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Research Article

The Net Laplacian Spectra of Signed Complete Graphs

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Abstract: The net Laplacian matrix of a signed graph $\dot{G}$ is defined to be $N(\dot{G}) = D^{\pm}(\dot{G}) - A(\dot{G})$, where $D^{\pm}(\dot{G})$ and $A(\dot{G})$ are the diagonal matrix of net-degrees and the adjacency matrix of $\dot{G}$, respectively. In this paper, we prove that n (resp., $-n$) is a net Laplacian eigenvalue of the signed complete graph with the multiplicity at least t if there are t vertices whose all incident edges are positive (resp., negative). We establish a relationship between the net Laplacian eigenvalues of the signed complete graph $\dot{K}_n$ and the graph consisting of negative edges of $\dot{K}_n$. Additionally, we characterize signed complete graphs which have just two distinct net Laplacian eigenvalues.

Keywords: signed graph, complete graph, net Laplacian eigenvalues

MSC: 05C22, 05C50

1. Introduction

A signed graph $\dot{G}$ is an ordered pair (G, σ), where $G = (V, E)$ is a simple graph, called the underlying graph, and $\sigma : E \rightarrow \{1, -1\}$ is a sign function. A signed graph is positive (resp., negative) if all of its edges are positive (resp., negative) and denoted by $(G, +)$ (resp., $(G, -)$). Throughout the paper, we interpret an unsigned graph as a signed graph with all its edges being positive. The order of G is $|V(G)|$. The complement of G is denoted by G^c. Let $S^+(\dot{G})$ (resp., $S^-(\dot{G})$) denote the set of vertices of $\dot{G}$ whose all incident edges are positive (resp., negative). More notions and applications about signed graphs see [1-2].

For any $v \in V(\dot{G})$, the number of positive (resp., negative) edges incident with v is called the positive (resp., negative) degree of v and denoted by d_v^+ (resp., d_v^-). The net-degree of v is defined to be $d_v^{\pm} = d_v^+ - d_v^-$. The net-degree matrix of a signed graph $\dot{G}$ is a diagonal matrix $D^{\pm}(\dot{G})$, whose i-th diagonal entry is $d_i^{\pm}$. The degree matrix of a underlying graph G is a diagonal matrix $D(G)$, whose i-th diagonal entry is degree $d_i = d_i^+ + d_i^-$. The adjacency matrix of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all is that correspond to negative edges. The net Laplacian matrix of a signed graph $\dot{G}$ is a symmetric matrix $N(\dot{G}) = D^{\pm}(\dot{G}) - A(\dot{G})$. The Laplacian matrix of $\dot{G}$ is defined $L(\dot{G}) = D(G) - A(\dot{G})$. In the case of $\dot{G} = (G, +)$, we have $L(\dot{G}) = N(\dot{G})$. The adjacency matrix and the Laplacian matrix of a signed graph have been received a great deal of attention in the theory of spectra of signed graphs. The net Laplacian matrix appears very recently and there are few results about it. In [3] Stanić gave some basic results on the spectrum of the net Laplacian matrix of a signed graph and obtained the applications of the net

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Laplacian eigenvalues in control theory [4]. It is well-known that two switching equivalent signed graph have the same spectrum of adjacency matrices and Laplacian matrices. However, the net Laplacian spectra of two switching equivalent signed graphs are different. For example, $(C_4, +)$ and $(C_4, -)$, which are switching equivalent, but have distinct net Laplacian spectra.

As $N(\dot{G})$ is a symmetric matrix, its eigenvalues are real number and denote them by $\mu_1, \mu_2, \cdots, \mu_n$, of course, which include possible repetitions. We also assume that $\mu_1 = 0$ and do not assume any ordering of the remaining ones. If the distinct net Laplacian eigenvalues of $\dot{G}$ are $\mu_1, \cdots, \mu_m$ and their multiplicities are $m(\mu_1), \cdots, m(\mu_m)$, respectively, then we use $Spec(\dot{G}) = \{\mu_1^{m(\mu_1)}, \cdots, \mu_m^{m(\mu_m)}\}$ to denote the net Laplacian spectrum of the signed graph $\dot{G}$.

Let G be a (signed) graph and H be a subgraph of G. Then $G\backslash V(H)$ denotes the subgraph of G by removing all vertices of H. Let K_n be the complete graph of order n and $K_{r,s}$ be the complete bipartite graph with parts of size r and s. The matrix $J_{r \times s}$ is an all-one matrix of size $r \times s$.

Let $\dot{K}_n = (K_n, -H_k)$ be a signed complete graph whose negative edges induce a graph H_k of order k, then H_k has no isolated vertex. If $\dot{K}_n = (K_n, +)$ then H_k is empty. In this paper, all signed complete graphs are assumed to contain at least a negative edge and hence $2 \le k \le n$. In [5] Akbar, Dalvandi, Heydari, and Maghasedi investigated the adjacency eigenvalues of signed complete graphs.

