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Research Article

Differential Subordination Results of Multivalent Functions Involving High-Order Derivatives

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Abstract: We study differential subordinations for normalized higher-order derivatives of p-valent analytic functions in the unit disk. The method is based on Miller–Mocanu first-order differential subordination principles applied to auxiliary normalized functions formed from $f^{(r)}$ and non-vanishing analytic factors. We derive verifiable sufficient conditions under which these normalized derivatives are subordinate to prescribed univalent, convex, or starlike dominant functions, and we identify the best dominant whenever the hypotheses of the corresponding Miller–Mocanu lemma are satisfied. The framework includes Janowski-type, convex-dominant, and Ma–Minda-type specializations and recovers several established multivalent results. Particular attention is given to the admissibility of the auxiliary functions used in the proofs and to the role of the higher-order derivative index r, which permits one formulation to cover a hierarchy of multivalent differential subordinations.

Keywords: analytic functions, multivalent functions, differential subordination, higher-order derivatives, analytic dominant function

MSC: 30C45, 30C50, 30C80

1. Introduction

Differential subordination is a central tool in geometric function theory for converting analytic differential inequalities into geometric inclusion relations. Miller and Mocanu developed the foundational theory of first- and second-order differential subordination, including the best-dominant principle, in their seminal paper and monograph [1, 2]. For higher-order subordination problems, Ponnusamy and Juneja developed third-order differential inequalities [3], while Antonino and Miller later formulated a systematic third-order subordination framework [4]. Atshan et al. introduced a fourth-order subordination and superordination setting for multivalent analytic functions [5], and Al-Ameedee et al. obtained further higher-order results for multivalent functions [6]. More recently, Tayyah et al. studied third-order subordination and superordination for p-valent analytic functions involving a fractional derivative operator [7].

The multivalent setting is emphasized because a p-valent function can assume a value up to p times in the unit disk, so it naturally contains the univalent case $p = 1$ while retaining the multiplicity parameter in the normalization of

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higher derivatives. This additional parameter allows treatment of geometric subclasses whose behavior is not visible under the one-valent normalization. Multivalent subordination also interacts naturally with integral and convolution operators; for example, recent work has produced sandwich results for multivalent functions and related operators [8]. In a complementary direction, Hadi et al. derived sandwich results for symmetric p-valent meromorphic functions with an application to radial heat diffusion [9], while Darweesh et al. obtained third-order subordination, superordination, and sandwich results for holomorphic functions associated with a logarithmic integral operator [10]. These developments motivate a careful higher-derivative framework in which the analytic hypotheses needed by the differential subordination lemmas are explicit.

Let $\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\}$ denote the open unit disk. An analytic function f on $\mathbb{D}$ is said to be subordinate to another analytic function g on $\mathbb{D}$, written $f \prec g$, if there exists a Schwarz function ω, which is analytic on $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$, such that $f(\xi) = g(\omega(\xi))$ for all $\xi \in \mathbb{D}$. If g is univalent on $\mathbb{D}$, then subordination reduces to the geometric condition $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

For $p \in \mathbb{N}$, let A_p denote the class of so-called p-valent analytic functions in $\mathbb{D}$ having the form

$$f(\xi) = \xi^p + \sum_{n=p+1}^{\infty} a_n \xi^n. \tag{1}$$

In particular, the class A_1 is the standard class of normalized analytic functions with $f(0) = 0$ and $f'(0) = 1$. Many classical subclasses of analytic functions, including multivalent starlike and convex functions, arise as special cases of appropriate differential subordinations satisfied by members of A_p.

Differentiating (1) r times yields

$$f^{(r)}(\xi) = \Upsilon(r, p)\, \xi^{\,p-r} + \sum_{n=p+1}^{\infty} \Upsilon(r, n)\, a_n\, \xi^{n-r}, \tag{2}$$

where $\Upsilon(r, p)$ and $\Upsilon(r, n)$ are the factorial coefficients

$$\Upsilon(r, p) = \frac{p!}{(p-r)!} \quad \text{and} \quad \Upsilon(r, n) = \frac{n!}{(n-r)!}, \quad 0 \le r \le p < n, \quad \text{for} \quad p \in \mathbb{N}. \tag{3}$$

A p-valent function $f \in A_p$ is said to be *starlike* and *convex* if the real parts satisfy the following conditions, respectively:

$$\frac{1}{p}\Re\left[\frac{\xi f'(\xi)}{f(\xi)}\right] > 0 \quad \text{and} \quad \frac{1}{p}\Re\left[1 + \frac{\xi f''(\xi)}{f'(\xi)}\right] > 0 \qquad \xi \in \mathbb{D}.$$

Recently, subclasses of starlike and convex functions associated with various geometric domains have received considerable attention. In particular, Khan et al. [11] introduced and investigated a subclass of symmetric starlike functions in A_1 associated with a balloon-shaped domain, obtaining sharp coefficient estimates, Fekete–Szegö inequalities, and bounds for Hankel, inverse, and logarithmic coefficients.

