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Research Article

Existence of Solutions for a Quasilinear System with Gradient Terms

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Abstract: In this paper, it is considered the existence of solutions for a quasilinear system involving the p-Laplacian operator and gradient terms. The approach is based on sub-supersolution arguments and the Schauder's Fixed Point Theorem. The results in this paper allow to consider several growth conditions in the gradient and complete some recent contributions.

Keywords: p-Laplacian, gradient term, convection term, dependence of the gradient, quasilinear system, sub-supersolutions

MSC: 35A15, 35J60

1. Introduction

In the last decades, elliptic problems involving gradient terms have been attracting the attention of several researchers due to interesting difficulties which arise when one intends to consider this kind of problem, see for instance [1-3].

For example in [1], it was applied the Galerkin method to show the existence of a solution for the elliptic problem given by

$$\begin{cases} -\Delta u = h(x,u) + \lambda g(x, \nabla u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda \geq 0$ is a parameter, h is a function with singular and sublinear terms and g is a continuous function satisfying certain conditions.

In [3], it is considered the problem

$$\begin{cases} -\Delta_p u = h(u)|\nabla u|^p + g(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $-\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$ and h and g are functions satisfying certain technical conditions. By means

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of a Kazdan-Kramer change of variable, the equation is reduced to a quasilinear one without gradient term which is approachable by variational methods. Such a method allowed the authors to obtain several existence and multiplicity results.

In the reference [2], through the sub-supersolution technique and the Schauder's Fixed Point Theorem, it was obtained the existence of positive solutions for the following problem

$$\begin{cases} -\Delta_p u = \lambda h(x,u) + \beta f(x,u,\nabla u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where h is a sublinear function and f may exhibit a growth higher than p in the gradient variable. For other interesting papers which consider problems involving gradient terms see [4-8] and the references therein.

In this work, which was motivated by [2], we are interested to obtain the existence of positive solutions for the quasilinear system

$$\begin{cases} -\Delta_p u = \lambda h(x,u) + \beta f(x,u,\nabla u) \text{ in } \Omega, \\ -\Delta_q v = \lambda k(x,v) + \beta g(x,u,\nabla u) \text{ in } \Omega, \\ \quad u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{1,\xi}$ boundary, for some $\xi \in (0, 1)$, $1 < \min\{p, q\} \leq \max\{p, q\} < N$, λ and β are positive parameters, $h, k : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $f, g : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions satisfying

(H_1) There are $1 < r < p$, $1 < s < q$ with $\max\{r, s\} < \min\{p, q\}$, $0 \leq \omega_1(x)t^{r-1} \leq h(x, t) \leq \omega_2(x)t^{r-1}$ and $0 \leq w_1(x)t^{s-1} \leq k(x, t) \leq w_2(x)t^{s-1}$ for all $(x, t) \in \bar{\Omega} \times [0, +\infty)$;

(H_2) There exist $a, b, c, d > 0$ such that $0 \leq f(x, t, \xi) \leq \omega_3(x)t^a|\xi|^b$, $0 \leq g(x, t, \xi) \leq \omega_3(x)t^c|\xi|^d$ for all $(x, t, \xi) \in \bar{\Omega} \times [0, +\infty) \times \mathbb{R}^N$,

where $\omega_i : \bar{\Omega} \to \mathbb{R}$, $i = 1, 2, 3$ are nonnegative continuous functions.

A weak solution for (1) is a pair $(u, v) \in W$, where $W := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, with $u(x), v(x) > 0$ a.e in Ω such that

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u . \nabla \varphi \, dx = \int_\Omega \lambda h(x,u)\varphi + \beta f(x,v,\nabla v)\varphi \, dx$$

and

$$\int_\Omega |\nabla v|^{q-2}\, \nabla v . \nabla \eta \, dx = \int_\Omega \lambda k(x,v)\eta + \beta g(x,u,\nabla u)\eta \, dx,$$

for all $(\varphi, \eta) \in W$.

An important fact is that a direct approach via Variational arguments is not applicable due to the presence of the convection terms in (1). Since we are also interested to study problems with a superlinear growth in the gradient, it is expected that the Galerkin method will not be effective due to technical difficulties that arise in concave-convex problems, see for instance the commentaries in [9]. The goal of this manuscript consists in to study (1) via a sub-supersolution approach by using the ideas of [2]. Note that additional mathematical difficulties arise due to the fact that we are considering the system version of the scalar problem considered in [2]. Such difficulties are handled by a careful application of Lemma 2.2 which allows constructing convenient sub-supersolutions. At this point, we quote that, to the best of our knowledge, it is the first time that the system (1) is considered under the conditions (H_1)-(H_2). This manuscript also completes the study done in [2] due to the fact that a system version of the problem in [2] is studied and the papers [1, 3-8] in the sense that different hypotheses can be considered to study systems with convection terms and involving the p-Laplacian operator.

