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Research Article

Solvability of Higher Order Iterative System with Non-Homogeneous Integral Boundary Conditions

Sreedhar Namburi[1*], **Kanakayya Namana**[1], **Rajendra Prasad Kapula**[2]

[1]Department of Mathematics, Institute of Science, Gandhi Institute of Technology and Management, Visakhapatnam, India
[2]Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India
E-mail: sreedharnamburi@rediffmail.com

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Abstract: The aim of this paper is to establish the existence of positive solutions by determining the eigenvalue intervals of the parameters $\mu_1, \mu_2, ..., \mu_m$ for the iterative system of nonlinear differential equations of order p

$$\left.\begin{array}{c} w_i^{(p)}(x) + \mu_i a_i(x) f_i(w_{i+1}(x)) = 0,\ 1 \le i \le m,\ x \in [0,1], \\ w_{m+1}(x) = w_1(x),\ x \in [0,1], \end{array}\right\}$$

satisfying non-homogeneous integral boundary conditions

$$\left.\begin{array}{c} w_i(0) = 0,\ w_i'(0) = 0,\ \ldots,\ w_i^{(p-2)}(0) = 0, \\ w_i^{(r)}(1) - \eta_i \int_0^1 g_i(\tau) w_i^{(r)}(\tau) d\tau = \lambda_i,\ 1 \le i \le m, \end{array}\right\}$$

where $r \in \{1, 2, ..., p-2\}$ but fixed, $p \ge 3$ and $\eta_i, \lambda_i \in (0, \infty)$ are parameters. The fundamental tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem to establish the existence of positive solutions of the problem for operators on a cone in a Banach space. Here the kernels play a fundamental role in defining an appropriate operator on a suitable cone.

***Keywords*:** differential equation, iterative system, integral boundary conditions, eigenvalues, kernel, positive solution

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1. Introduction

The theory of differential equations has been applied in the modeling of physical, biological and medical sciences as well as economics to determine the best investment strategies. In analyzing real life problems, many mathematical

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models generate either initial value problems or boundary value problems with two-point/multi-point boundary conditions. A particular class of problems involving integral boundary conditions arises in thermoelectricity, chemical engineering, plasma physics, and other fields. In these applied settings, only the positive solutions are relevant. To mention a few papers along these lines, see [1-5].

Many engineering applications involve complicated systems with several degrees of freedom that must be addressed as a system of ordinary differential equations which satisfy certain assumptions. The first challenging step is usually to develop a model for complicated systems and then investigate the existence of solutions for the model using various methods. Due to its importance in theory and applications in recent years, significant emphasis has been made in finding optimal eigenvalue intervals for the existence of positive solutions for the system of nonlinear boundary value problems by an application of the Guo-Krasnosel'skii fixed point theorem.

In 2007, Henderson and Ntouyas [6] determined values of the parameter λ for which there exist positive solutions for the system of nonlinear differential equations

$$u^{(n)} + \lambda a(t) f(v) = 0, 0 < t < 1,$$

$$v^{(n)} + \lambda b(t) g(u) = 0, 0 < t < 1,$$

satisfying

$$u(0) = 0,\ u'(0) = 0,\ \ldots,\ u^{(n-2)}(0) = 0,\ u(1) = \alpha u(\eta),$$

$$v(0) = 0,\ v'(0) = 0,\ \ldots,\ v^{(n-2)}(0) = 0,\ v(1) = \alpha v(\eta),$$

by using Guo-Krasnosel'skii fixed point theorem. In 2008, Henderson, Ntouyas and Purnaras [7] studied the existence of positive solutions by determining the values of parameter λ for the system of three-point boundary value problems

$$u'' + \lambda a(t) f(v) = 0,\ 0 < t < 1,$$

$$v'' + \lambda b(t) g(u) = 0, 0 < t < 1,$$

$$u(0) = \beta u(\eta), u(1) = \alpha u(\eta),$$

$$v(0) = \beta v(\eta), v(1) = \alpha v(\eta),$$

by using Guo-Krasnosel'skii fixed point theorem and in the same year, Henderson, Ntouyas and Purnaras [8] determined values of the parameters λ and μ for which there exist positive solutions for the system of four-point nonlinear boundary value problems