More generally, let φ be an analytic function on $\mathbb{D}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\Re(\varphi(\xi)) > 0$, mapping $\mathbb{D}$ conformally onto a region symmetric about the real axis and starlike with respect to 1. The classes $S_p^*(\varphi)$ of p-valent starlike functions and $C_p(\varphi)$ of p-valent convex functions are defined, respectively, by:

$$S_p^*(\varphi) = \left\{ f \in A_p : \frac{1}{p}\frac{\xi f'(\xi)}{f(\xi)} \prec \varphi(\xi) \right\}, \tag{4}$$

$$C_p(\varphi) = \left\{ f \in A_p : \frac{1}{p}\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) \prec \varphi(\xi) \right\}. \tag{5}$$

These classes were introduced and explored in [12]. The theory developed by Ma and Minda was originally formulated for the special case $p = 1$ in [13], which corresponds to the class of normalized analytic functions on $\mathbb{D}$.

The principal contribution of this paper is a unified normalization for higher-order derivatives of p-valent functions together with a collection of sufficient differential subordination criteria. Unlike a result tied to one fixed derivative order, the parameter r allows the same argument to be applied across successive derivatives. The key findings are: (i) best-dominant criteria for the normalized quantity $f^{(r)}(\xi)/(\xi^{p-r}\Upsilon(r,\, p)q_1q_2)$; (ii) Janowski and Ma–Minda-type consequences; and (iii) further criteria for the normalized logarithmic derivative $\xi f^{(r+1)}/f^{(r)}$. The proofs below explicitly verify the form of the Miller–Mocanu auxiliary functions and, where an auxiliary function would otherwise depend on ξ, impose the required constant-coefficient hypothesis.

Given $p \in \mathbb{N}$, a p-valent function $f \in A_p$, and an integer r with $0 \leq r < p$, we consider differential expressions of the form

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,\, p)\, q_1(\xi)q_2(\xi)},$$

and investigate differential subordination criteria associated with them. Here, q_1 and q_2 are non-vanishing analytic functions on $\mathbb{D}$ with $q_1(0) = q_2(0) = 1$ chosen to unify various geometric subclasses. Furthermore, we set

$$q(\xi) = \frac{\xi q_1'(\xi)}{q_1(\xi)} + \frac{\xi q_2'(\xi)}{q_2(\xi)} + p - r, \tag{6}$$

which gives

$$q'(\xi) = \left[\xi \frac{q_1'(\xi)}{q_1(\xi)}\right]' + \left[\xi \frac{q_2'(\xi)}{q_2(\xi)}\right]'. \tag{7}$$

Our principal aim is to establish sufficient conditions, with best-dominant conclusions where justified, for differential subordinations of the forms

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,\, p)q_1(\xi)q_2(\xi)} \prec \Phi_p(\xi) \qquad \text{and} \qquad \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi) + 1 \prec \Phi_p(\xi),$$

where Φ_p is a univalent function on $\mathbb{D}$ belonging to the appropriate starlike or convex classes. Ali et al. developed subordination results for multivalent functions [12]; Irmak and Cho investigated a differential operator associated with higher-order derivatives and its applications to multivalently analytic functions [14]; and Özkan Kılıç obtained sufficient subordination conditions for multivalent functions [15]. The present formulation is designed to place related higher-order derivative consequences within a common normalization.

The paper is organized as follows. Section 2 presents the necessary preliminary lemmas needed in the sequel. Section 3 contains the main results of subordination, including sufficient conditions for various families of dominant functions. Section 4 provides concluding remarks and outlines potential directions for future investigations.

2. Preliminary lemmas

In this section, we recall two fundamental lemmas from the theory of differential subordination, which will be used throughout our main results. For convenience, we present here the statements that appear in the monograph of Miller and Mocanu [2] in a form adapted to our notation.

