Research Article

Structural Identifiability of Feedback Systems with Nonlinear Adulterating

Nikolay Karabutov

Department of Problems Control, MIREA-Russian Technological University, Moscow, Russia
E-mail: kn22@yandex.ru

Received: 24 November 2021; Revised: 7 June 2022; Accepted: 14 June 2022

Abstract: We consider the structural identifiability estimation problem of Nonlinear Feedback Systems (NFS) with nonlinear adulterating. The problem of NFS Structural Identifiability (NFSI) has not been studied. Studying the NFSI problem guarantees the possibility of nonlinearity identification under uncertainty. Two cases of the adulterate influence are analyzed: (i) additive effect of feedback nonlinearity on the nonlinearity in the straight chain (ii) nonlinearity argument nonlinear adulterating in straight chain of the system. The basis for the identifiability estimation is: (a) the Geometric Frameworks (GF) analysis method reflecting properties of the nonlinear system; (b) structural frequency diagrams, and (c) the hierarchical immersion method. We obtain conditions of identifiability, unidentifiability and local identifiability for NFS. The influence of the nonlinear argument is analyzed on the system identifiability. We propose conditions for system unidentifiability verifying with a nonlinear argument of the function. The influence of the nonlinear argument is analyzed for estimating the system identifiability. Results are applicable in the synthesis of nonlinear control systems.

Keywords: structure, identifiability, nonlinear system of the second class, nonlinear adulterating, framework

MSC: 93C10, 93C95

1. Introduction

The identifiability problem of dynamic systems is the relevant areas of study. Foundational results are obtained on the Parametric Identification (PI) of systems [1-3]. The main direction PI is the analysis of a priori identifiability. Parametric identification methods are the basis of a priori Identifiability (AI). As a rule, the model structure sets a priori. Therefore, identification AI and Structural Identifiability (SI), which are often used in publications, are incorrect. AI methods are transferred to the estimation problem of the nonlinear systems structural identifiability. This approach eliminates the estimation problem of the nonlinearity structure.

Approximation methods are applied for identifiability problem solution [4-10]. In [4], the approach based on the analysis of the system output sensitivity is used to study identifiability. The effectiveness of this approach shows when studying the identifiability of the system parameter combination. In [10], local Parametric Identifiability (IPI) condition is obtained for various types of experimental data. A critical analysis of approaches used to estimate the
identifiability of biological models is given in [8]. Methods for identifiability estimating nonlinear systems are based on the AI approaches. The requirements on data used for the PI task solution are given in [6]. The study of various types of identifiability (global, local, structural, and practical) are described in many works [4, 6, etc.]. Two main approaches (see [11]) are applied to estimate the structural identifiability of nonlinear dynamical systems. A priori methods (the first approach) only use the model definition (authors understand the definition as the structure choice). Posterior procedures (the second approach) apply the available data to find unidentifiable parameters. The basis of many a priori algorithms is the Lie group theory. A posteriori method uses the available data to perform identifiability analysis. They infer structural identifiability based on the adequacy of the model to experimental data (this is the stage of parametric estimation).

The method proposed to analyze the identifiability of closed-loop systems with nonlinear feedback in [12]. The method is based on the harmonic balance and the double Fourier Series (FS). FS approximates the nonlinearity. It shows that a multiplicative periodic reference signal enhances the identifiability of the system. Parametric identifiability is the main direction in the analysis of linear closed-loop systems (see for example [13-15]). PI guarantees the identifiability of the system. The structural identifiability problem is not solved by parametric methods under uncertainty. The parametric approach can only be applied to estimating nonlinearity in a specific area. The properties changing of the input may give different identification results. This is the main problem of the parametric approach. Works on PI problems do not consider the estimation problem of the system structure. Therefore, the concept of parametric identifiability does not reflect the essence of the structural identifiability problem. However, this terminology is actively using in the task of assessing identifiability. The NFS structural identifiability analysis requires the use of new approaches.