In this paper, we investigate the net Laplacian eigenvalues of signed complete graphs. In Section 2 we give some basic results on the net Laplacian eigenvalues of a signed complete graph, we prove that n (resp., $-n$) is a net Laplacian eigenvalue of the signed complete graph with the multiplicity at least t if there are t vertices whose all incident edges are positive (resp., negative). We establish a relationship between the net Laplacian eigenvalues of the signed complete graph $\dot{K}_n$ and the graph consisted by negative edges of $\dot{K}_n$. In Section 3, we characterize signed complete graphs with just two distinct net Laplacian eigenvalues.

2. The eigenvalues of a signed complete graph $\dot{K}_n$

In this section and next section we mention the eigenvalues of $\dot{G}$ are the net Laplacian eigenvalues of $\dot{G}$.

In this section, we give some basic results on the eigenvalues of a signed complete graph $\dot{K}_n$, such as the lower bound for the multiplicity of n (resp. $-n$) as an eigenvalue of $\dot{K}_n$, the relationship between eigenvalues of $\dot{K}_n$ and the graph H_k consisted by its negative edges.

The join of two signed graphs $\dot{G}_1$ and $\dot{G}_2$, denoted by $\dot{G}_1 \nabla^* \dot{G}_2$, is the signed graph obtained from the disjoint union of $\dot{G}_1$ and $\dot{G}_2$ by adding the edges $\{uv : u \in V(G_1), v \in V(G_2)\}$, where the signs of all adding edges are $*$ and $* \in \{+, -\}$. The next lemma gives the eigenvalues of the join $\dot{G}_1 \nabla^* \dot{G}_2$ of two signed graphs.

Lemma 2.1 ([4, Theorem 3]) Let $\dot{G}_1$ and $\dot{G}_2$ be two signed graphs whose eigenvalues are $v_1(\dot{G}_1), \cdots, v_{n_1}(\dot{G}_1) = 0$ and $v_1(\dot{G}_2), \cdots, v_{n_2}(\dot{G}_2) = 0$, respectively. For $* \in \{+, -\}$, we have that the eigenvalues of $\dot{G}_1 \nabla^* \dot{G}_2$ are $*(n_1 + n_2)$, $v_1(\dot{G}_1) * n_2, \cdots, v_{n_1-1}(\dot{G}_1) * n_2, v_1(\dot{G}_2) * n_1, \cdots, v_{n_2-1}(\dot{G}_2) * n_1, 0$.

Theorem 2.2 Let $\dot{K}_n$ be a signed complete graph. If there exist $|S^+(\dot{K}_n)|$ vertices for which all edges attached these vertices are positive. Then $m(n) \ge |S^+(\dot{K}_n)|$.

Proof. Let $t = |S^+(\dot{K}_n)|$ and $\dot{K}_{n-t} = \dot{K}_n \backslash S^+(\dot{K}_n)$. Let $(K_t, +)$ be a positive complete graph with the vertex set $V(K_t) = S^+(\dot{K}_n)$.

We can obtain that $\dot{K}_n = (K_t, +)\nabla^+ \dot{K}_{n-t}$. Let $\lambda_1 = 0, \lambda_2, \cdots, \lambda_{n-t}$ be the eigenvalues of $\dot{K}_{n-t}$. By Lemma 2.1 and $Spec(K_t, +) = \{0, t^{t-1}\}$, we have $Spec(\dot{K}_n) = \{0, n^t\} \cup \{\lambda_i + t \mid i = 2, \cdots, n - t\}$.

By Theorem 2.2, we have the following result.

Corollary 2.3 Let $\dot{K}_n = (K_n, -H_k)$ be a signed complete graph whose all negative edges induce a graph H_k of order $k(2 \le k \le n)$. Then $m(n) \ge n - k$.

Proof. Since H_k is an induced graph of the negative edges of $(K_n, -H_k)$ and $|V(H_k)| = k$, then $|S^+(\dot{K}_n)| = n - k$. By Theorem 2.2, the result follows.

Apply Theorem 2.2 to the signed graph $-\dot{K}_n$, we have

Corollary 2.4 Let $\dot{K}_n$ be a signed complete graph and there exist $|S^-(\dot{K}_n)|$ vertices whose all incident edges are negative. Then $m(-n) \geq |S^-(\dot{K}_n)|$.

Next we establish a relationship between the spectra of the signed complete graph $\dot{K}_n$ and graph H_k induced by all negative edges of $\dot{K}_n$.

Theorem 2.5 Suppose that $\dot{K}_n = (K_n, -H_k)$ is a signed complete graph whose all negative edges induce a graph H_k of order $k(2 \leq k \leq n)$. Then

(1). $m(n) \geq n - k$.