Lemma 2.1 (Miller–Mocanu [2, Theorem 3.4h, p.132, special case]). Let Φ_p be a univalent function on $\mathbb{D}$, and let κ be analytic in a domain U containing $\Phi_p(\mathbb{D})$ where $\kappa(w) \neq 0$. Suppose that the function $\xi\Phi_p'(\xi)\,\kappa(\Phi_p(\xi))$ is starlike on $\mathbb{D}$. If $P(\xi)$ is analytic on $\mathbb{D}$ with

$$P(0) = \Phi_p(0), \qquad P(\mathbb{D}) \subset U,$$

and

$$\xi P'(\xi)\,\kappa(P(\xi)) \prec \xi\Phi_p'(\xi)\,\kappa(\Phi_p(\xi)),$$

then

$$P(\xi) \prec \Phi_p(\xi) \quad \text{and} \quad \Phi_p(\xi) \text{ is the best dominant.}$$

Remark. Lemma 2.1 is a special case of Miller and Mocanu [2, Theorem 3.4h, p.132]. Indeed, in the theorem, taking $\theta(w) \equiv 0$ and $\phi(w) = \kappa(w)$, one obtains

$$Q(\xi) = \xi\Phi_p'(\xi)\kappa(\Phi_p(\xi)).$$

Hence, when Q is starlike on $\mathbb{D}$, the condition

$$\Re\left(\frac{\xi Q'(\xi)}{Q(\xi)}\right) > 0$$

is automatically satisfied. Therefore, the differential subordination

$$\xi P'(\xi)\kappa(P(\xi)) \prec \xi\Phi_p'(\xi)\kappa(\Phi_p(\xi))$$

implies $P \prec \Phi_p$, and Φ_p is the best dominant.

Lemma 2.2 (Miller–Mocanu [2, Theorem 3.4h, p.132]). Let Φ_p be univalent on $\mathbb{D}$, and let θ and ϕ be analytic in a domain U containing $\Phi_p(\mathbb{D})$, with $\phi(w) \neq 0$ for $w \in \Phi_p(\mathbb{D})$. Define

$$Q(\xi) = \xi \Phi_p'(\xi)\phi(\Phi_p(\xi)) \quad \text{and} \quad h(\xi) = \theta(\Phi_p(\xi)) + Q(\xi).$$

Assume that Q is starlike on $\mathbb{D}$ and

$$\Re\left(\frac{\xi h'(\xi)}{Q(\xi)}\right) > 0 \qquad \xi \in \mathbb{D}.$$

If P is analytic on $\mathbb{D}$, $P(0) = \Phi_p(0)$, $P(\mathbb{D}) \subset U$, and

$$\theta(P(\xi)) + \xi P'(\xi)\phi(P(\xi)) \prec h(\xi),$$

then $P \prec \Phi_p$, and Φ_p is the best dominant.

Remark. Lemma 2.2 is an application of the general differential subordination principle of Miller and Mocanu [2, Theorem 3.4h, p.132]. In the notation of the theorem, let

$$Q(\xi) = \xi \Phi_p'(\xi)\phi(\Phi_p(\xi)) \quad \text{and} \quad h(\xi) = \theta(\Phi_p(\xi)) + Q(\xi),$$

where θ and ϕ are analytic in a domain containing $\Phi_p(\mathbb{D})$ and $\phi(w) \neq 0$ on $\Phi_p(\mathbb{D})$. The hypotheses imposed in Lemma 2.2 ensure that the corresponding functions Q and h satisfy the conditions required in the Miller–Mocanu theorem. Consequently, whenever an analytic function P with $P(0) = \Phi_p(0)$ and $P(\mathbb{D}) \subset U$ satisfies

$$\theta(P(\xi)) + \xi P'(\xi)\phi(P(\xi)) \prec \theta(\Phi_p(\xi)) + \xi \Phi_p'(\xi)\phi(\Phi_p(\xi)),$$

the conclusion

$$P(\xi) \prec \Phi_p(\xi), \qquad \xi \in \mathbb{D},$$

follows, and Φ_p is the best dominant. This formulation also emphasizes that the auxiliary functions θ and ϕ depend only on their argument and not explicitly on the independent variable ξ.

Lemmas 2.1 and 2.2 provide the basic differential subordination tools used in the sequel. In each application, an appropriate auxiliary function P is introduced so that the differential subordination under consideration can be transformed into the standard Miller–Mocanu form. Once the analyticity and range conditions for the auxiliary functions, together with the required starlikeness or convexity assumptions on the dominant function, are verified, the corresponding lemma yields the subordination $P \prec \Phi_p$ and, under its stated hypotheses, identifies Φ_p as the best dominant. This procedure will be applied in Section 3 to higher-order derivatives of multivalent analytic functions and will also recover several known results as special cases.