So, the system identifiability understands as the possibility to estimate its parameters (parametric identification). PI methods based on the information matrix nondegeneracy estimation used. Similar results are obtained in the parametric estimation theory. They are based on checking the Constant Excitation (CE) condition for the input and output of the system. As a rule, the model structure sets prior, and the essence of the structural identifiability is not always clear. The identifiability of nonlinear system is reduced to the parametric problem. PI paradigm is not always applicable to a nonlinear systems class. These systems work under uncertainty. The task becomes more complicated if the system has several nonlinearities [16]. Therefore, the structural identification problem solving (structural identifiability) is an urgent task. Parametric identifiability methods do not provide a solution to the SI problem. This is fairly for evaluating the NFS structural identifiability. In [16, 17], the SI methodology proposes for nonlinear dynamical systems. It is based on the GF analysis under uncertainty. The problem relates to the structural identification task closely. The structural identification problem solution is the basis for the SI of nonlinear systems under uncertainty. The approach is based on the GF properties analysis. Using GF allows you to solve the NFS structural identifiability problem. The GF approach is applicable for NFS with nonlinear adulteration. Nonlinear adulterating is a NFS special case.

Next, we give the SI problem solution for the feedback systems with nonlinear adulterating. We do not consider the identifiability of the system linear part. This problem has been studied well (see for example [3]). We consider two classes of NFS. The first class is NFS with nonlinearity in the return loop and forward contour. The second class is NFS with a nonlinear argument of the system main nonlinearity. We give the main results for the GF analysis. The analysis allows you to decide about the NFS structural identifiability.

2. Problem statement

Consider the system

$$\dot{X} = AX + B_u \Phi(Y) + B_\mu u,$$

$$Y = C^T X,$$  \hspace{1cm} (1)

where $u \in \mathbb{R}$, $Y \in \mathbb{R}^g$ are input and output, $X \in \mathbb{R}^q$ is the state vector, $A \in \mathbb{R}^{q \times q}$, $B_u \in \mathbb{R}^{q \times 1}$, $B_\mu \in \mathbb{R}^{q \times k}$, $C \in \mathbb{R}^{g \times q}$ are matrices of corresponding dimensions, $\Phi(X) \in \mathbb{R}^k$ is vector nonlinear function. $A$ is Hurwitz matrix.

The nonlinear function $\varphi(\zeta) \in \Phi$ is smooth and satisfies the condition

Contemporary Mathematics 258 | Nikolay Karabutov
where \( \zeta \in \mathbb{R} \) is the input of a nonlinear element. \( \zeta \) is a linear combination of state variables. Information set \( S \) guarantees \( X \) are the left and right fragments \( \{ \phi_1, \phi_2 \} \).

\[
X \in F_p = \{ \gamma_1, \xi_1 \leq \phi_1(\xi) \leq \gamma_2, \xi_1 \neq 0, \phi_1(0) = 0, \gamma_1 \geq 0, \gamma_2 < \infty \},
\]

We assume that some state variable \( x \), may depend on the difference in the outputs of multiple nonlinearities. Problem: analyze the set \( I \) and estimate the structural identifiability of the system (1).

### 3. Preliminary concepts

Let \( \phi(\cdot) = \Phi(X) \), \( y \in \mathbb{R} \), \( x \in \mathbb{R} \). Apply the approach to the GF construction [3, 17] and synthesize a framework \( S_y \), \( S_y \) described by the function \( f_{y} : y \rightarrow e \), \( e \in \mathbb{R} \). \( e \) reflects nonlinear processes in the system.

**Assumption B1.** The input \( u(t) \) is constantly excited at the interval \( J \). B2. The input \( u(t) \) ensures an informative framework \( S_y \). B3. Let the framework \( S_y \) be closed, and the area \( S_y \) is not zero. Denote height \( S_y \) as \( h(S_y) \), where height is the distance between two points opposite sides of the framework \( S_y \). If \( u(t) \) satisfies B1 and B2 conditions, then the input \( u(t) \) is representative.

If the framework \( S_y \) satisfies the assumption B3, then \( S_y \) calls \( h \)-identifiable [17]. Consider the framework \( S_y \). Introduce notations: \( D_y = \text{dom}(S_y) \) is definitional domain \( S_y \), \( D_y = D_y = \max y(t) - \min y(t) \) is diameter \( D_y \). Let \( u(t) \in U \), where \( U \) is the acceptable set of inputs for the system (1). The set \( U \) contains representative inputs.