The main result of this manuscript is provided below.

Theorem 1.1 Suppose that h, k, f and g are continuous functions satisfying (H_1) and (H_2). There exists a set $\mathcal{R}$ in the $\lambda\beta$-plane such that, if $(\lambda, \beta) \in \mathcal{R}$, then the problem (1) has a solution.

2. Preliminaries

Before we present the proof of Theorem 1.1 it will be considered some auxiliary facts.

Unless otherwise stated Ω will denote a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{1,\xi}$ boundary, for some $\xi \in (0, 1)$. The next result, which can be found in [2], will be needed for our purposes.

Lemma 2.1 If $u \in W_0^{1,m}(\Omega)$, $1 < m < N$ is a solution of the problem

$$\begin{cases} -\Delta_m u = g \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g \in L^\infty(\Omega)$, then there exists a positive constant $\mathcal{K}_m$, depending only on m, N and Ω, such that

$$\|\nabla u\|_\infty \leq \mathcal{K}_m \|g\|_\infty^{\frac{1}{m-1}}. \tag{2}$$

Consider $\omega(x) := \max\{\omega_1(x), \omega_2(x), \omega_3(x)\}$, $x \in \bar{\Omega}$, the first eigenpair of the p-Laplacian with weight w_1 and the first eigenpair of the q-Laplacian with weight w_1; that is,

$$\begin{cases} -\Delta_p u_1 = \lambda_1 w_1 u_1^{p-1} \text{ in } \Omega, \\ \quad u_1 = 0 \text{ on } \partial\Omega, \end{cases} \text{ and } \begin{cases} -\Delta_q v_1 = \theta_1 w_1 v_1^{q-1} \text{ in } \Omega, \\ \quad v_1 = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

which satisfy $u_1(x), v_1(x) > 0$ a.e in Ω and $\|u_1\|_\infty, \|v_1\|_\infty = 1$. Let $\phi \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$, $\psi \in W_0^{1,q}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$, $\alpha \in (0, 1)$ be the solutions of the problems

$$\begin{cases} -\Delta_p \phi = \omega \text{ in } \Omega, \\ \quad \phi = 0 \text{ on } \partial\Omega, \end{cases} \text{ and } \begin{cases} -\Delta_q \psi = \omega \text{ in } \Omega, \\ \quad \psi = 0 \text{ on } \partial\Omega. \end{cases}$$

Define

$$\gamma := \max\left\{ \frac{\|w\|_\infty^{\frac{1}{p-1}} \mathcal{K}_p}{\|\phi\|_\infty}, \frac{\|w\|_\infty^{\frac{1}{q-1}} \mathcal{K}_q}{\|\psi\|_\infty}, 1 \right\}, \tag{4}$$

where $\mathcal{K}_p$ and $\mathcal{K}_q$ are the positive constants obtained by applying Lemma 2.1 with $m = p$ and $m = q$ respectively. We emphasize that γ depends only on ω, p, q, N, Ω and ω.

The next result, which is based on [2], will play an important role in our arguments.

Lemma 2.2 There exists a set $\mathcal{R}$ in the $\lambda\beta$-plane such that, if $\lambda, \beta > 0$ and $(\lambda, \beta) \in \mathcal{R}$, then

$$\begin{aligned} \lambda M^{r-1} + \beta \gamma^b M^{a+b} &\leq (M / \|\phi\|_\infty)^{p-1}, \\ \lambda M^{s-1} + \beta \gamma^c M^{c+d} &\leq (M / \|\psi\|_\infty)^{q-1}, \end{aligned} \tag{5}$$

for some constant $M > 0$.

Proof. Consider the function $\Psi(t) := \lambda A t^{l-k} + \beta B t^{n-k}$, $t > 0$, where $A := (\min\{(1/\|\phi\|_\infty)^{p-1}, (1/\|\psi\|_\infty)^{q-1}\})^{-1}$, $l := \max\{r-1, s-1\}$, $k := \min\{p-1, q-1\}$, $B := A\gamma^m$ with $m := \max\{b, c\}$ and $n := \max\{a+b, c+d\}$. It will be proved that $\Psi(M) \leq 1$ for a constant $M \geq 1$, which replies the result.