3. Main results

In this section, we obtain sufficient conditions for differential subordinations involving the higher-order derivatives of functions in the class A_p.

For convenience, we rewrite the basic subordination of interest

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)} \prec \Phi_p(\xi), \tag{8}$$

where q_1 and q_2 are non-vanishing analytic functions on $\mathbb{D}$ with $q_1(0) = q_2(0) = 1$, and Φ_p is a suitably chosen dominant function.

The results are divided into two categories:

(1) Sufficient conditions for differential subordinations of type (8).

(2) Sufficient conditions for differential subordinations of type (9),

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi) + 1 \prec \Phi_p(\xi), \tag{9}$$

where $q(\xi)$ is defined in (6).

3.1 *Sufficient conditions for differential subordinations of type (8)*

Theorem 3.1 Let Φ_p be univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$ and $0 \notin \Phi_p(\mathbb{D})$, and assume that

$$\frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)}$$

is starlike on $\mathbb{D}$. If $f \in A_p$ satisfies the differential subordination

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} \prec \frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)} + q(\xi), \tag{10}$$

then

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)} \prec \Phi_p(\xi), \tag{11}$$

and $\Phi_p(\xi)$ is the best dominant.

Proof. Define

$$P(\xi) = \frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)} \qquad \text{and} \qquad \kappa(\mu) = \frac{1}{\mu}. \tag{12}$$

Since the left-hand side of (10) is analytic on $\mathbb{D}$ and $f^{(r)}$ has no zeros in $\mathbb{D}\setminus\{0\}$, P is non-vanishing on $\mathbb{D}$ with $P(0)=1$. κ is also analytic in a domain containing $\Phi_p(\mathbb{D})$.

Logarithmic differentiation of (12) gives

$$\frac{P'(\xi)}{P(\xi)}=\frac{f^{(r+1)}(\xi)}{f^{(r)}(\xi)}-\frac{p-r}{\xi}-\frac{q_1'(\xi)}{q_1(\xi)}-\frac{q_2'(\xi)}{q_2(\xi)}.$$

Multiplication by ξ and the definition (6) therefore yield

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}=q(\xi)+\frac{\xi P'(\xi)}{P(\xi)}. \tag{13}$$

Subordination (10) becomes

$$\frac{\xi P'(\xi)}{P(\xi)}\prec\frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)}.$$

Hence

$$\xi P'(\xi)\,\kappa(P(\xi))\prec\xi\Phi_p'(\xi)\,\kappa(\Phi_p(\xi)),$$

and since the right-hand side is starlike, Lemma 2.1 implies $P\prec\Phi_p$. □

Corollary 3.2 Let $-1\le M<L\le 1$. If $f\in A_p$ satisfies

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\prec\frac{(L-M)\xi}{(1+M\xi)(1+L\xi)}+q(\xi), \tag{14}$$

then

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)}\prec\frac{1+L\xi}{1+M\xi}. \tag{15}$$

Proof. Let

$$\Phi_p(\xi)=\frac{1+L\xi}{1+M\xi}.$$

A direct calculation gives

$$K(\xi) := \frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)} = \frac{(L-M)\xi}{(1+M\xi)(1+L\xi)},$$

$$h(\xi) := \frac{\xi K'(\xi)}{K(\xi)} = \frac{1-LM\xi^2}{(1+M\xi)(1+L\xi)}.$$

Let $\xi = re^{i\theta}$, then

$$\Re h(re^{i\theta}) = \frac{(1-LMr^2)\left(1+LMr^2+(L+M)r\cos\theta\right)}{\left|(1+Mre^{i\theta})(1+Lre^{i\theta})\right|^2}.$$

Since

$$1+LMr^2+(L+M)\,r\cos\theta \geq (1-Lr)\,(1-Mr) > 0$$

for $(L+M) \geq 0$, and similarly

$$\left(1+LMr^2+(L+M)\,r\cos\theta\right) \geq (1+Lr)\,(1+Mr) > 0$$

for $(L+M) \leq 0$, it follows that $\Re h(\xi) > 0$, and hence $\dfrac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)}$ is starlike on $\mathbb{D}$.

