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A Note on Differential Identities in Prime and Semiprime Rings

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Abstract: Let R be a prime ring, I a nonzero ideal of R, d a derivation of R and m, n fixed positive integers. (i) If $(d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n - d(r \circ s))^m$ for all $r, s \in I$, then R is commutative. (ii) If $(d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n - d(r \circ s)^m \in Z(R)$ for all $r, s \in I$, then R satisfies s_4, the standard identity in four variables. Moreover, we also examine the case when R is a semiprime ring.
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1. Introduction

Throughout this paper, R always denotes an associative ring with center $Z(R)$, Q its Martindale quotient ring, and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [1] for these objects).

For each $r, s \in R$, the Lie commutator of r, s, is denoted by $[r, s]$ and defined by $[r, s] = rs - sr$ and the anticommutator of R is defined $r \circ s = rs + sr$. By d we mean a derivation of R, that is an additive mapping $d: R \rightarrow R$ satisfying $d(rs) = d(r)s + rd(s)$ for all $r, s \in R$. A derivation d is called Q-inner if it is inner induced by an element, say $\theta \in Q$ as an adjoint, that is, $d(r) = [\theta, r]$ for all $r.\in R$. A derivation which is not Q-inner is called a Q-outer derivation. The standard polynomial identity s_4 in four variables is defined as $s_4(r_1, r_2, r_3, r_4) = \sum_{\sigma \in s_4} (-1)^\sigma r_{\sigma(1)} r_{\sigma(2)} r_{\sigma(3)} r_{\sigma(4)}$ where $(-1)^\sigma$ is + or - according to σ being even or odd permutation in symmetric group s_4.

In [2], Ashraf and Rehman proved that "if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(r \circ s) = r \circ s$ for all $r, s \in I$, then R is commutative". In [3], Argaç and Inceboz generalized the above result as follows: " Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a nonzero derivation d with the property $(d(r \circ s))^n = r \circ s$ for all $r, s \in I$, then R is commutative".

In 1994 Bell and Daif, initiated the study of strong commutativity-preserving maps and proved that "a nonzero right ideal I of a semiprime ring is central if R admits a derivation which is scp on I ". In 2002 Ashraf and Rehman, proved that "if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(r) \circ d(s) = r \circ s$ for all $r, s \in I$, then R is commutative". The present paper is motivated by the previous results and we here generalized the results obtained in [3] and [2]. Moreover, we continue this line of investigation by examining what happens to a ring R (or an algebra A) if it satisfies the identity $(d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^m - d(r \circ s))^n \in Z(R)$, for all $r, s \in I$. We obtain some analogous results for semiprime rings in the case $I = R$.

2. The results in Prime Rings

Theorem 2.1. Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a nonzero derivation d such that $d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n = (d(r \circ s))^m$ for all $r, s \in I$, then R is commutative.

Proof. Since R is a prime ring and if R admits a derivation d, by the given hypothesis, we have

$$(d(rs + sr)(rs + sr) + (rs + sr)\, d(rs + sr))^n = (d(rs + sr))^m \text{ for all } r, s \in I.$$

Thus I satisfies the differential identity

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$$(d(r)\, s + sd(r) + rd(s) + d(s)\, r)^m = (d(r)\, s + sd(r) + rd(s) + d(s)r)(rs + sr) + (rs + sr)(d(r)s + sd(r) + rd(s) + d(s)r)^n \text{ for all } r, s \in I.$$

We divide the proof into two cases:

Case 1. If d is Q-outer derivation, then I satisfies the polynomial identity

$$(ss + ss + rt + tr)^m = ((ss + ss + rt + tr)(rs + sr) + (rs + sr)(ss + ss + rt + tr))^n, \text{ for all } r, s, s, t \in I.$$

In particular, for $s = 0$, I satisfies the blended component $(rt + tr)^m = 0$, for all $r, t \in I$. If char(R) $\neq 2$, then $(2r^2)^m = 0$ for all $r \in I$. By Xu [7], we get a contradiction. If char(R) = 2, then $(rt+tr)^m = 0 = [r, t]^m$ for all $r, t \in I$, and hence R is commutative by Herstein [8, Theorem 2].