(2). If the eigenvalues of H_k are $\mu_1 = 0, \mu_2, \cdots, \mu_k$. Then the eigenvalues of $(K_n, -H_k)$ are $\{0, n^{n-k}\} \cup \{n - 2\mu_i | i = 2, \cdots, k\}$.

Proof. Let H_k^c be the complement of H_k with respect to K_k and $K_{n-k} = K_n \backslash V(H_k)$. We have

$$A(K_n, -H_k) = \begin{bmatrix} A(H_k^c) - A(H_k) & J_{k\times(n-k)} \\ J_{(n-k)\times k} & (J-I)_{(n-k)\times(n-k)} \end{bmatrix}_{n\times n},$$

where $A(-H_k) = -A(H_k)$, $A(H_k^c) = J_{k\times k} - I_{k\times k} - A(H_k)$. The vertices of H_k denoted by $v_1, \cdots, v_k$ and corresponding degrees are $d_1, \cdots, d_k$. So the degrees of H_k^c are $k - 1 - d_1, \cdots, k - 1 - d_k$. Let G_1 be the signed complete graph which induced by $V(H_k)$. Then

$$D^{\pm}(G_1) = \begin{bmatrix} k-1-2d_1 & 0 & \cdots & 0 \\ 0 & k-1-2d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & k-1-2d_k \end{bmatrix} = (k-1)I_{k\times k} - 2D^{\pm}(H_k),$$

and $A(G_1) = A(-H_k) + A(H_k^c) = -A(H_k) + J_{k\times k} - I_{k\times k} - A(H_k) = J_{k\times k} - I_{k\times k} - 2A(H_k)$. Thus we have

$$\begin{aligned} N(G_1) &= D^{\pm}(G_1) - A(G_1) \\ &= (k-1)I_{k\times k} - 2D^{\pm}(H_k) - [J_{k\times k} - I_{k\times k} - 2A(H_k)] \\ &= kI_{k\times k} - 2[D^{\pm}(H_k) - A(H_k)] - J_{k\times k} \\ &= kI_{k\times k} - J_{k\times k} - 2N(H_k). \end{aligned}$$

Let $\beta_1 = j_k = (1, 1, \cdots, 1), \beta_2, \cdots, \beta_k$ be the mutual orthogonal eigenvectors of $N(H_k)$ and associated eigenvalues $\mu_1 = 0, \mu_2, \cdots, \mu_k$, respectively. Since $N(G_1)\beta_i = kI_{k\times k}\beta_i - J_{k\times k}\beta_i - 2N(H_k)\beta_i = (k - 2\mu_i)\beta_i$, $(i = 2, \cdots, k)$, we have $Spec(G_1) = \{0\} \cup \{k - 2\mu_i \mid i = 2, \cdots, k\}$.

If $k = n$, then $(K_n, -H_k) = (K_n, -H_n) = G_1$. Hence, $Spec((K_n, -H_k)) = \{0\} \cup \{n - 2\mu_i \mid i = 2, \cdots, k\}$.

If $2 \leq k < n$. Since $Spec(K_{n-k}, +) = \{0, (n-k)^{n-k-1}\}$ and $(K_n, -H_k) = G_1 \nabla^+ (K_{n-k}, +)$, it follows from Lemma 2.1 that

$$Spec(K_n,-H_k) = \{n,(n-k+k)^{(n-k-1)},0\} \cup \{k-2\mu_i+n-k \mid i=2,\cdots,k\}$$

$$= \{n^{n-k},0\} \cup \{n-2\mu_i \mid i=2,\cdots,k\},$$

This completes the proof.

Remark 2.6 Let $\dot{K}_n = (K_n, -H_n)$ be a signed complete graph whose negative edges induce a graph H_n. If the eigenvalues of H_n^c are $\mu_1' = 0,\ \mu_2' \le \cdots \le \mu_n'$. By Theorem 2.5 we conclude that the eigenvalues of $(K_n, -H_n)$ are 0, $2\mu_2' - n, \cdots, 2\mu_n' - n$.

Note that if $\dot{K}_n = (K_n, -H_k)$ is a signed complete graph and H_k or H_k^c is a graph of order k, then the spectrum of $\dot{K}_n$ can be obtained by Theorem 2.5. So we have the following results.

Corollary 2.7 Let $\dot{K}_n = (K_n, -H_k)$ be a signed complete graph and s be the number of connected component of H_k. Then the multiplicity of eigenvalue n of $(K_n, -H_k)$ is $s + n - k - 1$.

Corollary 2.8 Let $\dot{K}_n = (K_n, -H_k)$ be a signed complete graph and s be the number of connected component of H_k^c. Then the multiplicity of eigenvalue $-n$ of $(K_n, -H_k)$ is $s - 1$.

At the end of this section, we give the multiplicities of eigenvalue 0 of $(K_n, -H_k)$, which is also a direct result of Theorem 2.5.