Thus, the conditions of Theorem 3.1 are satisfied, and the result follows. □

Moreover, Theorem 3.1 produces two more corollaries in [12] by taking the following special conditions, respectively:

$$q_1(\xi) = \frac{1}{q_2(\xi)}, \quad r = 0, \quad \text{and} \quad \Phi_p(\xi) = \frac{h_{\varphi,\,p}(\xi)}{\xi^p}.$$

$$q_1(\xi) = \frac{1}{q_2(\xi)}, \quad r = 0, \quad \text{and} \quad \Phi_p(\xi) = \frac{k'_{\varphi,\,p}(\xi)}{\xi^p}.$$

Corollary 3.3 ([12]) If $f \in S_p^*(\varphi)$, then

$$\frac{f(\xi)}{\xi^p} \prec \frac{h_{\varphi,\,p}(\xi)}{\xi^p}.$$

Corollary 3.4 ([12]) If $f \in C_p(\varphi)$, then

$$\frac{f(\xi)}{\xi^p} \prec \frac{k'_{\varphi,\,p}(\xi)}{\xi^p}.$$

3.2 *Sufficient conditions with convex dominants*

Theorem 3.5 Let Φ_p be convex univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$. If $f \in A_p$ satisfies the differential subordination

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)}\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi)\right] \prec \xi\Phi_p'(\xi), \tag{16}$$

then

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)} \prec \Phi_p(\xi),$$

and $\Phi_p(\xi)$ is the best dominant.

Proof. Let P be defined by (12) and

$$\kappa(\mu) = 1.$$

Using identity (13), we obtain

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)\,q_1(\xi)\,q_2(\xi)}\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi)\right] = \xi P'(\xi).$$

Thus, the given subordination (16) reduces to

$$\xi P'(\xi) \prec \xi\Phi_p'(\xi),$$

which can be rewritten as

$$\xi P'(\xi)\,\kappa(P(\xi)) \prec \xi\Phi_p'(\xi)\,\kappa(\Phi_p(\xi)).$$

Since Φ_p is convex, $\xi\Phi_p'(\xi)$ is starlike, so Lemma 2.1 applies. □

Example 3.6 Take $q_1(\xi) = 1/q_2(\xi)$ and

$$\Phi_p(\xi) = 1 + \frac{\xi}{\Upsilon(r,p)}.$$

Then $q(\xi) = p - r$ and $P(\xi) = f^{(r)}(\xi)/(\Upsilon(r,p)\xi^{p-r})$. Hence the hypothesis of Theorem 3.5 becomes

$$\xi P'(\xi) \prec \frac{\xi}{\Upsilon(r,p)}.$$

In particular, the image-domain inclusion gives $|\xi P'(\xi)| < 1/\Upsilon(r,\, p)$. Multiplication by $\Upsilon(r,\, p)|\xi|^{p-r}$ therefore yields the left-hand estimate below. Likewise, the conclusion $P \prec 1+\xi/\Upsilon(r,\, p)$ gives $|P(\xi)-1| < 1/\Upsilon(r,\, p)$ and hence the right-hand estimate. Thus, this example illustrates a related differential operator approach to multivalent functions, as studied by Irmak and Cho [14].

$$\left| f^{(r)}(\xi) \left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - (p-r) \right] \right| < |\xi|^{p-r}$$

$$\text{implies} \quad \left| f^{(r)}(\xi) - \Upsilon(r,\, p)\, \xi^{p-r} \right| < |\xi|^{p-r}.$$

3.3 *Sufficient conditions of Ma–Minda type*

Theorem 3.7 Let Φ_p be convex univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$ and assume that $q(\xi) \equiv p-r$. If $f \in A_p$ satisfies the differential subordination

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,\, p) q_1(\xi) q_2(\xi)} \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} \prec \xi \Phi_p'(\xi) + (p-r)\Phi_p(\xi), \tag{17}$$

then

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,\, p) q_1(\xi) q_2(\xi)} \prec \Phi_p(\xi),$$

and Φ_p is the best dominant.

Proof. Let P be defined by (12). From (13) and $q = p-r$,

$$P(\xi) \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} = (p-r)P(\xi) + \xi P'(\xi).$$

Thus, (17) is equivalent to

$$\theta(P(\xi)) + \xi P'(\xi)\phi(P(\xi)) \prec \theta(\Phi_p(\xi)) + \xi \Phi_p'(\xi)\phi(\Phi_p(\xi)),$$

with $\theta(\mu) = (p-r)\mu$ and $\phi(\mu) = 1$. Here $Q(\xi) = \xi\Phi_p'(\xi)$ is starlike because Φ_p is convex, and

$$\Re \frac{\xi h'(\xi)}{Q(\xi)} = \Re \left(p-r+1+\frac{\xi \Phi_p''(\xi)}{\Phi_p'(\xi)} \right) > 0$$

as $\Re(p-r) = p-r > 0$. Lemma 2.2 therefore gives $P \prec \Phi_p$ and the best-dominant assertion. □

The following corollary follows from Theorem 3.7 by taking $q_1(\xi) = \dfrac{1}{q_2(\xi)}$ and $r = 0$.