$$u'' + \lambda a(t) f(v) = 0, 0 < t < 1,$$

$$v'' + \mu b(t) g(u) = 0, 0 < t < 1,$$

$$u(0) = \alpha u(\xi),\ u(1) = \beta u(\eta),$$

$$v(0) = \alpha v(\xi),\ v(1) = \beta v(\eta).$$

Later, in 2013, Prasad, Sreedhar and Kumar [9] dealt with the existence of positive solutions by determining the eigenvalues λ_i, $1 \le i \le n$, for the iterative system of three-point nonlinear boundary value problems

$$y_i''(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) = 0,\ 1 \le i \le n,\ y_{n+1}(t) = y_1(t),\ t \in [t_1, t_3],$$

$$\alpha_i y_i(t_1) - \beta_i y_i'(t_1) = 0,\ \gamma_i y_i(t_3) + \delta_i y_i'(t_3) = y_i'(t_2),\ 1 \le i \le n,$$

using Guo-Krasnosel'skii fixed point theorem. In 2020, Prasad, Rashmita and Sreedhar [10] determined intervals of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ for which the existence of positive solutions of the boundary value problems

$$y_i^{(n)}(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) = 0,\ 1 \le i \le m,\ y_{m+1}(t) = y_1(t),\ t \in [0,1],$$

$$y_i(0) = y_i'(0) = \cdots = y_i^{(n-2)}(0) = 0,\ \alpha_i y_i^{(n-2)}(1) - \beta_i y_i^{(n-2)}(\eta) = \mu_i,\ 1 \le i \le m.$$

Here the results are now extended to the iterative system of higher order differential equations with non-homogeneous integral boundary conditions.

In this paper, we determine eigenvalue intervals of the parameter $\mu_1, \mu_2, ..., \mu_m$ for which the existence of positive solutions of the iterative system of nonlinear differential equations of order p

$$\left.\begin{aligned} w_i^{(p)}(x) + \mu_i a_i(x) f_i(w_{i+1}(x)) = 0,\ 1 \le i \le m,\ x \in [0,1], \\ w_{m+1}(x) = w_1(x),\ x \in [0,1], \end{aligned}\right\} \tag{1}$$

satisfying non-homogeneous integral boundary conditions

$$\left.\begin{aligned} w_i(0) = 0,\ w_i'(0) = 0,\ \ldots,\ w_i^{(p-2)}(0) = 0, \\ w_i^{(r)}(1) - \eta_i \int_0^1 g_i(\tau) w_i^{(r)}(\tau) d\tau = \lambda_i,\ 1 \le i \le m, \end{aligned}\right\} \tag{2}$$

where $r \in \{1, 2, ..., p-2\}$ but fixed, $p \ge 3$ and $\eta_i, \lambda_i \in (0, \infty)$ are parameters for $1 \le i \le m$, by an application of Guo-Krasnosel'skii fixed point theorem on a cone in a Banach space. Eloe and Henderson [11] studied the existence of positive solutions for higher order two-point boundary value problem for $\eta_1 = 0$ and $\lambda_1 = 0$. Sun and Li [12] addressed the third order problem with integral boundary conditions and established the existence of positive solutions for $p = 3$, $\eta_1 = 1$ and $\lambda_1 = 0$. The results in the paper [10] are the particular case of this paper by taking $r = p - 2$. The following conditions are true throughout the paper:

(C1) $f_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $w_i \in C([0, 1], \mathbb{R}^+)$ and $g_i \in C([0, 1], \mathbb{R}^+)$ for $1 \le i \le m$,

(C2) $a_i(x) \in C([0, 1], \mathbb{R}^+)$ and $a_i(x)$ do not vanish identically on any closed subinterval of $[0, 1]$ for $1 \le i \le m$,

(C3) $1 - \eta_i\theta_i > 0$, where $\theta_i = \int_0^1 g_i(\tau)\tau^{p-r-1} d\tau$ for $1 \le i \le m$,

(C4) each of

$$f_{i0} = \lim_{w \to 0^+} \frac{f_i(w)}{w} \text{ and } f_{i\infty} = \lim_{w \to \infty} \frac{f_i(w)}{w}$$

exists as positive real number for $1 \le i \le m$.

The remaining part of the article is arranged as follows. The solution to the problem (1) and (2) is written as an analogous integral equation in terms of kernels, and bounds for kernels are determined in Section 2. In Section 3, we use the Guo-Krasnosel'skii fixed point theorem to establish the existence of at least one positive solution of the problem (1) and (2) by determining the eigenvalues $\mu_1, \mu_2, ..., \mu_m$. At the end, we provide examples to demonstrate our results.