Corollary 2.9 Let the multiplicity of eigenvalue 0 of $(K_n, -H_k)$ be $m(0)$ and the multiplicity of eigenvalue $\frac{n}{2}$ of H_k be $m_{H_k}(\frac{n}{2})$. Then $m(0) = 1 + m_{H_k}(\frac{n}{2})$. Moreover, $1 \le m(0) \le n - 1$, $m(0) = 1$ if and only if $m_{H_k}(\frac{n}{2}) = 0$ and $m(0) = n - 1$ if and only if $H_k = K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$.

3. $\dot{K}_n$ with two distinct net Laplacian eigenvalues

In this section, by the statement "$\dot{G}$ has k eigenvalues" we mean that $\dot{G}$ has exactly k distinct eigenvalues.

The signed graphs with two adjacency and Laplacian eigenvalues have been studied in [6-7] and [8], respectively. In this section, we would like to characterize the signed complete graphs with two net Laplacian eigenvalues.

By Theorem 2.5, if the signed complete graph $(K_n, -H_k)$ has two eigenvalues, then H_k has at most three eigenvalues. Furthermore, it is well known that a connected graph has one eigenvalue if and only if it is an empty graph. It has two eigenvalues if and only if it is a complete graph. Additionally, a connected graph with just three distinct Laplacian eigenvalues has been studied in [9].

A graph G has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have μ common neighbors. G has constant μ and $\bar{\mu}$ if G has constant $\mu = \mu(G)$, and its complement G^c has constant $\bar{\mu} = \mu(G^c)$. Next lemma gives basic facts about the connected graph with constant μ and $\bar{\mu}$.

Lemma 3.1 ([9, Theorem 2.1]) Let G be a connect graph on n vertices. Then G has constant μ and $\bar{\mu}$ if and only if G has three distinct Laplace eigenvalues 0, θ_1 and θ_2. If so then only two vertex degrees k_1 and k_2 can occur, and $\theta_1 + \theta_2 = k_1 + k_2 + 1 = \mu + n - \bar{\mu}$ and $\theta_1\theta_2 = k_1k_2 + \mu = \mu_n$.

Moreover, we also need the following Lemmas to make further discussion.

Lemma 3.2 ([10, Proposition 7.3.3]) Let G be a graph with n vertices. Then $v_1(G) \le n$, where $v_1(G)$ is the largest Laplacian eigenvalues of G.

Lemma 3.3 If $\dot{G}$ is a signed graph with n vertices. Then $Spec(\dot{G}) = \{0, n^{n-1}\}$ if and only if $\dot{G} = (K_n, +)$.

Proof. If $\dot{G} = (K_n, +)$ then is clear that $Spec(\dot{G}) = \{0, n^{n-1}\}$. If $Spec(\dot{G}) = \{0, n^{n-1}\}$ then $\Sigma_{i=1}^{n} d_i^{\pm} = n(n-1)$. Hence $d_i^{\pm} = n - 1$ and $\dot{G} = (K_n, +)$.

Theorem 3.4 Let $\dot{K}_n$ be a signed complete graph in which the all negative edges induce a graph H_k of order k ($2 \le k \le n$). Then $\dot{K}_n$ has two eigenvalues if and only if H_k is one of the following five cases:

(1). $k = n$ and $H_n = K_n$,

(2). $k = \frac{n}{2}$ and $H_{\frac{n}{2}} = K_{\frac{n}{2}}$,

(3). $k = n$ and $H_n = K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$,

(4). $k = n$ and H_n is a strongly regular graph with parameter $(n, \frac{2n-h}{4} - \frac{1}{2}, \frac{n-h}{4} - 1, \frac{n-h}{4})$, where h is non-zero eigenvalue of $\dot{K}_n$.

(5). $k = n$ and H_n is a connected graph with two distinct degrees and H_n has three eigenvalues 0, $\frac{n}{2}$ and $\frac{n-h}{2}$,where h is non-zero eigenvalue of $\dot{K}_n$.

Proof. Recall that $S^+(\dot{K}_n)$ is set of vertices of $\dot{K}_n$ whose all incident edges are positive, and we divide three cases in term of the number $|S^+(\dot{K}_n)|$.

Case 1: $|S^+(\dot{K}_n)| \geq 2$.

In this case, we have that $2 \leq k \leq n - 2$ and $m(n) \geq |S^+(\dot{K}_n)| \geq 2$ by Theorem 2.2. Then n is an eigenvalue of $\dot{K}_n$, and two distinct eigenvalues of $\dot{K}_n$ are 0, n.

Let $\mu_1 = 0, \mu_2, \cdots, \mu_k (k \leq n - 2)$ be the eigenvalues of H_k. By Theorem 2.5 we have $Spec((K_n, -H_k)) = \{0, n^{n-k}\} \cup \{n - 2\mu_i | i = 2, \cdots, k\}$. We obtain that $\mu_i \in \{0, \frac{n}{2}\}$ $(i = 2, \cdots, k)$.

