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On Simon's Conjecture

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Abstract: We prove Simon's conjecture for 3-manifolds.

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MSC: 57M50, 57M10, 20F67

1. Introduction

A manifold M is called a missing boundary manifold if it can be embedded in a compact manifold $\bar{M}$ such that $\bar{M}-\bar{M}$ is a closed subset of the boundary of $\bar{M}$. J. Simon gave the following definition in his venerable paper [1].

Definition 1. [1] p.251. If a 3-manifold M has the property that for any finitely generated subgroup H of $\pi_1(M)$ the cover of M corresponding to H is a missing boundary manifold, then we shall say that M has almost-compact coverings.

On page 245 of the same paper [1], Simon conjectured that every P^2-irreducible, compact, connected 3-manifold has almost-compact coverings.

Thurston's Geometrization Conjecture, proved by Perelman ([2] and [3]), implies that every compact orientable 3-manifold admits a decomposition along with a (possibly empty) collection of spheres, discs, incompressible annuli, and incompressible tori, such that each of the resulting 3-manifolds is either a Seifert manifold or hyperbolic.

Using modern terminology, Simon proved (Theorem 2.5 in [1] p.249) that if M is an orientable compact 3-manifold satisfying Thurston's Geometrization Conjecture and all submanifolds of M obtained from the decomposition of M given by the Geometrization Conjecture have almost-compact coverings then M has almost-compact coverings.

Canary in [4] gave proof of Simon's conjecture based on the unpublished work of Long and Reid. His argument used Marden's conjecture proved by Agol in a preprint entitled "Tameness of hyperbolic 3-manifolds" and, independently, by Calegari and Gabai in [5].

In this paper, we present a different proof of Simon's conjecture which is short and does not use the proof of Marden's conjecture. Our proof uses geometric group theory. Marden's conjecture is a much stronger result than is needed here. It asserts that any hyperbolic 3-manifold with finitely generated fundamental group is almost compact, whereas we only need to prove this result for such manifolds which cover the interior of compact ones.

We combine our new 3-manifolds results, stated below as Theorem 1, Theorem 2, and Theorem 3 with our new geometric group technique, introduced in Section 4 of this paper, to give a complete short proof of Simon's conjecture.

Theorem 1. Seifert manifolds have almost-compact coverings.

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Theorem 2. Compact 3-manifolds with hyperbolic interiors have almost-compact coverings.

Theorem 3. The cover of a finite volume hyperbolic 3-manifold corresponding to a geometrically finite subgroup of its fundamental group is a missing boundary manifold.

Theorem 1, Theorem 2, and Theorem 3 are proved in Section 2, Section 3, and Section 4 of this paper.

2. Proof of Theorem 1

The following definition is given by P. Scott in [6] p.429.

A Seifert fibered space is a 3-manifold M with a decomposition of M into disjoint circles, called fibers, such that each circle has a neighborhood in M, which is the union of fibers and is isomorphic to a fibered solid torus or Klein bottle.

An orientable 3-manifold cannot contain a solid Klein bottle, as that is not orientable, so in the orientable case, the preceding definition can be shortened as follows.

Definition 2. cf. [6] p.429. An orientable Seifert manifold is a 3-manifold M foliated by disjoint circles, such that each circle has a fibered neighborhood in M homeomorphic to a fibered solid torus.

Proof of Theorem 1.

Any compact orientable Seifert manifold is finitely covered by a bundle over an orientable compact surface with fiber a circle. Any cover of such a bundle is either a bundle over an orientable surface with fiber a circle or an orientable bundle over an orientable surface with fiber $\mathbf{R}$. If the fundamental group of the bundle is finitely generated, so is the fundamental group of the base space as explained in Lemma 3.2 of [6] p.432. The relevant part of this lemma is the following statement.

Let M be a Seifert fibered space with base X. There exists a short exact sequence

$$1 \to K \to \pi_1(M) \to \pi_1(X) \to 1$$

where K denotes the cyclic subgroup of the fundamental group of M generated by a regular fiber.