Corollary 3.8 ([15]) If Φ_p is convex and $f \in A_p$ satisfies

$$\frac{f'(\xi)}{\xi^{p-1}} \prec \xi\Phi_p'(\xi) + p\Phi_p(\xi),$$

then

$$\frac{f(\xi)}{\xi^p} \prec \Phi_p(\xi).$$

In particular, if $p = 1$, then Corollary 3.8 reduces to the corollary below.

Corollary 3.9 ([15]) If Φ_1 is convex and $f \in A_1$ satisfies

$$f'(\xi) \prec \xi\Phi_1'(\xi) + \Phi_1(\xi),$$

then

$$\frac{f(\xi)}{\xi} \prec \Phi_1(\xi).$$

3.4 *Sufficient conditions with starlike quotients*

Theorem 3.10 Let Φ_p be convex univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$ and $0 \notin \Phi_p(\mathbb{D})$. Assume that

$$\frac{\xi\Phi_p'(\xi)}{\Phi_p^2(\xi)}$$

is starlike. If $f \in A_p$ satisfies the differential subordination

$$\frac{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)}{f^{(r)}(\xi)}\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi)\right] \prec \frac{\xi\Phi_p'(\xi)}{\Phi_p^2(\xi)}, \tag{18}$$

then

$$\frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r,p)q_1(\xi)q_2(\xi)} \prec \Phi_p(\xi),$$

and $\Phi_p(\xi)$ is the best dominant.

Proof. Let $P(\xi)$ be defined by (12) and

$$\kappa(\mu) = \frac{1}{\mu^2}.$$

The analyticity of the left-hand side of (18) ensures that P is non-vanishing on $\mathbb{D}$. It follows from the identity (13) that

$$\frac{\xi^{p-r}\Upsilon(r,p)\,q_1(\xi)\,q_2(\xi)}{f^{(r)}(\xi)}\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi)\right] = \frac{1}{P(\xi)}\frac{\xi P'(\xi)}{P(\xi)} = \frac{\xi P'(\xi)}{P^2(\xi)}.$$

Thus, the given subordination (18) can be replaced with

$$\frac{\xi P'(\xi)}{P(\xi)^2} \prec \frac{\xi \Phi_p'(\xi)}{\Phi_p(\xi)^2},$$

which yields

$$\xi P'(\xi)\,\kappa(P(\xi)) \prec \xi \Phi_p'(\xi)\,\kappa(\Phi_p(\xi)).$$

Since $\dfrac{\xi \Phi_p'(\xi)}{\Phi_p^2(\xi)}$ is starlike by hypothesis, Lemma 2.1 now applies. □

3.5 *Sufficient conditions for differential subordinations of type (9)*

We now consider the following normalized differential subordination of type (9)

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - q(\xi) + 1 \prec \Phi_p(\xi).$$

The following theorems provide sufficient conditions that ensure the differential subordination (9), where $q_1 q_2 = 1$, equivalently, $q(\xi) = p - r$. Define

$$P(\xi) = \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - (p-r) + 1. \tag{19}$$

By (2), the quotient function $\xi f^{(r+1)}(\xi)/f^{(r)}(\xi)$ has a removable singularity at the origin with limiting value $p-r$, whereas $f^{(r+1)}(\xi)/f^{(r)}(\xi)$ generally has a pole at $\xi = 0$. Thus, the factor ξ in the quotient is essential and gives the normalization $P(0) = 1$. In the results below, we additionally assume that P is analytic on $\mathbb{D}$.

Theorem 3.11 Let $f \in A_p$ and Φ_p be univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$, and assume that $\Phi_p(\mathbb{D})$, $P(\mathbb{D}) \subset \mathbb{C} \setminus \{0,\, 1 - p + r\}$ for P defined by (19). If

$$\frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)\left[\Phi_p(\xi)+(p-r)-1\right]}$$

is starlike on $\mathbb{D}$ and f satisfies the differential subordination

$$\frac{1+\dfrac{\xi f^{(r+2)}(\xi)}{f^{(r+1)}(\xi)}-\dfrac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}}{P(\xi)} \prec \frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)\left[\Phi_p(\xi)+(p-r)-1\right]}, \tag{20}$$

then

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}-q(\xi)+1 \prec \Phi_p(\xi),$$

and Φ_p is the best dominant.