**Definition 1** The input \( u(t) \in U \) is the S-synchronizing system (1) if the domain \( D_y \) has a maximum diameter \( D_y \) on set \( \{ y(t), t \in J \} \).

Consider a reference framework \( S_y^{ref} \). \( S_y^{ref} \) is the framework \( S_y \) reflecting all properties of the function \( \phi(y) \). Designate by the diameter \( D_y(S_y^{ref}) \) as \( D_y^{ref} \). \( D_y^{ref} \) exists if the input the system (1) is S-synchronizing.

Definitions 1 show if \( S_y \equiv S_y^{ref} \), then \( |D_y - D_y^{ref}| \leq \varepsilon_y \), where \( \varepsilon_y \geq 0 \). \( \equiv \) is the proximity sign. Properties \( U_S \)

\[
|D_y(S_y(u(t)|_{u(t)}) - D_y^{ref})| \leq \varepsilon_y.
\]

The fulfillment of condition \( d_{h,y} = \max_{u_{y}} D_y \) guarantees \( h \)-identifiability of the system. The conditions for \( h \)-identifiability have the form

\[
|D_y(S_y(u(t)|_{u(t)}) - d_{h,y})| \leq \varepsilon_y.
\]

If \( u_{y}(t) \not\in U \), then the condition of the unidentifiable or insignificant framework \( NS_y \not\equiv S_y \) is

\[
|D_y(S_y(u(t)|_{u(t)}) - d_{h,y})| \geq \varepsilon_y.
\]

Let the input \( u_{y}(t) \) synchronize the set \( D_y \). If \( u(t) \) is S-synchronizing, then we write \( u_{y}(t) \in S \). Note that the finite set \( \{ u_{y}(t) \} \in S \) exists for system (1). The choice of optimal \( u_{y}(t) \) depends on \( d_{h,y} \) and (5). Provision of condition (5) is one of the SI conditions for the system (1).

Let \( S_y = F_{y}^{l} \cup F_{y}^{r} \), where \( F_{y}^{l} \), \( F_{y}^{r} \) are the left and right fragments \( S_y \). Secants for \( F_{y}^{l} \), \( F_{y}^{r} \)
\[ \gamma'_s = a' y + b', \quad \gamma''_s = a y + b'. \] (7)

**Definition 2** If \( S_o \) is \( h \)-identified and conditions \( \| \alpha' - \alpha' \| \leq \delta_h \), (4) satisfies, then the \( S_o \) or the system (1) is structurally identified or \( h_o \)-identifiable.

Let \( c_s \) is the centre of the framework \( S_o \) on the set \( J_o = \{ y(t) \} \), \( c_h \) is the centre of the area \( D_h \).

**Theorem 1** [3, 17]. Let the set \( U_j \) of synchronizing inputs \( u(t) \) consider for the system (1) and (i) \( e \geq 0 \) exists such that \( |c_S - c_h| \leq e \); (ii) \( \| \alpha' - \alpha' \| \leq \delta_h \), where \( \alpha', \alpha' \) are coefficients of secants (7) for \( (F'_o, F''_o) \subset S_o \). Then the system (1) is \( h_o \)-identifiable, and the input \( u_h(t) \in S \).

We do not consider the method of constructing \( S_o \). It is described in [3, 17]. Examples of the framework \( S_o \) obtain is given in section 4. Let the hypothetical framework \( S_o \) (the framework \( S''_o \)) the have the diameter \( d_h \).

**Definition 3** The framework \( S_o \) has \( d_{h,x} \)-optimality property on the set \( U_k \) if \( e_k > 0 \) exists such that \( |d_{h,x} \leq d_{h,x} | \) \( \forall t \in \mathbb{R} \) \( U_k \). The \( S_{o,h}(u_{h,k}) \)-framework is locally structurally identifiable on the set \( U_k \) if has the \( d_{h,x} \)-optimality property.

There may be some subset \( \{ u_{h,i}(t) \} \subset U_k \subset U(\geq 1) \) whose elements have the S-synchronizability property. Each \( u_{h,i}(t) \) corresponds to the framework \( S_{o,h}(u_{h,i}) \) with the diameter \( D_{j,s} \) of the definition area \( D_{j,s} \).