Case 1 $n > k$. In such case, we obtain that $\lim_{t \to 0^+} \Psi(t) = \lim_{t \to +\infty} \Psi(t) = +\infty$, which implies that Ψ has a minimum value.

Since the only critical point M of Ψ is given by

$$M := \left[\frac{\lambda A(k-l)}{\beta B(n-k)}\right]^{\frac{1}{n-l}},$$

we obtain for such point that

$$\begin{aligned}\Psi(M) &= \lambda A\left[\frac{\lambda A(k-l)}{\beta B(n-k)}\right]^{\frac{l-k}{n-l}} + \beta BM^{n-k} = \frac{\beta B(n-k)M^{n-k}}{k-l} + \beta BM^{n-k} \\ &= \beta BM^{n-k}\left(\frac{n-l}{k-l}\right) \le \Psi(t),\end{aligned} \tag{6}$$

for all $t \ge 0$.

In order to have $M \ge 1$, it is necessary that

$$\frac{\lambda}{\beta} \ge \frac{B(n-k)}{A(k-l)} =: C.$$

Note that since $M \ge 1$ the inequalities in (5) will occur simultaneously. Now we need to obtain conditions on λ and β to get $\Psi(M) \le 1$ or, equivalently,

$$\beta B\left[\frac{\lambda A(k-l)}{\beta B(n-k)}\right]^{\frac{n-k}{n-l}}\left(\frac{n-l}{k-l}\right) \le 1.$$

Rewriting the above inequality it follows that

$$\lambda^{n-k}\beta^{k-l} \le \left(\frac{n-k}{A}\right)^{n-k}\left(\frac{k-l}{B}\right)^{k-l}\frac{1}{(n-l)^{n-l}} =: K.$$

Case 2 $n = k$. In this case we have that $\Psi(t) := \lambda At^{l-k} + \beta B$ is a positive and strictly decreasing function, which satisfies

$$\lim_{t\to 0^+}\Psi(t) = +\infty \text{ and } \lim_{t\to+\infty}\Psi(t) = \beta B.$$

To obtain $\Psi(M) \le 1$ for some $M \ge 1$, we need that $\beta B < 1$. If

$$\lambda > 0 \text{ and } \beta < B^{-1},$$

it is possible to choose $M > 0$ such that $\Psi(M) = 1$, that is,

$$M = \left(\frac{\lambda A}{1-\beta B}\right)^{\frac{1}{k-l}}.$$

Note that $M \geq 1$, for $\lambda A + \beta B \geq 1$.

Case 3 $n < k$. In this case we have that Ψ is a strictly decreasing function with $\lim_{t\to 0^+} \Psi(t) = +\infty$ and $\lim_{t\to +\infty} \Psi(t) = 0$. Then, we conclude that for any positive parameters λ and β, there is $M \geq 1$ with $\Psi(M) \leq 1$.

Thus, it follows that there is a constant $M \geq 1$ such that $\Psi(M) \leq 1$ whenever $(\lambda, \beta) \in \mathcal{R}$, where

$$\mathcal{R} := \begin{cases} \{\lambda, \beta > 0; \lambda \geq \beta C \text{ and } \lambda^{n-k}\beta^{k-l} \leq K\} & \text{if } n > k, \\ \{\lambda, \beta > 0; \beta < B^{-1} \text{ and } \lambda A + \beta B \geq 1\} & \text{if } n = k, \\ \{\lambda, \beta > 0\} & \text{if } n < k. \end{cases} \tag{7}$$

For each pair $(u, v) \in X$, where $X := C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$, define the continuous nonlinearities

$$F_v^u(x,t) := \lambda\omega_1 t^{r-1} + \lambda(h(x,u(x)) - w_1 u(x)^{r-1}) + \beta f(x,v,\nabla v),$$

$$G_v^u(x,t) := \lambda\omega_1 t^{s-1} + \lambda(k(x,v(x)) - w_1 v(x)^{s-1}) + \beta g(x,u,\nabla u).$$

Consider $(u, v) \in X$ and the system

$$\begin{cases} -\Delta_p w = F_v^u(x,w) & \text{in } \Omega, \\ -\Delta_q z = G_v^u(x,z) & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where a solution for (8) is understood to be a pair $(w,z) \in Z := (W_0^{1,p}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,q}(\Omega) \cap L^\infty(\Omega))$ such that

$$\int_\Omega |\nabla w|^{p-2}\, \nabla w.\nabla\varphi = \int_\Omega F_v^u(x,w)\varphi$$

and

$$\int_\Omega |\nabla z|^{q-2}\, \nabla z.\nabla\eta\, dx = \int_\Omega G_v^u(x,z)\eta\, dx,$$

for all $(\varphi, \eta) \in W$.