Proof. Set

$$\kappa(\mu)=\frac{1}{\mu(\mu+(p-r)-1)}. \tag{21}$$

Since

$$P(\xi)+(p-r)-1=\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)},$$

logarithmic differentiation gives

$$\frac{P'(\xi)}{P(\xi)+(p-r)-1}=\frac{1}{\xi}+\frac{f^{(r+2)}(\xi)}{f^{(r+1)}(\xi)}-\frac{f^{(r+1)}(\xi)}{f^{(r)}(\xi)}.$$

Hence

$$\frac{\xi P'(\xi)}{P(\xi)[P(\xi)+(p-r)-1]}=\frac{1+\dfrac{\xi f^{(r+2)}(\xi)}{f^{(r+1)}(\xi)}-\dfrac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}}{P(\xi)}. \tag{22}$$

Therefore (20) is precisely

$$\xi P'(\xi)\kappa(P(\xi)) \prec \xi\Phi_p'(\xi)\kappa(\Phi_p(\xi)).$$

The range assumptions make κ analytic on $\mathbb{C}\setminus\{0,\ 1-p+r\}$, and the right-hand side is starlike by hypothesis. Lemma 2.1 yields $P \prec \Phi_p$ and the best-dominant conclusion. □

For clarity, define the Janowski-type class $S^*[L, M]$ by

$$S^*[L, M] := \left\{ f \in A_1 : \frac{\xi f'(\xi)}{f(\xi)} \prec \frac{1+L\xi}{1+M\xi} \right\}, \qquad -1 \le M < L \le 1.$$

Applying Theorem 3.11 with $p = 1$, $r = 0$, and

$$\Phi_1(\xi) = \frac{1+L\xi}{1+M\xi}, \qquad -1 \le M < L \le 1.$$

Since $\dfrac{\xi\Phi_1'(\xi)}{\Phi_1^2(\xi)} = \dfrac{(L-M)\xi}{(1+L\xi)^2}$ is starlike on $\mathbb{D}$, we have the following direct specialization.

Corollary 3.12 ([16]) Let $-1 \le M < L \le 1$ and $f \in A_1$. Assume that $P(\xi) = \xi f'(\xi)/f(\xi)$ extends analytically to $\mathbb{D}$, $P(0) = 1$, and $P(\mathbb{D}) \subset \mathbb{C}\setminus\{0\}$. If

$$\frac{1+\dfrac{\xi f''(\xi)}{f'(\xi)} - \dfrac{\xi f'(\xi)}{f(\xi)}}{\dfrac{\xi f'(\xi)}{f(\xi)}} \prec \frac{(L-M)\xi}{(1+L\xi)^2},$$

then $f \in S^*[L, M]$.

Theorem 3.13 Let $f \in A_p$ and Φ_p be univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$, and assume that $\Phi_p(\mathbb{D})$, $P(\mathbb{D}) \subset \mathbb{C}\setminus\{0, r-p\}$ for P defined by (19). If

$$\frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)[\Phi_p(\xi)+(p-r)]}$$

is starlike on $\mathbb{D}$ and f satisfies the differential subordination

$$\frac{\xi\left[\dfrac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\right]'}{[P(\xi)+(p-r)]P(\xi)} \prec \frac{\xi\Phi_p'(\xi)}{\Phi_p(\xi)[\Phi_p(\xi)+(p-r)]}, \tag{23}$$

then

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - (p-r) + 1 \prec \Phi_p(\xi), \tag{24}$$

and Φ_p is the best dominant.

Proof. Set

$$\kappa(\mu) = \frac{1}{\mu(\mu + (p-r))}, \tag{25}$$

then κ is analytic in a domain containing $\Phi_p(\mathbb{D})$.

By (19)

$$P(\xi) + (p-r) = \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} + 1,$$

we have

$$P'(\xi) = \left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\right]'.$$

Thus, (23) becomes

$$\frac{\xi P'(\xi)}{P(\xi)[P(\xi) + (p-r)]} \prec \frac{\xi \Phi_p'(\xi)}{\Phi_p(\xi)[\Phi_p(\xi) + (p-r)]},$$

or equivalently

$$\xi P'(\xi)\kappa(P(\xi)) \prec \xi \Phi_p'(\xi)\kappa(\Phi_p(\xi)).$$

Hence, Lemma 2.1 gives the conclusion. □

Theorem 3.14 Let Φ_p be convex univalent on $\mathbb{D}$ with $\Phi_p(0) = 1$. If $f \in A_p$ satisfies the differential subordination

$$\xi\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\right]' + \frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} \prec \xi \Phi_p'(\xi) + \Phi_p(\xi) + (p-r) - 1, \tag{26}$$

then

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} - (p-r) + 1 \prec \Phi_p(\xi), \tag{27}$$

and Φ_p is the best dominant.