Since H_k has at least an edge, the largest eigenvalue of H_k is positive. Without loss of generality, let $Spec(H_k) = \{0^s, (\frac{n}{2})^{k-s}\}$, then s is the number of the component of H_k.

Since $k \leq n - 2$. We claim that H_k is connected. Otherwise $s \geq 2$. Clearly, H_k must have a connected component, say G_1, whose order is less than $\frac{n}{2}$. By Lemma 3.2 $v_1(G_1) < \frac{n}{2}$, which leads to a contradiction with the spectrum of H_k. It means that $s = 1$ and $Spec(H_k) = \{0, (\frac{n}{2})^{k-1}\}$, then $k = \frac{n}{2}$ and $H_{\frac{n}{2}} = K_{\frac{n}{2}}$.

Case 2: $|S^+(\dot{K}_n)| = 1$.

In this case, the signed graph $\dot{K}_n$ has exactly one vertex v_1 whose all incident edges are positive and $k = n - 1$. Then $(K_n, -H_k) = v_1 \nabla^+ (\dot{K}_n \backslash v_1)$. By Lemma 2.1, n is an eigenvalue of graph $(K_n, -H_k)$. Hence, the two distinct eigenvalues of $\dot{K}_n$ are 0 and n .

Let $\mu_1 = 0, \mu_2, \cdots, \mu_{n-1}$ be the eigenvalues of H_{n-1}. By Theorem 2.5, we have $Spec(K_n, -H_{n-1}) = \{0, n, n - 2\mu_2, \cdots, n - 2\mu_{n-1}\}$ and $Spec(\dot{K}_n \backslash v_1) = \{0, n - 1 - 2\mu_2, \cdots, n - 1 - 2\mu_{n-1}\}$ for the negative edges of $\dot{K}_n \backslash v_1$ induce the graph H_{n-1}. Thus $n - 2\mu_i = 0$ or n $(i = 2, \cdots, n - 1)$. So the eigenvalues of $\dot{K}_n \backslash v_1$ are 0 (which is simple), possible -1 and $n - 1$.

If $\dot{K}_n \backslash v_1$ has two eigenvalues 0 and $n - 1$, then $Spec(\dot{K}_n \backslash v_1) = \{0, (n - 1)^{n-2}\}$, and $\dot{K}_n \backslash v_1 = (K_{n-1}, +)$ by Lemma 3.3. Then $|S^+(\dot{K}_n)| = n$. Which is a contradiction with $|S^+(\dot{K}_n)| = 1$.

If $\dot{K}_n \backslash v_1$ has two eigenvalues 0 and -1, then $Spec(\dot{K}_n \backslash v_1) = \{0, (-1)^{n-2}\}$ and $\mu_2 = \cdots = \mu_{n-1} = \frac{n}{2}$. So $n - 2 + 1 = \frac{n}{2}$. Hence $n = 2$ and $H_2 = K_2$, which contradicts to $|S^+(\dot{K}_n)| = 1$.

If $\dot{K}_n \backslash v_1$ has three eigenvalues 0, $n - 1$ and -1, then $\mu_i \in \{0, \frac{n}{2}\}$ $(i = 2, \cdots, n - 1)$. Let the number of connected component of H_{n-1} be s then $1 \leq s \leq \frac{n}{2}$ for H_{n-1} has no isolated vertex. If $s = 1$, then $Spec(H_{n-1}) = \{0, (\frac{n}{2})^{n-2}\}$. Hence $n - 1 = \frac{n}{2}$ and $n = 2$, a contradiction. If $s \geq 2$ then $Spec(H_{n-1}) = \{0^s, (\frac{n}{2})^{n-s-1}\}$. Then we can find a connected component (say F) of H_{n-1} such that the order of F is less than $\frac{n}{2}$. By Lemma 3.2, $v_1(F) < \frac{n}{2}$, which leads to a contradiction.

Case 3: $|S^+(\dot{K}_n)| = 0$.

In this case, there is no vertex with all incident edges are positive, and $k = n$. Let $\mu_1 = 0, \mu_2, \cdots, \mu_n$ be the eigenvalues of H_n. By Theorem 2.5, $Spec(K_n, -H_n) = \{0, n - 2\mu_2, \cdots, n - 2\mu_n\}$. Since $(K_n, -H_n)$ has two eigenvalues, then we have $\mu_i \in \{\frac{n}{2}, b\}$ $(i = 2, \cdot\cdot\cdot, n)$, thus $b \neq \frac{n}{2}$ and H_n has at most three distinct eigenvalues $0, \frac{n}{2}, b$, hence $\frac{n}{2}, b$, are integers as they are rational eigenvalues of H_n. In order to make sure that $(K_n, -H_n)$ has two eigenvalues, we discuss the conditions with two parts as follows:

If $\mu_i = b$ for all $i \in \{2, \cdots, n\}$, that is $Spec(H_n) = \{0, b^{n-1}\}$. Then it is easy to know that $b = n$ and $H_n = K_n$, by Lemma 3.3.