### 4. Additive adulterate

Consider the system (1) with feedback (self-oscillation system)

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
S_\phi \colon &:
\begin{cases}
\dot{x}_3 = c_1 (\phi_1(u) - \phi_2(x_4)) + c_2 x_4, \\
\dot{x}_4 = \phi_1(u) - \phi_2(x_4), \\
y = x_4,
\end{cases}
\end{align*}
\] (8)

where \( u = -k_1 x_2 - k_2 x_3 \) is input, \( y \in \mathbb{R} \) is the output, \( c_1, c_2, k_1, k_2 \) are system parameters, \( \phi_1(x) = c \sign(x) \) is sign function, \( c > 0 \),

\[
\phi_2(x_4) = \begin{cases}
c, & \text{if } x_4 > b, \\
0, & \text{if } x_4 \in [-b, b], \\
-c, & \text{if } x_4 < -b,
\end{cases}
\]

We see the output of function \( \phi_2(u) \) adulterates by function \( \phi_2(x_4) \). (8) is the nonlinear system of the second class [18]. Harmonic linearization [18] does not apply to \( \phi_1(u), \phi_2(x_4) \). Find structural identifiability conditions for (8) using the results of section 3.

Determine the system \( S_o \) solution under the following initial conditions and parameters: \( x_1(0) = 2, x_2(0) = 1, x_3(0) = 0, x_4(0) = 1, c = 2, b = 1.5, c_1 = 2, c_2 = 2, k_1 = 1, k_2 = 1.5 \). Form the set (3). The system (8) phase portrait shows in Figure 1.

We see (Figure 1) that oscillations (variable \( x_3 \)) emerge at the system input, and limited oscillations (variable \( x_4 \)) occur at the output. They are the result of the action of nonlinearity.

Presented frameworks show processes do not have distortions (the left part of the portrait corresponds to the right). Therefore, the system input is S-synchronizing (see section 3). The S-synchronizability condition is a prerequisite for the analysis of the system structural identifiability. We see (Figure 1) that the conclusion cannot make about the structural identifiability of the system based on the mapping \( \gamma_{n,x_2} : x_1 \to x_2 \).

Apply the hierarchical immersion method [17] for further analysis. It gives mappings that reflect the nonlinear system (8) state.

Contemporary Mathematics 260 | Nikolay Karabutov
**Definition 4** System (8) is locally identifiable (LI) on the set \( \{ x_1, x_2 \} \) if the framework \( S_{x_1, x_2} \) described by the mapping \( \gamma_{x_1, x_2} : x_1 \rightarrow x_2 \) has the maximum diameter of the definition domain.

Figure 1 shows that the framework \( S_{x_1, x_2} \) is LI. The analysis \( S_{u, x} \) does not give the nonlinearity Structural Parameters (SP) estimate from adulteration. Go to a specific structural space. The mathematical model design is based on the relationship between variables.

Synthesize models on sets \( \{ u, \dot{x}_4 \} \) and \( \{ x_1, \dot{x}_4 \} \)

\[
\hat{x}_{4, u} = 2.234u - 0.004, \quad \hat{x}_{4, x_1} = -0.0712x_1 + 0.003. \tag{9}
\]

Determine residuals \( e_{4, u} = \dot{x}_4 - \hat{x}_{4, u}, \quad e_{4, x_1} = \dot{x}_4 - \hat{x}_{4, x_1} \) and introduce frameworks \( S_{u, e_4, u}, S_{u, e_4, x_1} \) which are described by functions \( \gamma_{e_4, u, u} : e_4, u \rightarrow \dot{x}_4 \) and \( \gamma_{e_4, u, x_1} : e_4, x_1 \rightarrow \dot{x}_4 \) (Figure 2).
Frameworks $S_{x_4, x_{4a}}$, $S_{x_4, x_{4a}}$ reflect the influence of nonlinearities $\phi_1$ and $\phi_2$. We see (Figure 2) that frameworks have saturation equalling to 2 and -2, which coincides with the initial data of the system (8). Switching points of nonlinear functions shifted because the adulterating effect.

Construct secants for $S_{x_4, x_{4a}}$, $S_{x_4, x_{4a}}$

$$\gamma_{x_4, x_{4a}} = a_{x_4} x_{4a}, \quad \gamma_{x_4, x_{4a}} = a_{x_4} x_{4a} + b_{x_4},$$

(10)

where $a_{x_4} = 1$, $a_{x_4} = 1.027$, $b_{x_4} = 0.22$.