Case 2. Let now d be Q-inner derivation induced by an element $\phi \in Q$, that is, $d(r) = [\phi, r]$ for all $r \in R$. It follows that,

$$([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m = (([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)\, (rs + sr) + ((rs + sr)([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n.$$

for any $r, s \in I$. By Chuang [11, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), hence we have

$$([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m = (([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)\, (rs + sr) + ((rs + sr)([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n, \text{ for all } r, s \in Q$$

Moreover, if C is infinite, we have

$$([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m = (([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s]+[\phi, s]\, r)\, (rs + sr) +((rs + sr)([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n,$$

for all $r, s \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [12, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus, we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed

$$([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m = (([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)\, (rs + sr) +((rs + sr)([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n, \text{ for all } r, s \in R$$

By Martindale [13, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with C as the associated division ring. Hence, by Jacobson's theorem [14, p.75], R is isomorphic to a dense ring of linear transformations of some vector space V over C and H consists of the finite rank linear transformations in R.

Assume first that $\dim_C V \geq 3$.

First of all we want to show that v and ϕv are linearly C-dependent for all $v \in V$. Since if $\phi v = 0$ then $\{v, \phi v\}$ is linearly C-dependent, suppose that $\phi v \neq 0$. If v and ϕv are linearly C-independent, since $\dim_C V \geq 3$, there exists $w \in V$ such that $\{v, \phi v, w\}$ are also linearly C-independent. By the density of R, there exist $r, s \in R$ such that:

$rv = 0$, $r\phi v = w$, $rw = 0$

$sv = 0$, $s\phi v = 0$, $sw = v$.

This implies that

$$v = (([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)^m)v$$
$$= (([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)(rs + sr)$$
$$+ (rs + sr)([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r))^n)v$$
$$= 0, \text{ a contradiction.}$$

So we conclude that $\{v, \phi v\}$ are linearly C-dependent, for all $v \in V$.

Our next goal is to show that there exists $\alpha \in C$ such that $\phi v = v\alpha$, for any $v \in V$. In fact, choose $v, w \in V$ linearly independent. Since $\dim_C V \geq 3$, there exists $u \in$ V such that v, w, u are linearly independent, and so there exist $\alpha_v, \alpha_w, \alpha_u \in C$ such that

$$\phi v = v\alpha_v,\ \phi w = w\alpha_w,\ \phi u = u\alpha_u \text{ that is } \phi(v + w + u) = v\alpha_v + w\alpha_w + u\alpha_u.$$

Moreover $\phi(v + w + u) = (v + w + u)\alpha_{v+w+u}$, for a suitable $\alpha_{v+w+u} \in C$. Then

$$0 = v(\alpha_{v+w+u} - \alpha_v) + w(\alpha_{v+w+u} - \alpha_w) + u(\alpha_{v+w+u} - \alpha_u),$$

and because v, w, u are linearly independent, $\alpha_u = \alpha_w = \alpha_v = \alpha_{v+w+u}$, that is, α does not depend on the choice of v. Hence we have $\phi v = \alpha v$ for all $v \in V$.

Now for any $r \in R$, $v \in V$. By Step 2, $\phi v = v\alpha$, $r(\phi v) = r(v\alpha)$, and also $\phi(rv) = (rv)\alpha$. Thus, $0 = [\phi, r]v$, for any $v \in V$, that is $[\phi, R]V = 0$. Since V is a left faithful irreducible R-module, hence $[\phi, R] = 0$, i.e., $\phi \in Z(R)$ and so a contradiction.

Suppose now that $\dim_C V$ must be ≤ 2. In this case, R is a simple GPI-ring with 1, so it is a central simple algebra of finite dimensional over its center. By Lanski [16, Lemma 2], it follows that there exists a suitable filed F such that $R \subseteq M_k(\mathrm{F})$, the ring of all $k \times k$ matrices over F, and moreover $M_k(\mathrm{F})$ satisfies the same GPI as R.

Assume $k \geq 3$, then by the same argument as above, we get a contradiction.

