let (χ, α) be a pair consisting of an isomorphism $\chi : \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathbb{Z}/(n+1)\mathbb{Z}$, where L is a cyclic Galois extension L/K of degree $n+1$, and an element $\alpha \in K^*$. Let $\{l_1, l_2, \ldots, l_{n+1}\}$ be a normal basis of L/K. Then, the Brauer-Severi variety associated with (χ, α) as in Theorem 2.1 and Proposition 2.2 is birationally equivalent over K to

$$\alpha \, \mathrm{N}_{L/K}(l_1x_1 + \ldots + l_{n+1}x_{n+1}) = x_0^{n+1}.$$

Proof. First of all, notice that this variety is birationally equivalent to B_n in Lemma 6.1 over L. In particular, it is birationally equivalent to $\mathbb{P}^n$ over L, and hence birationally equivalent to a Brauer-Severi variety over K. We will see that indeed, it is birationally equivalent over K to the Brauer-Severi variety associated with (χ, α) as in Theorem 2.1 and Proposition 2.2. We will see that a birational map from $\alpha \, \mathrm{N}_{L/K}(l_1x_1 + \ldots + l_{n+1}x_{n+1}) = x_0^{n+1}$ to B_n is given by the matrix

$$\tilde{\phi} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & l_1 & l_2 & \ldots & l_n \\ 0 & \alpha l_n & \alpha l_1 & \ldots & \alpha l_{n-1} \\ 0 & \alpha l_{n-1} & \alpha l_n & \ldots & \alpha l_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \alpha l_2 & \alpha l_3 & \ldots & \alpha l_1 \end{pmatrix}.$$

Let us call ϕ_n to the composition of $\tilde{\phi}$ with the inverse of the birational map ψ in Lemma 6.1 between B_n and $\mathbb{P}^n$. Then the cocycle defined by $\tilde{\xi}(\sigma) = \phi_n{}^{\sigma}\phi_n^{-1}$ is not equal, but equivalent, to the cocycle in proposition 2.2 defining the Brauer-Severi variety attached to (χ, α). Indeed, take

$$f(x_0 : x_1 : \ldots : x_n) = (P : P_0 : P_1 : \ldots : P_{n-1}) \in \mathrm{Aut}_{birat}(\mathbb{P}^n),$$

where $P = x_0 \cdots x_n$ and $P_i = P\frac{x_{i-1}}{x_i}$. It is straightforward to check that $f\xi(\sigma)^{\sigma}f^{-1} = \tilde{\xi}(\sigma)$.

Remark 6.3 A weaker version of Theorem 6.2 appears in [20]. However, notice that the first version of Theorem 6.2 appeared in ArXiv two years before than the first draft of the aforementioned reference.

Corollary 6.4 Let $\mathcal{B}$ be a Brauer-Severi surface dened over $\mathbb{Q}$. It corresponds to a pair (χ, α) where $\chi : \mathrm{Gal}(L/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z}$ is an isomorphism, L is a cyclic Galois extension of degree 3 and $\alpha \in \mathbb{Q}^*$. Write $L = \mathbb{Q}(l_1, l_2, l_3)$ with l_i a normal basis for $L/\mathbb{Q}$ with minimal polynomial $x^3 + Ax^2 + Bx + C$. Then $\mathcal{B}$ is given by the singular model

$$\alpha \, \mathrm{N}_{L/\mathbb{Q}}(l_1x_1 + l_2x_2 + l_3x_3) = x_0^3 \subseteq \mathbb{P}^3,$$

or equivalently by

$$-C(x_1^3 + x_2^3 + x_3^3) + D_1(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) + D_2(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) +$$

$$+(3AB - A^3)x_1x_2x_3 = \alpha^{-1}x_0^3 \subseteq \mathbb{P}^3,$$

where D_1 is the rational number $l_1^2l_2 + l_2^2l_3 + l_3^2l_1$ and $D_2 = l_1l_2^2 + l_2l_3^2 + l_3l_1^2$. [The discriminant of the polynomial $x^3 + Ax^2 + Bx + C = 0$ is $(D_1 - D_2)^2$ and $D_1 + D_2$ is a symmetric function on l_1, l_2, l_3, hence D_1 and D_2 are writtenable in terms of A, B and C].

Example 6.5 The singular model for the Brauer-Severi surface in subsection 5.2 is

$$2(x_1^3 + x_2^3 + x_3^3) - 12(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) + 6(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) = x_0^3 \subseteq \mathbb{P}^3.$$

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