If $\mu_{i1} = \mu_{i2} = \cdots = \mu_{i(j-1)} = \frac{n}{2}$ and $\mu_{ij} = \mu_{i(j+1)} = \cdots = \mu_{i(n-1)} = b$, then $Spec(H_n) = \{0, (\frac{n}{2})^{j-1}, (b)^{n-j}\}$.

If $b = 0$, then $Spec(H_n) = \{0^{n-j+1}, (\frac{n}{2})^{j-1}\}$. Let H_n have s connected components ($1 \le s \le \frac{n}{2}$). If $s = 1$, then $Spec(H_n) = \{0, (\frac{n}{2})^{n-1}\}$ and hence $\frac{n}{2} = n$. Thus $n = 0$, which leads to a contradiction. If $s = 2$, then $Spec(H_n) = \{0^2, (\frac{n}{2})^{n-2}\}$. We can find that $H_n = K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$. If $s \ge 3$, then $Spec(H_n) = \{0^s, (\frac{n}{2})^{n-s}\}$. Thus H_n has a connected component G_1 with the order less than $\frac{n}{2}$. By lemma 3.2, $v_1(G_1) < \frac{n}{2}$, which leads to a contradiction with the spectrum of H_n.

If $b \neq 0$, then $Spec(H_n) = \{0, (\frac{n}{2})^{j-1}, (b)^{n-j}\}$ ($b \neq \frac{n}{2}$). In other words, H_n is a connected graph with exactly three distinct eigenvalues. By Theorem 2.5, then $Spec(K_n, -H_n) = \{0^j, (n-2b)^{n-j}\}$. Let $h = n - 2b$, then $b = \frac{n-h}{2}$ and $Spec(H_n) = \{0, (\frac{n}{2})^{j-1}, (\frac{n-h}{2})^{n-j}\}$. Hence, H_n has three eigenvalues $0, \frac{n}{2}, \frac{n-h}{2}$. By Lemma 3.1, we have $\frac{n}{2} + \frac{n-h}{2} = k_1 + k_2 + 1 = \mu + n - \bar{\mu}$, and $\frac{n}{2} \cdot \frac{n-h}{2} = k_1 k_2 + \mu = \mu n$.

If H_n is a d-regular, then $k_1 = k_2 = d$. By some simple computations we obtain that $\mu = \frac{n-h}{4}$, $d = \frac{2n-h}{4} - \frac{1}{2}$ and $\bar{\mu} = \frac{n+h}{4}$.

Let N_v and $\bar{N}_v$ denote the neighbors of vertex v in graph H_n and H_n^c, respectively. Hence, we have that $|\bar{N}_{v_i} \cap \bar{N}_{v_j}| = \bar{\mu}$, for $v_i v_j \in E(H_n)$. Then $|N_{v_i} \cup N_{v_j}| = n - \bar{\mu}$, for $v_i v_j \in E(H_n)$. Since $|N_{v_i} \cup N_{v_j}| = d_i + d_j - |N_{v_i} \cap N_{v_j}|$, then we get that $|N_{v_i} \cap N_{v_j}| = 2d - (n - \bar{\mu}) = \frac{n-h}{4} - 1$, for $v_i v_j \in E(H_n)$. It follows that H_n is a strongly regular graph with parameter $(n, \frac{2n-h}{4} - \frac{1}{2}, \frac{n-h}{4} - 1, \frac{n-h}{4})$.

If H_n is non-regular, then H_n has exactly two distinct vertex degrees and three distinct eigenvalues. Obviously, H_n has constant $\mu = \frac{n-h}{4}$ and $\bar{\mu} = \frac{n+h}{4}$. The result follows.

Remark 3.5 We can find some examples of Theorem 3.3 (4) and (5) in Table 1 of [9]. Furthermore, the Peterson graph (srg(10, 3, 0, 1)) and its complement (srg(10, 6, 3, 4)), the line graph of $K_{4,4}$ (srg(16, 6, 3, 2)) and its complement (srg(16, 9, 4, 6)) are the graph of Theorem 3.3 (4).

Table 1. The spectra of completed signed graph in Theorem 3.4

H_k	Spectra of H_k	Spectra of $\dot{K}_n = (K_n, -H_k)$
K_n	$\{0, n^{n-1}\}$	$\{0, (-n)^{n-1}\}$
$K_{\frac{n}{2}}$	$\{0, \frac{n}{2}^{\frac{n}{2}-1}\}$	$\{0^{\frac{n}{2}}, n^{\frac{n}{2}}\}$
$K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$	$\{0^2, \frac{n}{2}^{n-2}\}$	$\{0^{n-1}, n\}$
$K_{\frac{n}{2}, \frac{n}{2}}$	$\{0, n, (\frac{n}{2})^{n-2}\}$	$\{0^{n-1}, -n\}$
$K_{\frac{n}{2}} \nabla^+ (K_{\frac{n}{2}})^c$	$\{0, n^{\frac{n}{2}}, \frac{n}{2}^{\frac{n}{2}-1}\}$	$\{0^{\frac{n}{2}}, (-n)^{\frac{n}{2}}\}$

We give an example of Theorem 3.4 (4) as shown in Figure 1 and an example of Theorem 3.3 (5) as shown in Figure 2, respectively. In the following figures, the solid line represents the positive edge and the dotted line represents the negative edge.