Obviously, if $k = 1$, then R is commutative. Thus we may assume that $k = 2$, i.e., $R \subseteq M_2(\mathrm{F})$, where $M_2(\mathrm{F})$ satisfies

$$([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)^m = (([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)(rs + sr)$$
$$+ ((rs + sr)([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r))^n,$$
$$\text{for all } r, s \in M_2(\mathrm{F}).$$

Denote e_{ij} the usual unit matrix with 1 in (i, j)-entry and zero elsewhere. Let $r = e_{11}$, $s = e_{12}$, then we get,

$$0 = ([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)^m$$
$$= (([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r)(rs + sr)$$
$$+ ((rs + sr)([\phi, r]s + s[\phi, r] + r[\phi, s] + [\phi, s]r))^n$$
$$= (\phi e_{12} - e_{12}\phi)^m.$$

In any case, we have at $0 = e_{12}(\phi e_{12})^m$. Set $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$. By calculation, we can have $\begin{pmatrix} 0 & \phi_{21}^m \\ 0 & 0 \end{pmatrix} = 0$ which implies that $\phi_{21} = 0$. In the same manner, we can see that $\phi_{12} = 0$. Therefore, ϕ is a diagonal in $M_2(\mathrm{F})$. Let $\theta \in Aut(M_2(\mathrm{F}))$. Since

$$([\theta(\phi), \theta(r)]\theta(s) + \theta(s)[\theta(\phi), \theta(r)] + \theta(r)[\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\theta(r))^m$$
$$= ([\theta(\phi), \theta(r)]\theta(s) + \theta(s)[\theta(\phi), \theta(r)]$$
$$+ \theta(r)[\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\theta(r))(\theta(r)\theta(s) + \theta(s)\theta(r))$$
$$+ (\theta(r)\theta(s) + \theta(s)\theta(r))([\theta(\phi), \theta(r)]\theta(s) + \theta(s)[\theta(\phi), \theta(r)]$$
$$+ \theta(r)[\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\theta(r)))^n$$

So, $\theta(\phi)$ must be a diagonal matrix in M_2(F). In particular, let $\theta(r) = (1 - e_{ij})\, r\, (1 + e_{ij})$ for $i \neq j$. Then $\theta(\phi) = \phi + (\phi_{ii} - \phi_{ij})\, e_{ij}$, that is $\phi_{ii} = \phi_{ij}$ for $i \neq j$. This implies that ϕ is central in M_2(F), which leads to $d = 0$ a contradiction. This completes the proof.

Now, we prove our next theorem for the central case:

Theorem 2.2. Let R be a prime ring with characteristics different from 2, center $Z(R)$, I a nonzero ideal of R, and m, n are fixed positive integers. If R admits a nonzero derivation d such that $(d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n - d(r \circ s))^m \in Z(R)$, for all $r, s \in I$. Then R satisfies s_4, the standard identity in four variables.

Proof. On the contrary, suppose that R does not satisfy s_4. Since R is prime ring and d is a nonzero derivation of R. If $d(rs + sr)(rs + sr) + (rs + sr)\, d(rs + sr))^n = (d(rs + sr))^m$ for all $r, s \in I$, then R is commutative by Theorem 2.1. Otherwise, we have $I \cap Z(R) \neq 0$ by our assumptions. Let now J be a nonzero two-sided ideal of R_Z, the ring of the central quotients of R. Since $J \cap R$ is an ideal of R, then $J \cap R \cap Z(R) \neq 0$. Hence, that is, J contains an invertible element in R_Z, and so R_Z is simple with 1. By the hypothesis for any $r, s \in I$ and $z \in R$, thus I satisfies the differential identity

$$\begin{aligned}[&(d(r)s + sd(r) + rd(s) + d(s)r)(rs + sr) \\ &\quad + (rs + sr)(d(r)s + sd(r) + rd(s) + d(s)r))^n \\ &\quad - (d(r)s + sd(r) + rd(s) + d(s)r)^m, z] = 0.\end{aligned} \tag{1}$$