2. Kernels and its bounds

The solution of the problem (1) and (2) is expressed as an analogous integral equation involving kernels and several inequalities are established for kernels in this section.

Lemma 2.1 Assume that the condition ($C3$) is fulfilled. If $\varphi(x) \in C([0, 1], \mathbb{R}^+)$ then the problem

$$w_i^{(p)}(x)+\varphi(x)=0,\ 1\le i\le m,\ x\in[0,1], \tag{3}$$

with (2) has a unique solution and is

$$w_i(x)=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}+\int_0^1\left[R(x,\vartheta)+\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]\varphi(\vartheta)d\vartheta, \tag{4}$$

$$R(x,\vartheta)=\frac{1}{(p-1)!}\begin{cases}[x^{p-1}(1-\vartheta)^{p-r-1}-(x-\vartheta)^{p-1}], & 0\le\vartheta\le x\le 1,\\ x^{p-1}(1-\vartheta)^{p-r-1}, & 0\le x\le\vartheta\le 1,\end{cases} \tag{5}$$

and

$$S(\tau,\vartheta)=\begin{cases}\tau^{p-r-1}(1-\vartheta)^{p-r-1}-(\tau-\vartheta)^{p-r-1}, & 0\le\vartheta\le\tau\le 1,\\ \tau^{p-r-1}(1-\vartheta)^{p-r-1}, & 0\le\tau\le\vartheta\le 1.\end{cases} \tag{6}$$

Proof. Let $w_i(x)$, $1 \le i \le m$, be the solution of the problem (3) and (2). Then an equivalent integral equation of (3) is

$$w_i(x)=d_0+d_1x+d_2x^2+\cdots+d_{p-1}x^{p-1}-\frac{1}{(p-1)!}\int_0^x(x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta.$$

By applying conditions (2), one can get

$$d_j=0,\ \text{for}\ j=0,1,\ldots,p-2,$$

and

$$d_{p-1}=\frac{\lambda_i(p-r-1)!}{(p-1)!(1-\eta_i\theta_i)}+\frac{1}{(p-1)!(1-\eta_i\theta_i)}\left\{\int_0^1(1-\vartheta)\varphi(\vartheta)d\vartheta\right.$$

$$\left.-\eta_i\int_0^1 g_i(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau\right\}.$$

Then, the unique solution of (3) and (2) is

$$w_i(x)=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}+\frac{x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}\int_0^1(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta$$

$$-\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 g_i(\tau)\left[\int_0^\tau(\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau$$

$$-\frac{1}{(p-1)!}\int_0^x (x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta$$

$$=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}+\frac{x^{p-1}(1-\eta_i\theta_i+\eta_i\theta_i)}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 (1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta$$

$$-\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 g_i(\tau)\left[\int_0^\tau (\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau$$

$$-\frac{1}{(p-1)!}\int_0^x (x-\vartheta)^{p-1}\varphi(\vartheta)d\vartheta$$

$$=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}+\frac{1}{(p-1)!}\left[\int_0^x \left(x^{p-1}(1-\vartheta)^{p-r-1}-(x-\vartheta)^{p-1}\right)\varphi(\vartheta)d\vartheta\right.$$

$$\left.+\int_x^1 x^{p-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]+\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}$$

$$\int_0^1\int_0^1 g_i(\tau)\tau^{p-r-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\tau d\vartheta-\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}$$

$$\int_0^1 g_i(\tau)\left[\int_0^\tau (\tau-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau$$

$$=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}+\int_0^1 R(x,\vartheta)\varphi(\vartheta)d\vartheta+\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}$$

$$\int_0^1 g_i(\tau)\left[\int_0^\tau \left(\tau^{p-r-1}(1-\vartheta)^{p-r-1}-(\tau-\vartheta)^{p-r-1}\right)\varphi(\vartheta)d\vartheta\right.$$

$$\left.+\int_\tau^1 \tau^{p-r-1}(1-\vartheta)^{p-r-1}\varphi(\vartheta)d\vartheta\right]d\tau$$

$$=\frac{\lambda_i(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}$$

$$+\int_0^1\left[R(x,\vartheta)+\frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]\varphi(\vartheta)d\vartheta.$$