An orientable bundle over an orientable surface F with fiber $\mathbf{R}$ is homeomorphic to $F \times \mathbf{R}$. As $\pi_1(F)$ is finitely generated, F is a missing boundary surface with compactification which we denote by $\bar{F}$. So $F \times \mathbf{R}$ is a missing boundary manifold with compactification $\bar{F} \times \mathbf{I}$. A circle bundle over a missing boundary surface F extends to a circle bundle over $\bar{F}$, which is compact. Therefore Seifert manifolds have almost-compact coverings, proving Theorem 1.

3. Proof of Theorem 2

Definition 3. A hyperbolic 3-manifold is the quotient of $\mathbf{H}^3$ by a discrete group of isometries acting freely.

Definition 4. The convex hull of a hyperbolic 3-manifold M is the minimal convex submanifold $C(M)$ of M. It is characterized by the following property: for any pair of points x and y in $C(M)$ any geodesic segment connecting x and y is contained in $C(M)$.

Proof of Theorem 2.

If a compact 3-manifold with hyperbolic interior has a boundary component which is not a torus, then it has infinite volume. The fundamental group of such a manifold has the finitely generated intersection property (f.g.i.p.), cf. Theorem 1.3 in [7] p.19, so the result follows from Theorem 3.7 in [1], p.253.

If all the boundary components of a compact 3-manifold with hyperbolic interior are tori, then it has finite volume. In this case, we use the following result, stated by *W*. Thurston in [8] and R. Canary in [4] Corollary 8.1 and [9].

If M is a finite volume hyperbolic 3-manifold and H is a finitely generated subgroup of the fundamental group of M, then H is either geometrically finite or a virtual fiber group.

If H is a virtual fiber group, i.e. M is finitely covered by a bundle over $\mathbf{S}^1$ with fiber F, and $\pi_1(F) = H$, then the cover of M corresponding to H is homeomorphic to $F \times \mathbf{R}$. Hence that cover is a missing boundary manifold with compactification $\bar{F} \times \mathbf{I}$.

If H is geometrically finite, i.e. an ϵ-neighborhood of the convex core of the cover of M corresponding to H has finite volume for all positive ϵ, then Theorem 3 implies that the cover of M corresponding to H is a missing boundary manifold, proving Theorem 2.

4. Proof of Theorem 3

Let H be a subgroup of a group G given by the presentation $G = \langle X | R \rangle$. Let K be the standard presentation 2-complex of G, i.e. K has one vertex, K has an edge, which is a loop, for every generator $x \in X$, and K has a 2-cell for every relator $r \in R$. The Cayley complex of G, denoted by $Cayley_2(G)$, is the universal cover of K. Denote by $Cayley_2(G, H)$ the cover of K corresponding to a subgroup H of G.

Definition 5. cf. [10], [11], [12], and [13].

A finitely generated subgroup H of a finitely presented group G is tame in G if for any finite subcomplex C of $Cayley_2(G, H)$ and for any component C_0 of $Cayley_2(G, H) - C$ the group $\pi_1(C_0)$ is finitely generated.

T. Tucker proved the following result in [14] p.267.

Let M_0 be a compact orientable irreducible 3-manifold, and M be the cover of M_0 corresponding to a finitely generated subgroup of $\pi_1(M_0)$. Then M is a missing boundary manifold if and only if $\pi_1(M)$ is tame in $\pi_1(M_0)$.

Recall that an automatic structure on a group G generated by a finite set A is a set of the following finite-state automata.

(1) The word-acceptor, which accepts for every element of G at least one word in A^* representing it.

(2) For each $a \in A \cup 1$ a right multiplier, which accepts a pair $(w1, w2)$ of words accepted by the word-acceptor if and only if $w_1 a = w_2$ in G.

The property of having an automatic structure on a group G does not depend on a finite generating set A, so a group having an automatic structure is called an automatic group.

Proof of Theorem 3.

M. Mihalik proved the following result in [13] Theorem 4, p.365.

If H is a quasiconvex subgroup of the automatic group G, then H is tame in G.

Theorem 11.4.1 proven on page 266 of the classical text [15] and discussions on pages 271-314 of the same book demonstrate that any geometrically finite sub-group of the fundamental group of a geometrically finite hyperbolic 3-manifold is quasiconvex.

Therefore Theorem 4 follows from Theorem 6.

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Conflict of interest

The author declare that there is no personal or organizational conflict of interest with this work.

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