**Definition 5** The system (8) is structurally unidentifiable on the set $\{u(t), x_1(t)\}$ if the input is S-synchronizing, and secant (10) coefficients satisfy the condition $|a_{x_1} - a_{x_4}| \leq \delta$, where $\delta \geq 0$.

Definition 5 is applied to nonlinearities $\phi_1$, $\phi_2$ with $\delta = 0.005$. This conclusion is confirmed by Figure 3 (centered diagram), which shows the variables $x_{4a}$ (grey) and $x_{4a}$ (white).

![Figure 3. Comparison of variables $\epsilon_{x_{4a}}, \epsilon_{x_{4a}}$.](image)

The nonlinearity $\phi_1$ dominates over $\phi_2$, but its switching occurs at $x_1 = \pm 1.5$, which corresponds to parameters $\phi_2$. SI is the confirmation of structural identification. Therefore, we conclude that adulteration complicates the identifiability of the system (8) structure.

Consider the space $\{\epsilon_{x_{4a}}, \epsilon_{x_{4a}}\}$, where $\epsilon_{x_{4a}}$ forms similarly to $\epsilon_{x_{4a}}$, and decides on $\phi_1$. Obtained results show that the system (8) structural identification possible on the nonlinearity $\chi(\phi_1, \phi_2)$ depended on $\phi_1, \phi_2$.

**Statement 1** The system (8) nonlinear part $\chi(\phi_1, \phi_2)$ is locally structurally identified on the set $\{u(t), x_2(t), x_4(t)\}$ if the input $u(t) \in S$.

**Proof statement 1** It follows from (8) and Figure 1 as the input is S-synchronizing. S-synchronizability guarantees the maximum definitional domain diameter of frameworks $S_{x_4, x_{4a}}$, $S_{x_4, x_{4a}}$. Therefore, $\chi(\phi_1, \phi_2)$ is $h$-identifiable. The system output depends on the difference $\phi_1(u), \phi_2(x_4)$. The system (8) structure depends on structural parameters of the final function $\chi(\phi_1, \phi_2)$. ■

**Corollary of statement 1** Functions $\phi_1(\cdot), \phi_2(\cdot)$ are structurally unidentifiable on the set $I_o$.

Figure 4 gives indirect confirmation of statement 1, where the framework presents reflecting the function $\chi(\phi_1, \phi_2)$ influence, $\hat{\chi}(\phi_1, \phi_2)$ is function $\chi(\phi_1, \phi_2)$ estimation based on the analysis the set $I_o$.
Figure 4 shows that the system has the saturation region, which corresponds to parameters of initial nonlinearities. The further analysis clarifies some parameters of initial nonlinearities.

Introduce the variable \( v = u - x_2 \) and synthesis the model for \( \dot{x}_4 \)

\[
\dot{x}_4 = a_{4,0} + a_{4,1} v, \tag{11}
\]

where \( a_{4,0} = 0.125, a_{4,1} = 1.45 \). Find the residual \( \delta_4 = \dot{x}_4 - \hat{x}_4 \) and construct the framework \( S_{4,4} \) described by function \( \gamma_{4,4, \delta_4} : \delta_4 \rightarrow \dot{x}_4 \). Determine for \( S_{4,4, \delta_4}, S_{4,4, \delta_n} \) secants

\[
\gamma_{4,4, \delta_4} = a_{4,0} \delta_4 + a_{4,1}, \quad \gamma_{4,4, \delta_n, \delta_4} = \mu_{4,0} \delta_n + \mu_{4,1}, \tag{12}
\]

where \( a_{4,0} = 0.996, a_{4,1} = -0.06, \mu_{4,0} = 1.028, \mu_{4,1} = -0.219 \).

**Theorem 2** If the input \( u \in S \), and coefficients \( a_{4,0}, \mu_{4,0} \) of secants (12) for frameworks \( S_{4,4, \delta_4}, S_{4,4, \delta_n} \) satisfy the condition

\[
|a_{4,0} - \mu_{4,0}| \leq \delta_{\alpha, \mu}, \tag{13}
\]

where \( \delta_{\alpha, \mu} > 0 \) is specified value, then the function \( \chi(\phi_1, \phi_2) \) is structurally identifiable on the set \( \{ \delta_4(t), \dot{x}_4(t) \} \).