□

Lemma 2.2 If the condition ($C3$) is fulfilled, then the kernels $R(x, \vartheta)$ and $S(x, \vartheta)$ satisfy the following inequalities:
(i) $R(x, \vartheta) \geq 0$ and $S(x, \vartheta) \geq 0$ for all $x, \vartheta \in [0, 1]$,
(ii) $R(x, \vartheta) \leq R(1, \vartheta)$ for all $x, \vartheta \in [0, 1]$,
(iii) $R(x, \vartheta) \geq \frac{1}{4^{p-1}} R(1, \vartheta)$ for all $x \in I$ and $\vartheta \in [0, 1]$, where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Proof. The above inequalities are obtained by algebraic computations as shown in [13]. □

We note that an m-tuple $(w_1(x), w_2(x), ..., w_m(x))$ is a solution of the boundary value problem (1) and (2) if, and only if $w_i(x)$ satisfies the following equations

$$w_i(x) = \frac{\lambda_i (p-r-1)!x^{p-1}}{(p-1)!(1-\eta_i\theta_i)} + \mu_i \int_0^1 \left[R(x,\vartheta) + \frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)} \int_0^1 S(\tau,\vartheta) g_i(\tau) d\tau \right]$$

$$a_i(\vartheta) f_i(w_{i+1}(\vartheta)) d\vartheta, \ 1 \le i \le m, \ x \in [0,1],$$

and

$$w_{m+1}(x) = w_1(x), \ x \in [0,1],$$

so that, in particular,

$$w_1(x) = \frac{\lambda_1 (p-r-1)!x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x,\vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) f_1 \left(\frac{\lambda_2 (p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \right. \right.$$

$$\left. \int_0^1 S(\tau,\vartheta_2) g_2(\tau) d\tau \right] a_2(\vartheta_2) \cdots f_{m-1} \left(\frac{\lambda_m (p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} + \mu_m \int_0^1 \left[R(\vartheta_{m-1},\vartheta_m) \right. \right.$$

$$\left. \left. \left. + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau,\vartheta_m) g_m(\tau) d\tau \right] a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) \cdots d\vartheta_2 \right) d\vartheta_1.$$

The Guo-Krasnosel'skii fixed point theorem described below will serve as the foundation for presenting our main findings.

Theorem 2.3 [14, 15] Let $\mathfrak{B}$ be a Banach Space and ρ be a cone in $\mathfrak{B}$. Suppose Ω_1 and Ω_2 are any two open subsets of $\mathfrak{B}$ such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose further the completely continuous operator $\mathfrak{D} : \rho \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \rho$ satisfy the following conditions either

(i) $\|\mathfrak{D}w\| \le \|w\|$, for $w \in \rho \cap \partial\Omega_1$ and $\|\mathfrak{D}w\| \ge \|w\|$, for $w \in \rho \cap \partial\Omega_2$, or

(ii) $\|\mathfrak{D}w\| \ge \|w\|$, for $w \in \rho \cap \partial\Omega_1$ and $\|\mathfrak{D}w\| \le \|w\|$, for $w \in \rho \cap \partial\Omega_2$.

Then the operator $\mathfrak{D}$ has a fixed point in $\rho \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3. Positive solutions in a cone

In this section, we develop criteria to determine the eigenvalues for which the iterative system (1) and (2) have at least one positive solution in a cone.

For this, let $\mathfrak{B} = \{w : w \in C([0, 1], \mathbb{R})\}$ be the Banach space with the norm

$$\| w \| = \max_{x \in [0,1]} | w(x) |.$$

Define a cone ρ in $\mathfrak{B}$ as

$$\rho = \left\{ w \in \mathfrak{B} : w(x) \geq 0 \text{ on } x \in [0,1] \text{ and } \min_{x \in I} w(x) \geq \frac{1}{4^{p-1}} \| w \| \right\}.$$

Let us define an operator $\mathfrak{D} : \rho \to \mathfrak{B}$ for $w_1 \in \rho$ as

$$\left.\begin{aligned}
\mathfrak{D}w_1(x) = {} & \frac{\lambda_1 (p-r-1)!\, x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x,\vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1) g_1(\tau) d\tau \right] \\
& a_1(\vartheta_1) f_1 \left(\frac{\lambda_2 (p-r-1)!\, \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \right.\right. \\
& \left. \int_0^1 S(\tau,\vartheta_2) g_2(\tau) d\tau \right] a_2(\vartheta_2) \cdots f_{m-1} \left(\frac{\lambda_m (p-r-1)!\, \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} + \mu_m \int_0^1 \left[R(\vartheta_{m-1},\vartheta_m) \right.\right. \\
& \left.\left.\left. + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau,\vartheta_m) g_m(\tau) d\tau \right] a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) \cdots d\vartheta_2 \right) d\vartheta_1 .
\end{aligned}\right\} \tag{7}$$

Lemma 3.1 The operator $\mathfrak{D} : \rho \to \mathfrak{B}$ defined in the equation (7) is a self map on the cone ρ.