We say that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in Z$ is a sub-supersolution for (8) if $0 < \underline{u}(x) \leq \overline{u}(x)$, $0 < \underline{v}(x) \leq \overline{v}(x)$ a.e in Ω and the inequalities

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2}\nabla\underline{u}.\nabla\varphi\, dx & \leq & \int_\Omega F_v^u(x,\underline{u})\varphi\, dx, \\ \int_\Omega |\nabla \underline{v}|^{q-2}\nabla\underline{v}.\nabla\eta\, dx & \leq & \int_\Omega G_v^u(x,\underline{v})\eta\, dx \end{cases} \text{ and } \begin{cases} \int_\Omega |\nabla \overline{u}|^{p-2}\nabla\overline{u}.\nabla\varphi\, dx & \geq & \int_\Omega F_v^u(x,\overline{u})\varphi\, dx \\ \int_\Omega |\nabla \overline{v}|^{q-2}\nabla\overline{v}.\nabla\eta\, dx & \geq & \int_\Omega G_v^u(x,\overline{u})\eta\, dx, \end{cases}$$

hold for all nonnegative functions $\varphi \in W_0^{1,p}(\Omega)$ and $\eta \in W_0^{1,q}(\Omega)$.

Now we are in position to prove Theorem 1.1.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Consider $(\lambda, \beta) \in \mathcal{R}$, where the set $\mathcal{R}$ was defined in (7). Let $M, \gamma > 0$ be the constants given in Lemma 2.2 and in the equation (4) respectively. Define the set

$$\mathcal{U} := \{(u,v) \in X; \epsilon u_1 \le u \le (M/\|\phi\|_\infty)\phi, \epsilon v_1 \le v \le (M/\|\psi\|_\infty)\psi, \|\nabla u\|_\infty \le M\gamma \text{ and } \|\nabla v\|_\infty \le M\gamma\},$$

where u_1 and v_1 are given in (3) and

$$0 < \epsilon < \min\{1, (\lambda/\lambda_1)^{\frac{1}{p-r}}, (\lambda/\theta_1)^{\frac{1}{q-s}}, \|u_1\|_\infty^{-1}, \|v_1\|_\infty^{-1}, (M\lambda_1^{-\frac{1}{p-1}})/\|\phi\|_\infty, (M\theta_1^{-\frac{1}{q-1}})/\|\psi\|_\infty\} := \epsilon_0.$$

The proof will be considered in some steps.

Step 1 It will be proved for each $(u, v) \in \mathcal{U}$ there is a unique pair of positive functions $(w, z) \in X$ which solves the system (8) and satisfies

$$\epsilon u_1 \le w \le (M/\|\phi\|_\infty)\phi \text{ and } \epsilon v_1 \le z \le (M/\|\psi\|_\infty)\psi. \tag{9}$$

The uniqueness follows from [10], due to the fact that (8) consists of two sublinear equations at w and z respectively for each pair $(u, v) \in \mathcal{U}$. Regarding the existence of the solution, it will be proved that $(\underline{u}, \underline{v}) = (\epsilon u_1, \epsilon v_1)$ and $(\overline{u}, \overline{v}) = ((M/\|\phi\|_\infty)\phi, (M/\|\psi\|_\infty)\psi)$ is a sub-supersolution and then by applying [11], we will obtain a solution $(w, z) \in Z$ for (8) satisfying (9). From Lemma 2.2 and the fact that $\|\overline{u}\|_\infty = \|\overline{v}\|_\infty = M$ we have

$$\begin{aligned}
F_v^u(x, \overline{u}) &\le \lambda\omega_1\overline{u}^{r-1} + \lambda(h(x,u) - \omega_1 u^{r-1}) + \beta f(x, v, \nabla v) \\
&\le \lambda\omega_1\overline{u}^{r-1} + \lambda(\omega_2 - \omega_1)u^{r-1} + \beta\frac{M^a}{\|\psi\|_\infty^a}\psi^a\gamma^b M^b \\
&\le \omega(\lambda M^{r-1} + \beta\gamma^b M^{a+b}) \le (M/\|\phi\|_\infty)^{p-1}\omega \\
&= -\Delta_p\overline{u}
\end{aligned}$$

and $G_v^u(x, \overline{v}) \le -\Delta_q\overline{v}$ in Ω. From the definition of $\epsilon_0 > 0$ we obtain that

$$-\Delta_p\underline{u} = \lambda_1\omega_1\underline{u}^{p-1} \le \lambda_1\omega_1\epsilon^{p-r}\underline{u}^{r-1} \le \lambda\omega_1\underline{u}^{r-1} \le F_v^u(x, \underline{u})$$