**Proof theorem 2** The input of the system (8) is \( S \)-synchronizing. According to statement 1, the framework \( S_{4,4, \delta_4, \delta_n} \) has the maximum diameter of the definition domain. Therefore, the function \( \chi(\phi_1, \phi_2) \) is locally structurally identifiable. The framework \( S_{4,4} \) has the maximum diameter of the definition domain also. We construct for \( S_{4,4, \delta_4}, S_{4,4, \delta_n} \) secants (12). The condition (13) satisfied with \( \delta_{\alpha, \mu} = 0.04 \). Hence, the function \( \chi(\phi_1, \phi_2) \) is locally structurally identifiable on the set \( \{ \delta_4(t), \dot{x}_4(t) \} \).

Theorem 2 application simplifies the identifiability estimation of the system (8).

Figure 5 confirms the validity of theorem 2, where we present frameworks \( S_{4,4}, S_{4,4, \delta_n} \) and the adequacy of considered frameworks in space \( (\delta_4, \delta_n) \). The coefficient of determination is 99\% between \( \delta_4 \) and \( \delta_n \).

So, we show that system (8) is structurally unidentifiable on set \( I_0 \). Go into space \( (\delta_4(t), \dot{x}_4(t)) \) and obtain estimates for functions \( \phi_1, \phi_2 \).
Apply the variable permutation procedure. Let $\dot{x}_4$ be the input, and $x_1$, $x_2$ outputs. This approach almost eliminates the adulteration influence. Construct frameworks $S_{x_1,\dot{x}_4}$, $S_{x_2,\dot{x}_4}$. Since, $u(t) \in S$, frameworks $S_{x_1,\dot{x}_4}$, $S_{x_2,\dot{x}_4}$ are significant [3, 17] and have the maximum diameter of the definitional domain. Apply theorem 2 and confirm by SI ($h_\delta$-identifiability) of the system (8). Corresponded frameworks show in Figure 6. They give to evaluate properties for nonlinearities on the set of function switching intervals. Figure 6 shows that two nonlinearities are in the system. One nonlinearity (framework $S_{x_2,\dot{x}_4}$) is saturation, and the framework $S_{x_1,\dot{x}_4}$ is the saturation function with Dead Space (DS). DS does not coincide with the original area from the specific of the system. The saturation level is 2.

The relationship between $u$ and $x_2$ (the determination coefficient is 0.98) effects the nonlinear properties of the system (8). Consequently, the system (8) and the nonlinearity $\chi(\phi_1, \phi_2)$ are structurally identifiable. Parameters $\chi(\phi_1, \phi_2)$ depend on the connection $\phi_1$, $\phi_2$ into the system.

So, we have

**Statement 2** If: (i) $u \in S$ and $\dot{x}_4 \in S$; (ii) frameworks $S_{x_1,\dot{x}_4}$, $S_{x_2,\dot{x}_4}$ described by functions $\gamma_{\dot{x}_4, x_1} : \dot{x}_4 \rightarrow x_1$ and
have $d_{k,z}$-optimal properties, then system (8) and function $\chi(\varphi_1, \varphi_2)$ are structurally identifiable.

5. Nonlinear adulterate of nonlinearity argument

Consider the self-oscillation generation system with the Nonlinear Feedback (NF). NF adulterates the nonlinear function argument in a straight chain. The system contains the actuator with parameters $k_4$, $T_4$ and output $y$; gain element with parameters $k_3$, $T_1$; nonlinear feedback on $x_2$ with the function $f_2(x_2)$; linear feedback on $x_2$ with parameters $k_1$, $k_5$:

\[
S_f : \begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = x_3, \\
\dot{x}_3 = -\frac{1}{T_1 T_2} x_2 - \frac{T_1 + T_2}{T_1 T_2} x_3 + \frac{k_4}{T_2} f_1(u, b, c), \\
\dot{x}_4 = -k_3 x_3, \\
y = x_3,
\end{cases}
\tag{14}
\]

where $y$ is the rotation angle of the actuator shaft, $u = x_4 - k_{ad} f_2(x_2)$ is function $f_i$ argument, $f_2(x_2) = (x_2)^\gamma \text{sign}(x_2),
\]

\[
f_i(x_4) = \begin{cases}
c, \text{ if } x > b \\
0, \text{ if } x \in [-b, b], \text{ b > 0.} \\
-c, \text{ if } x < -b
\end{cases}
\tag{15}
\]

We have the adulterated argument of the function $f_i$. Hence, itself argument $u(x_2, x_4)$ is the nonlinear function. It complicates the SI assessment of the system (14).