Since I and Q satisfy the same differential identities [11, Theorem 1], we may assume that Q satisfies (1). Now consider two cases:

Case 1. If d is not Q-inner. By Kharchenko's theorem [5], Q satisfies the same polynomial identity,

$$\begin{aligned}[&((ss + ss + rt + tr)(rs + sr) + (rs + sr)(ss + ss + rt + tr))^n \\ &\qquad - (ss + ss + rt + tr)^m, z] = 0.\end{aligned}$$

This is a polynomial identity and hence there exists a field F such that $Q \subseteq M_k(F)$ with $k > 1$ and Q, $M_k(F)$ satisfy the same polynomial identity [16]. Now choose $r = e_{12}$, $t = 0$, $s = e_{21}$, $w = e_{13}$, $s = e_{12}$ one can get,

$$\begin{aligned}0 &= ((ss + ss + rt + tr)(rs + sr) + (rs + sr)(ss + ss + rt + tr))^n \\ &\quad - (ss + ss + rt + tr)^m, z] \\ &= 2^n e_{13} - e_{13}, \text{ a contradition.}\end{aligned}$$

Case 2. If d is a Q-inner derivation induced by an element $\phi \in Q$, such that $d(r) = [\phi, r]$ for all $r \in R$. Then by (1) we have

$$\begin{aligned}&(([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)(rs + sr) \\ &\quad + (rs + sr)\, ([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n \\ &\quad - ([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m, z] = 0,\end{aligned} \tag{2}$$

for all $r, s \in I$ and $z \in R$. By Chuang [11], Q satisfy (2). By localizing R at $Z(R)$ it follows that

$$\begin{aligned}&(([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)(rs + sr) \\ &\quad + (rs + sr)\, ([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r))^n \\ &\quad - ([\phi, r]\, s + s\, [\phi, r] + r\, [\phi, s] + [\phi, s]\, r)^m \in Z(R_Z),\end{aligned}$$

for all $r, s \in R_Z$. Since R and R_Z satisfy the same polynomial identities, by our assumption, we have that R_Z does not satisfy s_4. Thus, replacing R with R_Z, we may assume that R is a simple ring with 1. By Martindale theorem [13], R is a primitive ring with the minimal right ideal, whose commuting ring C is a division ring that is finite dimensional over $Z(R)$. However, since R is simple with 1, R must be Artinian. Hence $R = D_S$, the $s \times s$ matrices over C, for some $s \geq 1$. By [16, Lemma 2], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F, with k > 1, and $M_k(F)$ satisfies (2.2) that is,

$$\begin{aligned}&(([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)(rs + sr)\\&\quad + (rs + sr) ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r))^n\\&\quad - ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)^m \in Z(M_k(F)) = F.I_k.\end{aligned}$$

If $k \geq 2$, now let $\phi = (\phi_{ij})_{k\times k}$. By assumption, for all $r, s \in R$,

$$\begin{aligned}&(([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)(rs + sr)\\&\quad + (rs + sr) ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r))^n\\&\quad - ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)^m\end{aligned}$$

is zero or invertible. We choose $r = e_{ii}$, $s = e_{ij}$ for any $i \neq j$. Then we have

$$\begin{aligned}&(([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)(rs + sr)\\&\quad + (rs + sr) ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r))^n\\&\quad - ([\phi, r] s + s [\phi, r] + r [\phi, s] + [\phi, s] r)^m\\&\quad = -[\phi, e_{ij}]^m\end{aligned}$$

Since rank of $[\phi, e_{ij}]^m$ is ≤ 2, it cannot be invertible in R and so $[\phi, e_{ij}]^m = 0$. By solving above and left multiplying by e_{ij}, one can get

$$0 = e_{ij} (\phi e_{ij})^m = e_{ij} \phi_{ji}^m$$

implying $\phi_{ji} = 0$. Thus, for any $i \neq j$, $\phi_{ji} = 0$, ϕ is diagonal. Now set $\phi = \sum_t \phi_{tt} e_{tt}$ with $\phi_{tt} \in F$. For any F-automorphism θ of R, we have