and $-\Delta_p\underline{v} \le G_v^u(x, \underline{v})$ in Ω for all $0 < \epsilon < \epsilon_0$. Since $0 < \epsilon < \epsilon_0$, we get

$$-\Delta_p\underline{u} \le \epsilon^{p-1}\lambda_1\omega_1 \le \epsilon^{p-1}\lambda_1\omega \le (M/\|\phi\|_\infty)^{p-1}\omega = -\Delta_p\overline{u}$$

and $-\Delta_p\underline{v} \le -\Delta_p\overline{v}$ in Ω. From the weak comparison principle it follows that $\underline{u}(x) \le \overline{u}(x)$ and $\underline{v}(x) \le \overline{v}(x)$ a.e in Ω.

Step 2 We will prove that the solution $(w, z) \in X$ obtained in the last step satisfies $\|\nabla w\|_\infty, \|\nabla z\|_\infty \le \gamma M$. From Lemma 2.1 we have $\|\nabla w\|_\infty^{p-1} \le \mathcal{K}_p^{p-1}\|F_v^u(x, w)\|_\infty$ and $\|\nabla z\|_\infty^{q-1} \le \mathcal{K}_q^{q-1}\|G_v^u(x, z)\|_\infty$. Applying (H_1), (H_2), Lemma 2.2 and the definition of γ, we have

$$\begin{aligned}
0 \le F_v^u(x, w) &\le \lambda\omega_1 w^{r-1} + \lambda(h(x,u) - \omega_1 u^{r-1}) + \beta f(x, v, \nabla v) \\
&\le \lambda\omega_1 w^{r-1} + \lambda(\omega_2 - \omega_1)u^{r-1} + \beta\omega_3 v^a |\nabla v|^b \\
&\le \lambda\omega_2(M\phi/\|\phi\|_\infty)^{r-1} + \beta\omega_3(M\phi/\|\phi\|_\infty)^a(\gamma M)^b
\end{aligned}$$

$$\leq \omega(\lambda M^{r-1} + \beta\gamma^b M^{a+b}) \leq \| \omega\|_\infty (M/\| \phi\|_\infty)^{p-1}$$

$$\leq (\gamma M / \mathcal{K}_p)^{p-1}$$

and $0 \leq G_v^u(x,z) \leq (\gamma M / \mathcal{K}_q)^{q-1}$ in Ω, which imply that $\|\nabla w\|_\infty, \|\nabla z\|_\infty \leq \gamma M$.

Step 3 From the regularity results of [12] and the uniform boundedness of w, $|\nabla w|$, z and $|\nabla z|$ in $L^\infty(\Omega)$, which depends only on the pair $(\lambda, \beta) \in \mathcal{R}$, it follows that $w, z \in C^{1,\theta}(\bar{\Omega})$, for some $0 < \theta < 1$.

Step 4 Consider X endowed with the norm $|\cdot| := \|\cdot\| + \|\cdot\|$, where $\|\cdot\|$ denotes the usual norm in $C^1(\bar{\Omega})$. The previous steps imply that the operator

$$T : \mathcal{U} \subset X \to X$$

$$(u,v) \to (w,z), \tag{10}$$

where $(w, z) \in X$ is the unique solution of (8) is well defined. In fact, from Step 1 we have that for each $(u, v) \in \mathcal{U}$ there is a unique pair of positive functions $(w, z) \in X$ which solves the system (8). The regularity obtained in Step 3 implies that such solution belongs to X and the claim is verified. From Step 2 we have that $T(\mathcal{U}) \subset \mathcal{U}$. Recall from Step 3 that we have uniform boundedness of w, $|\nabla w|$, z and $|\nabla z|$ in $L^\infty(\Omega)$, which depends only on the pair $(\lambda, \beta) \in \mathcal{R}$ and implies the boundedness of the set $\mathcal{U}$. The compact embedding $C^{1,\theta}(\bar{\Omega}), \to C^1(\bar{\Omega})$ provides that T is continuous and compact. Thus, by the Schauder's Fixed Point Theorem it follows that T admits a fixed point, which is a solution for (1).

Remark 3.1 We quote that the problem (1) admits a solution for $\lambda, \beta > 0$ small enough in all the cases considered in Lemma 2.2. In fact, note that for a fixed $M > 0$ the inequality (5) holds for $\lambda, \beta > 0$ small. Repeating the arguments of the proof of Theorem 1.1, we have the result.

Conflicts of interest

The authors declare no competing financial interest.

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