Consider the system $S_f$ with two sets of parameters

\[
P_i = \{k_i(0), k_i(1), k_i(4), k_{ad(i)}, T_{1(i)}, T_{2(i)}, b_i(0), c_i(0)\}, \ i = 1, 2,
\tag{16}
\]

where

\[
P_1 = \{0.35, 10, 1, 1.25, 0.005, 2.5, 0.4, 0.45, 0.75\}, \ P_2 = \{0.35, 10, 1, 0.55, 0.005, 2.5, 0.4, 0.25, 2\}.
\tag{17}
\]

These sets guarantee self-oscillation in the system. Therefore, the input is constantly excited for $P_i$, and frameworks $S_{opt(i)}$ must be h-identifiable. However, the adulterating effect can give false h-identifiable frameworks $S_{opt(i)}$ (denote these frameworks and the system as $LS_{opt(i)}$ and $LS_{(i)}$). Extend the frameworks $S_{opt(i)}$ set and their derivatives (frameworks $KS_{opt(i)}$) to select the reference information set $I_0$ for analysis SI.

Consider the variable $u(t)$ and construct the structurally frequency diagram $H_u$ for it. $H_u$ reflects the distribution $u$ according to the change in the system output.

**Definition 6** If $H_u$ contains areas conforming to system (14) features, then the variable $u(t)$ is structurally significant or $u \in H_u$.

**Definition 7** $S_{opt(i)}$ framework based on $u \in H_u$ is structurally significant or $S_{opt(i)} \in SS_{opt(i)}$. The system $S_{(i)}$ with $u \in H_u$ is called structurally significant or $S_{(i)} \in SS_{(i)}(H_u)$. If $u \notin H_u$, then $S_{(i)} \notin SS_{(i)}(H_u)$.

Consider frameworks $S_{x(1), x(2)}$, $S_{x(3), x(4)}$ is described by functions

\[
\eta_{x(1), x(2)} : x(1) \rightarrow x(2), \ \eta_{x(3), x(4)} : x(3) \rightarrow x(4).
\tag{18}
\]
Present frameworks and their structural-frequency analysis [19] on sets (17) in Figures 7 and 8.

**Theorem 3** Consider the system $S_f$ with sets $P_1, P_2$ and (i) input $u(i)$ is constantly excited; (ii) the system state reflects by the frameworks $S_{u(i),x(i)}$, $S_{u(i),x(i)}$ and the structural-frequency diagrams $H_{ii}(i = 1, 2)$. Then frameworks defined on the set $P_i$ are structurally significant, i.e. they are applied for system identifiability analysis.

**Proof theorem 3** The system (14) is designed to generate self-oscillations. Therefore, the input $u(i)$ satisfies the condition of constant excitation for the given $P_i$. Apply results of section 3 and obtain SI of the frameworks under consideration. Consider the Structural-frequency Diagrams (SDS) presented in Figures 7 and 8. The system $S_f$ contains two nonlinearities. Therefore, SDS reflects their features. Consider frameworks $S_{x(i),x(i)}$. We see that the framework $S_{x(i),x(i)}$ contains fragments (see the distribution $n_{x(i)}$ in Figure 7), which not represented in $S_{x(i),x(i)}$. Compare distributions $n_{x(i)}$. We see the input $u(i)$ is more structurally significant than $u(2)$. Consequently, $u(i) \in H_{ii}$, and

![Figure 7. System (14) phase portraits with parameters $P_i$](image-url)
Thus, the system (14) with the set $P_1$ is the candidate for the analysis of the system $S_f$ identifiability. Confirm these conclusions through the analysis frameworks $S_{x(2),x(1)}$ (Figure 8). Consequently, further SI analysis is based on the use of the information set for the system $S_f(P_1)$. Then $S_f(P_1) \in SS_{H(H_u)}$, and $S_f(P_2) \in SIS_{H(H_u)}$. ■

Figure 8. System (14) phase portraits with parameters $P_2$.