$$\begin{aligned}&([\theta(\phi), \theta(r)]\, \theta(s) + \theta(s)[\theta(\phi), \theta(r)] + \theta(r)\, [\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\, \theta(r))^m\\&\quad = ([\theta(\phi), \theta(r)]\, \theta(s) + \theta(s)[\theta(\phi), \theta(r)]\\&\qquad + \theta(r)\, [\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\, \theta(r))\, (\theta(r)\theta(s) + \theta(s)\, \theta(r))\\&\qquad + (\theta(r)\theta(s) + \theta(s)\theta(r))\, ([\theta(\phi), \theta(r)]\, \theta(s) + \theta(s)[\theta(\phi), \theta(r)]\\&\qquad + \theta(r)\, [\theta(\phi), \theta(s)] + [\theta(\phi), \theta(s)]\, \theta(r)))^n\end{aligned}$$

is zero or invertible for every $r, s \in R$. By the above argument $\theta(\phi)$ must be diagonal. Therefore, for each $j \neq i$, we have $\theta(\phi) = (1 + e_{ij})\phi(1 - e_{ij}) = \sum_{i=1}^{k} \phi_{ii} e_{ii} + (\phi_{jj} - \phi_{ii}) e_{ij}$ is diagonal. Therefore, $\phi_{jj} = \phi_{ii}$ and so $\phi \in F.I_k$, and hence $d = 0$, which is a contradiction and completes the proof.

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1. Let S be any ring, $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$ be a nonzero ideal of R. We define a map $d: R \rightarrow R$ by $d(a) = e_{11}a - ae_{11}$. Then it is easy to see that d is a nonzero derivation. It is straightforward to check that d satisfies the property $d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n = (d(r \circ s))^m$. However, R is not commutative.

Example 2.2. Let S be any ring, $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$ be a nonzero ideal of R. Define a a map d: $R \to R$ by $d(a) = [a, e_{11} + e_{12}]$. It is easy to see that d is a nonzero derivation and satisfies the property, $d(r \circ s)(r \circ s) d(r \circ s)(r \circ s) + (r \circ s)\, d(r \circ s)^n = (d(r \circ s))^{m.}$, but R is not commutative.

3.The case: *R* a semiprime ring

In this section, we extended Theorem 2.1 and Theorem 2.2 to the semiprime ring. Let R be a semiprime ring and U be its left Utumi quotient ring. Then $C = Z(U)$ is extended centroid of R (see [17, p-38]). It is well known that "any derivation of semiprime ring R can be extended to a derivation of its left Utumi quotient ring U and so any derivation of R can be defined on the whole of U" [6, Lemma 2].

Theorem 3.1. Let R be a semiprime ring, U the left Utumi quotient ring of R, and m, n are fixed positive integers. If R admits a nonzero derivation d such that $d(x \circ y)(x \circ y) + (x \circ y)\, d(x \circ y)^n = (d(x \circ y))^m$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, d vanishes identically on eU and the ring $(1 - e)U$ is commutative.

Proof. by Beidar [1] "any derivation of a semiprime ring R can be defined on the whole U", the Utmi quotient ring R. In view of Lee [6], R and U satisfy the same differential identities, hence $d(x \circ y)(x \circ y) + (x \circ y)\, d(x \circ y)^n = (d(x \circ y))^m$ for all $x, y \in U$.

Let B be the complete boolean algebra of idempotents in C and let M be any maximal ideal of B. Due to Chuang [17, p. 42], U is an orthogonal complete B-algebra and MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore, $\overline{d}$ has in $\overline{U}$ the same property as d in U. In particular, $\overline{U}$ is prime and so, by Theorem 2.1, we have either $\overline{U}$ is commutative or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case, $d(U)[U, U] \subseteq MU$, for all M, where MU runs over all prime ideals of U. Therefore, $d(U)[U, U] \subseteq \bigcap_M MU = 0$, we obtain $d(U)[U, U] = 0$. By using the theory of orthogonal completion for semiprime rings [1, Chapter 3], it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, d vanishes identically on eU and the ring $(1 - e)U$ is commutative. This completes the proof of the theorem.