By the proofs of Theorem 3.4 (4) and (5), it is easy to know that n is not an eigenvalues of $\dot{K}_n = (K_n, -H_n)$. Furthermore, we would like to characterize the graph H_k such that n or $-n$ is an eigenvalues of $(K_n, -H_k)$.

Corollary 3.6 The signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues 0 and n if and only if $H_k = K_{\frac{n}{2}}$ or $(K_{\frac{n}{2}} \cup K_{\frac{n}{2}})$. The signed graph $\Sigma = (K_n, -H_k)$ has two eigenvalues 0 and $-n$ if and only if $H_k = K_n$ or $K_{\frac{n}{2}} \nabla^+ \frac{n}{2} K_1$ or $K_{\frac{n}{2}, \frac{n}{2}}$.

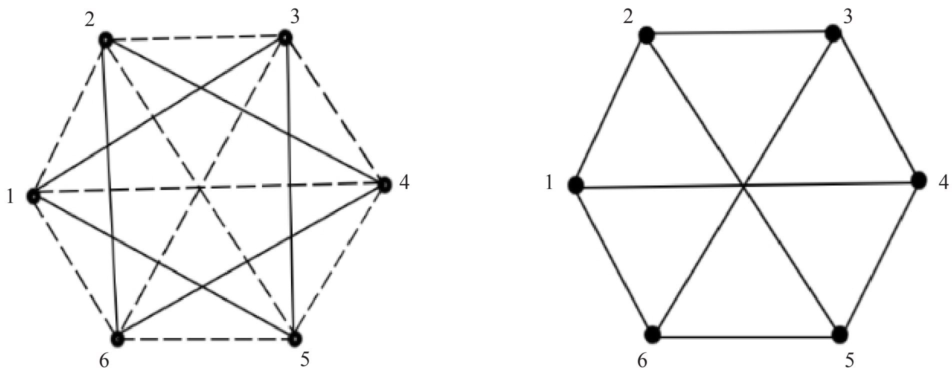

Figure 1. The signed graph $\dot{K}_6$ (the left) and the graph H_6 (the right) induced by its negative edges, $H_6 = K_{3,3}$

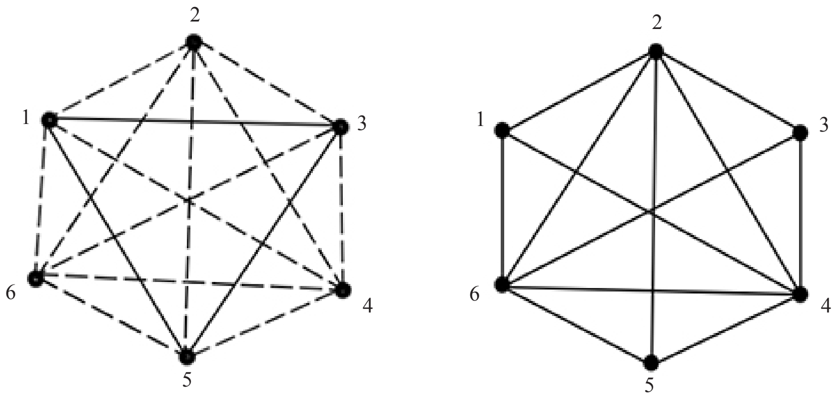

Figure 2. The signed graph $\dot{K}_6$ (the left) and the graph H_6 (the right) induced by its negative edges, H_6 is a non-regular graph with two distinct vertex degrees 5 and 3

Proof. If the signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues 0 and n. By Theorem 3.4, then H_k must be one of $K_{\frac{n}{2}}$ or $K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$. If $H_k = K_{\frac{n}{2}}$ or $K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$, By Table 1, the signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues 0 and n.

If the signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues 0 and $-n$. By $Spec(N(-\dot{K}_n)) = -Spec(N\dot{K}_n))$. Note that if $\dot{K}_n = (K_n, -H_k)$ then $-\dot{K}_n = (K_n, -H_k^c)$, where H_k^c is the complement of H_k in K_n. Thus result follows from $(K_{\frac{n}{2}} \cup K_{\frac{n}{2}})^c = K_{\frac{n}{2}, \frac{n}{2}}$ and $(K_{\frac{n}{2}})^c = K_{\frac{n}{2}} \nabla^{+} \frac{n}{2} K_1$.