Proof. Define

$$\theta(\mu) = \mu + (p-r) - 1 \qquad \text{and} \qquad \phi(\mu) = 1. \tag{28}$$

P defined by (19) gives

$$P(\xi)+(p-r)-1=\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)} \qquad \text{and} \qquad \xi P'(\xi)=\xi\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\right]'. \tag{29}$$

Thus, (26) is exactly

$$\theta(P(\xi))+\xi P'(\xi)\phi(P(\xi)) \prec \theta(\Phi_p(\xi))+\xi\Phi_p'(\xi)\phi(\Phi_p(\xi)). \tag{30}$$

For Lemma 2.2,

$$Q(\xi)=\xi\Phi_p'(\xi), \qquad h(\xi)=\Phi_p(\xi)+(p-r)-1+\xi\Phi_p'(\xi).$$

Since Φ_p is convex, Q is starlike and

$$\Re\left(\frac{\xi h'(\xi)}{Q(\xi)}\right)=\Re\left(2+\frac{\xi\Phi_p''(\xi)}{\Phi_p'(\xi)}\right)>0.$$

Thus, Lemma 2.2 yields $P \prec \Phi_p$ and the best-dominant assertion. □

Theorem 3.15 Let Φ_p be convex univalent on $\mathbb{D}$ with $\Phi_p(0)=1$. If $f \in A_p$ satisfies the differential subordination

$$\xi\left[\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}\right]' \prec \xi\Phi_p'(\xi), \tag{31}$$

then

$$\frac{\xi f^{(r+1)}(\xi)}{f^{(r)}(\xi)}-(p-r)+1 \prec \Phi_p(\xi), \tag{32}$$

and Φ_p is the best dominant.

Proof. Applying (29) for P defined by (19), (31) is precisely

$$\xi P'(\xi) \prec \xi\Phi_p'(\xi).$$

Since Φ_p is convex, $\xi\Phi_p'$ is starlike. Applying Lemma 2.1 with $\kappa(\mu)=1$ yields $P \prec \Phi_p$ and the best dominant conclusion. □

4. Conclusion

This paper develops a higher-order differential subordination framework for p-valent analytic functions. The main device is the normalized auxiliary function

$$P(\xi) = \frac{f^{(r)}(\xi)}{\xi^{p-r}\Upsilon(r, p)q_1(\xi)q_2(\xi)},$$

whose logarithmic derivative converts conditions on $\xi f^{(r+1)}/f^{(r)}$ into standard Miller–Mocanu subordinations. This yields sufficient criteria with best dominants for starlike-quotient, convex-dominant, Janowski, and Ma–Minda-type settings. In contrast with first-order results, the index r retains the multiplicity information of the original p-valent function and permits one argument to cover a hierarchy of higher derivatives. The proofs also make explicit that auxiliary functions used in the Miller–Mocanu lemmas must depend only on the dependent variable; consequently, the results requiring a coefficient of the form $q(\xi)$ are stated under the mathematically necessary constant-coefficient hypothesis. Several known multivalent results are recovered by taking $r = 0$ and $q_1q_2 = 1$.

The framework suggests two natural extensions. First, one may seek genuine variable-coefficient analogues using admissibility theory designed for coefficients depending explicitly on ξ. Second, pairing the present best-dominant results with corresponding superordination theorems may yield sandwich-type results comparable to recent developments for p-valent and operator-defined classes. Further directions include investigating coefficient estimates and extending to meromorphic p-valent functions through differential operators arising from q-calculus. In this direction, Al-Shbeil et al. [17] employed a Sălăgean-type q-differential operator to study subclasses of meromorphic p-valent q-starlike functions in the Janowski domain and obtained, among other results, coefficient estimates, distortion properties, partial-sum results, and radii of starlikeness.

Ethics statement

This article does not contain any studies with human participants or animals performed by any of the authors.

AIGC usage statement

The authors confirm that no artificial intelligence tools were used in the preparation of this manuscript.

Author contributions

Methodology, MG.K., H.A., J.G., and M.H.; formal analysis, MG.K., H.A., J.G., and M.H.; writing—original draft, MG.K., H.A., J.G., and M.H.; writing—review & editing, MG.K., H.A., J.G., and M.H. All authors have read and approved the final version of the manuscript. All authors contributed equally to the final version of the manuscript.

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

The authors declare that they have no conflict of interest.

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