Theorem 3.2. Let R be a semiprime ring with characteristics different from 2, U the left Utumi quotient ring of R and m, n are fixed positive intemgers. If R admits a nonzero derivation d such that $d(x \circ y)(x \circ y) + (x \circ y)\, d(x \circ y)^n - (d(x \circ y))^m \in Z(R)$ for all $x, y \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, d vanishes identically on eU and the ring $(1 - e)U$ satisfies s_4, the standard identity in four variables.

Proof. Since "any derivation d can be uniquely extended to a derivation in U, and U and R satisfy the same differential identities" (see [6]), then $d(x \circ y)(x \circ y) + (x \circ y)\, d(x \circ y)^n - (d(x \circ y))^m \in Z(R)$ for all $x, y \in U$.

Let B be the complete boolean algebra of idempotents in C and let M be nay maximal ideal of B. Due to Chuang [17, p. 42] U is an orthogonal complete B-algebra and MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore $\overline{d}$ has in $\overline{U}$ the same property as d in U. In particular, $\overline{U}$ is prime and so, by Theorem 2.2, either $\overline{U}$ satisfies s_4 or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $s_4(x_1, x_2, x_3, x_4) \subseteq MU$, for all $x_1, x_2, x_3, x_4 \in U$. In any case $d(U)s_4(x_1, x_2, x_3, x_4) \subseteq \bigcap_M MU = 0$. From [1, Chapter 3], there exists a central idempotent element e of U, the left Utumi quotient ring of R, such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, $d(eU) = 0$ and the ring $(1 - e)U$ is satisfies s_4.

References

[1] Beidar, K.I. Martindale III, W.S. Mikhalev, A.V. Rings with Generalized Identities. *Pure and AppliedMathematics* Marcel Dekker 196, New York, 1996.

[2] Ashraf, M. Rehman, N. On commutativity of rings with derivations. *Results Math*. 2002; (42): No1-2, 3-8.

[3] Arga¸c, N. Inceboz, H.G. Derivation of prime and semiprime rings. *J. Korean Math. Soc* 2009(46): No.5, 997-1005.

[4] Bell, H. E., Daif, M. N. On commutativity and strong commutativity-preserving maps. *Cand. Math. Bull*.1994; (37): 443-447

[5] Kharchenko, V. K. Differential identities of prime rings. *Algebra Logic.* 1979; (17): 155-168.

[6] Lee, T. K. Semiprime rings with differential identities. *Bull. Inst. Math., Acad. Sin*. 1992; (20): 27-38.

[7] Xu, X. W. The power values properties of generalized derivations. *Doctor Thesis of Jilin University*. Changchun, 2006.

[8] Herstein, I. N. Center-like elements in prime rings. *J. Algebra.* 1979; (60): 567-574.

[9] Mayne, J. H. Centralizing mappings of prime rings. *Canad. Math. Bull.* 1984; 27(1): 122-126.
[10] Chuang, Ch.-L. Hypercentral derivations. *J. Algbera*. 1994; 166(1): 34-71
[11] Chuang, Ch.-L. GPIs having coeffcients in Utumi quotient rings. *Proc. Am. Math. Soc*.1988; (103): 723-728.
[12] Erickson, T. S. Martindale III, W. Osborn J. M. Prime nonassociative algebras. *Pac.J. Math*. 1975; (60): 49-63.
[13] Martindale III, W.S. Prime rings satisfying a generalized polynomial identity. *J. Algebra.* 1969; (12): 576-584.
[14] Jacobson, N. Structure of Rings. Colloquium Publications 37. *Am. Math. Soc*. VII,Provindence, RI, 1956.
[15] Carini, L. Filippis, V. D. Commuatators with power central values on a Lie ideal. *Pacific J. Math.* 2000; 193(2): 269-278.
[16] Lanski, C. An Engel condition with derivation. Proc. *Am. Math. Soc*. 1993; (118): 731-734.
[17] Chuang, Ch.-L. Hypercentral derivations. *J.Algebra.* 1994; (166), 34-71.