The framework $S_{x(2),x(1)}$ does not reflect the system (14) specifics. Explain it by the nonlinear feedback action. Consider the system (14) defined on the set of parameters

$$P_1 = \left\{0.35, 1.2, 1.55, \left\{k_{w_3,1}\right\}, 3, 0.4, 0.5, 1.5\right\}$$

with different values $k_{w_3}$. 

Volume 3 Issue 2[2022] 267 Contemporary Mathematics
Statement 3 If the system (14) is defined on the set (19), then increasing the parameter $k_\alpha$ changes the domains of
the function $f_1$ in the coordinate origin.

Proof of statement 3 Consider a neighborhood $O_{f_i}$ of the function $f_i$ in coordinate origin. Let $k_\alpha^* = 0.0025$ be the
reference value for $k_\alpha$. Increasing $k_\alpha$ gives a value increase in the function $f_2$. Growing $f_2$ gives change to
the argument $u(k_\alpha)$ and the function $f_1$ compares to values of $f_1(u(k_\alpha^*))$ at the same $t$. Therefore, the domain of values
$O_{f_1(u(k_\alpha))}$ will not coincide with the domain $O_{f_1(u(k_\alpha^*))}$ of the same function $f_1$.

Denote the values domain and the diameter of neighborhood $O_{f_1(u(k_\alpha))}$ as $\text{Im}(O_{f_1(u(k_\alpha))})$.

Definition 8 Call the input $u(t)$ of the system (14) $U_h$-identifying if it minimizes $\text{Im}(O_{f_1(u(k_\alpha))})$.

$$u^* = \min_u D_{O_{f_1(u)}}.$$ 

Denote the set of $U_h$-identifying inputs as $U_{U_h}$.

Theorem 4 If the input $u(t)$ is constantly exciting and $U_h$-identifying, then the system (14) and the function $f_1$ are
$h_\delta$-identifiable.

The proof of theorem 4 is based on the construction of the framework $S_{(x_3,x_1)}$ and the section 3 results application.

The domain $O_{f_1(u)}$ at $u \not\in U_{U_h}$ can determine by the histogram or the framework $S_{(x_3,x_1)}$. The region $O_{f_1(u)}$ changes the
interval $[b; b]$ of the function $f_1$. It leads to the change in the definition area and the domain of values $f_1$. According to
theorem 4, this violates the $h_\delta$-identifiability property of the system (14).

Consider input satisfying conditions of theorem 4.

Theorem 5 If the input $u \in U_{U_h}$ is constantly exciting, then the function $f_2(x_2)$ of the system $S_f$ is $h_\delta$-identifiable.

The proof of theorem 5 follows from theorem 3 conditions fulfilment. Show the function $f_2(x_2)$ estimation in Figure 9.

![Figure 9. Change estimation of function $f_2$.](image)

6. Conclusion

The structural identifiability problem of nonlinear feedback systems considers with adulterating nonlinearity. We
apply the analysis of geometric frameworks and hierarchical immersion method for identifiability estimation study
under uncertainty. The nonlinear adulterating of the nonlinear function argument considered. Structural identifiability
conditions are obtained for this case. We study the input influence on the nonlinearity structural identifiability in the
system straight chain. We define a class of inputs that gives the solution to the SI problem. The adulteration influence of
nonlinear argument shows on the nonlinearity parameters estimation. Structurally-frequency diagrams are the basis for
the analysis of these systems. Conditions for unidentifiability verification are proposed at the nonlinear mixing of the
argument. We consider the additive effect of feedback nonlinearity on the nonlinearity in the straight chain. It shows that the synchronizing input provides the solution to the structural identifiability problem. We have shown that this system is structurally unidentifiable according to obtained experimental data. A subset of the system states on which the system is locally structurally identifiable obtained.

**Conflict of interest**

The author declares that there is no conflict of interest regarding the publication of this paper.

**References**