Proof. From the positivity of kernels $R(x, \vartheta)$, $S(x, \vartheta)$ and for $w_1 \in \rho$, $\mathfrak{D}w_1(x) \geq 0$ on $x \in [0, 1]$. Now, for $w_1 \in \rho$ and by Lemma 2.2, one can obtain

$$\begin{aligned}
\mathfrak{D}w_1(x) = {} & \frac{\lambda_1 (p-r-1)!\, x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x,\vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1) g_1(\tau) d\tau \right] \\
& a_1(\vartheta_1) f_1 \left(\frac{\lambda_2 (p-r-1)!\, \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \right.\right. \\
& \left. \int_0^1 S(\tau,\vartheta_2) g_2(\tau) d\tau \right] a_2(\vartheta_2) \cdots f_{m-1} \left(\frac{\lambda_m (p-r-1)!\, \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} + \mu_m \int_0^1 \left[R(\vartheta_{m-1},\vartheta_m) \right.\right. \\
& \left.\left.\left. + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau,\vartheta_m) g_m(\tau) d\tau \right] a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) \cdots d\vartheta_2 \right) d\vartheta_1 \\
\leq {} & \frac{\lambda_1 (p-r-1)!}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(1,\vartheta_1) + \frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1) g_1(\tau) d\tau \right] \\
& a_1(\vartheta_1) f_1 \left(\frac{\lambda_2 (p-r-1)!\, \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \right.\right.
\end{aligned}$$

$$\int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau\Bigg]a_2(\vartheta_2)\cdots f_{m-1}\Bigg(\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}$$

$$+\mu_m\int_0^1\Bigg[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\Bigg]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m\Bigg)\cdots d\vartheta_2\Bigg)d\vartheta_1,$$

so that

$$\left.\begin{aligned}
\|\mathfrak{D}w_1(x)\|\le{}&\frac{\lambda_1(p-r-1)!}{(p-1)!(1-\eta_1\theta_1)}+\mu_1\int_0^1\Bigg[R(1,\vartheta_1)+\frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)}\\
&\int_0^1 S(\tau,\vartheta_1)g_1(\tau)d\tau\Bigg]a_1(\vartheta_1)f_1\Bigg(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}\\
&+\mu_2\int_0^1\Bigg[R(\vartheta_1,\vartheta_2)+\frac{\eta_2\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}\int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau\Bigg]\\
&a_2(\vartheta_2)\cdots f_{m-1}\Bigg(\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\\
&+\mu_m\int_0^1\Bigg[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\Bigg]\\
&a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m\Bigg)\cdots d\vartheta_2\Bigg)d\vartheta_1.
\end{aligned}\right\}\tag{8}$$

Next, if $w_1\in\rho$, from Lemma 2.2 and (8), we have that

$$\min_{x\in I}\mathfrak{D}w_1(x)=\min_{x\in I}\Bigg\{\frac{\lambda_1(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_1\theta_1)}+\mu_1\int_0^1\Bigg[R(x,\vartheta_1)+\frac{\eta_1x^{p-1}}{(p-1)!(1-\eta_1\theta_1)}$$

$$\int_0^1 S(\tau,\vartheta_1)g_1(\tau)d\tau\Bigg]a_1(\vartheta_1)f_1\Bigg(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}$$

$$+\mu_2\int_0^1\Bigg[R(\vartheta_1,\vartheta_2)+\frac{\eta_2\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}\int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau\Bigg]$$

$$a_2(\vartheta_2)\cdots f_{m-1}\left(\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\right.\right.$$

$$\left.\left.\left.\left.\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m\right)\cdots d\vartheta_2\right)d\vartheta_1\right\}$$

$$\geq\frac{1}{4^{p-1}}\left\{\frac{\lambda_1(p-r-1)!}{(p-1)!(1-\eta_1\theta_1)}+\mu_1\int_0^1\left[R(1,\vartheta_1)+\frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)}\right.\right.$$