If the signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues in which n and $-n$ are not the eigenvalues of $\dot{K}_n$, by Corollary 3.6, then H_k must be a graph in the case of Theorem 3.4 (4) and (5).

By the proof of Theorem 3.4 (3), if $\dot{K}_n$ has two distinct eigenvalues 0 and h then h is an integer and $h = n - 2b$ for some integer b. Moreover, $\frac{n}{2}$ is also a integer. Then n and $n - 2b$ are even. We have characterized signed complete graphs with two distinct eigenvalues 0, n (or 0, $-n$). Next we would like to investigative signed complete graph with two distinct eigenvalues 0, $n - 2i$ for $i \neq 0, n$.

Corollary 3.7 If the signed graph $\dot{K}_n = (K_n, -H_k)$ has two eigenvalues are 0 and $n - 2i$ ($i \neq 0, n$). Then $k = n$ and H_n is one of the following connected graph:

(1) $i = \frac{n \pm 2\sqrt{n-1}}{2}$ and H_n is a strongly regular with parameter $(n, \frac{n+\sqrt{n-1}-1}{2}, \frac{n+2\sqrt{n-1}}{4} - 1, \frac{n+2\sqrt{n-1}}{4})$ or $(n, \frac{n-\sqrt{n-1}-1}{2}, \frac{n-2\sqrt{n-1}}{4} - 1, \frac{n-2\sqrt{n-1}}{4})$;

(2) H_n has two distinct vertex degrees $k_1 = \frac{2i+n-2+2\sqrt{(\frac{n}{2}-i)^2+1-n}}{4}$, $k_2 = \frac{2i+n-2-2\sqrt{(\frac{n}{2}-i)^2+1-n}}{4}$ and with constant $\mu = \frac{i}{2}$ and $\bar{\mu} = \frac{n-i}{2}$.

Proof. Since $n - 2i \neq \pm n$, by Theorem 2.5, we can obtain that $k = n$ and the multiplicities of eigenvalues 0 of H_n is simple and H_n is a connected graph with distinct eigenvalues 0, i and $\frac{n}{2}$. Hence H_n is a connected graph. By Theorem 3.4, H_n is either a strongly regular with parameter $(n, \frac{n+2i-2}{4}, \frac{i}{2}-1, \frac{i}{2})$ or H_n has three eigenvalues and two distinct degrees. Let k_1 and k_2 be the only two distinct degrees of H_n. By Lemma 3.1, we have $\mu = \frac{i}{2}$, $\bar{\mu} = \frac{n-i}{2}$, $\frac{n}{2} + i = k_1 + k_2 + 1$, $\frac{n}{2} \cdot i = k_1 k_2 + \mu = \mu n$. By a simple computation we obtain that $k_1 = \frac{2i+n-2+2\sqrt{(\frac{n}{2}-i)^2+1-n}}{4}$, $k_2 = \frac{2i+n-2-2\sqrt{(\frac{n}{2}-i)^2+1-n}}{4}$.

If H_n is a strongly regular graph, then $(\frac{n}{2} - i)^2 + 1 - n = 0$, hence $i = \frac{n \pm 2\sqrt{n-1}}{2}$. Thus, H_n is a strongly regular with parameter $(n, \frac{n+\sqrt{n-1}-1}{2}, \frac{n+2\sqrt{n-1}}{4}-1, \frac{n+2\sqrt{n-1}}{4})$ or $(n, \frac{n-\sqrt{n-1}-1}{2}, \frac{n-2\sqrt{n-1}}{4}-1, \frac{n-2\sqrt{n-1}}{4})$.

If H_n is a non-regular graph with two distinct vertex degrees, then the two distinct vertex degrees k_1 and k_2 are as shown in the previous proof and H_n has constant $\mu = \frac{i}{2}$ and $\bar{\mu} = \frac{n-i}{2}$. The result follows.

Remark 3.8 We make a simple discussion on H_n in Corollary 3.7 where H_n is a strongly regular with parameter $(n, \frac{n+\sqrt{n-1}-1}{2}, \frac{n+2\sqrt{n-1}}{4}-1, \frac{n+2\sqrt{n-1}}{4})$ or $(n, \frac{n-\sqrt{n-1}-1}{2}, \frac{n-2\sqrt{n-1}}{4}-1, \frac{n-2\sqrt{n-1}}{4})$. Since these parameters should be integer, we would like find some examples to find its existence. When $n = 10$, we find that H_n is a strongly regular with parameter (10, 6, 3, 4) or (10, 3, 0, 1) (Peterson graph). Moreover, when $n = 26$, we find that H_n is a strongly regular with parameter (26, 10, 3, 4) or (26, 15, 8, 9).

Conflict of interest statement

The authors declared that they have no conflicts of interest to this work.

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