$$\left.\int_0^1 S(\tau,\vartheta_1)g_1(\tau)d\tau\right]a_1(\vartheta_1)f_1\left(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}\right.$$

$$+\mu_2\int_0^1\left[R(\vartheta_1,\vartheta_2)+\frac{\eta_2\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)}\int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau\right]$$

$$a_2(\vartheta_2)\cdots f_{m-1}\left(\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\right.\right.$$

$$\left.\left.\left.\left.\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m\right)\cdots d\vartheta_2\right)d\vartheta_1\right\}$$

$$\geq\frac{1}{4^{p-1}}\|\mathfrak{D}w_1(x)\|.$$

Therefore, $\mathfrak{D}:\rho\to\rho$ and the proof is complete. □

Furthermore, $\mathfrak{D}$ is completely continuous operator based on Arzela-Ascoli theorem [16]. Let us define the positive numbers F_1 and F_2 by

$$F_1=\max_{1\leq i\leq m}\left\{\left(\frac{f_{i\infty}}{4^{2p-2}}\int_{\vartheta\in I}\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_{\tau\in I}S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta\right)^{-1}\right\}\tag{9}$$

and

$$F_2=\min_{1\leq i\leq m}\left\{\left(2f_{i0}\int_0^1\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta\right)^{-1}\right\}.\tag{10}$$

Theorem 3.2 Suppose the conditions mentioned in ($C1$), ($C2$), ($C3$) and ($C4$) are fulfilled. Then, for each μ_1, μ_2, ..., μ_m satisfying

$$F_1<\mu_1<F_2,\ F_1<\mu_2<F_2,\ \ldots,\ F_1<\mu_m<F_2,\tag{11}$$

there exists an m-tuple (w_1, w_2, ..., w_m) satisfying (1) and (2) such that $w_i(x)>0$ on (0, 1] and $\lambda_i\in(0,\infty)$ is sufficiently small for $1 \le i \le m$.

Proof. Let μ_i, $1 \le i \le m$, be given as in (11). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\le i\le m}\left\{\left(\frac{(f_{i\infty}-\epsilon)}{4^{2p-2}}\int_{\vartheta\in I}\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_{\tau\in I}S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta\right)^{-1}\right\}$$

$$\le \min\{\mu_1,\mu_2,\ldots,\mu_m\}$$

and

$$\max\{\mu_1,\mu_2,\ldots,\mu_m\}$$

$$\le \min_{1\le i\le m}\left\{\left(2(f_{i0}+\epsilon)\int_0^1\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta\right)^{-1}\right\}.$$

Now, we seek fixed point of the completely continuous operator $\mathfrak{D}:\rho\to\rho$ defined by (7).

From the definition of f_{i0}, $1 \le i \le m$, there exists an $H_1 > 0$ such that, for each $1 \le i \le m$,

$$f_i(w) \le (f_{i0}+\epsilon)w,\ 0 < w \le H_1.$$

Let λ_i, $1 \le i \le m$, be such that

$$0 < \lambda_i \le \frac{(p-1)!(1-\eta_i\theta_i)H_1}{(p-r-1)!2}.$$

Let $w_1 \in \rho$ with $\| w_1\| = H_1$. By an application of Lemma 2.2 and the choice of ϵ, for $0 \le \vartheta_{m-1} \le 1$, we have

$$\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m$$

$$\le \frac{\lambda_m(p-r-1)!}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(1,\vartheta_m)+\frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)(f_{m0}+\epsilon)w_1(\vartheta_m)d\vartheta_m$$

$$\le \frac{H_1}{2}+\mu_m\int_0^1\left[R(1,\vartheta_m)+\frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)d\vartheta_m(f_{m0}+\epsilon)\| w_1\| \le \frac{H_1}{2}+\frac{H_1}{2}=H_1.$$

It follows in a similar manner from Lemma 2.2 and the choice of ϵ that, for $0 \le \vartheta_{m-2} \le 1$,

$$\frac{\lambda_{m-1}(p-r-1)!\vartheta_{m-2}^{p-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} + \mu_{m-1}\int_0^1 \left[R(\vartheta_{m-2}, \vartheta_{m-1}) + \frac{\eta_{m-1}\vartheta_{m-2}^{p-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} \right.$$

$$\left. \int_0^1 S(\tau, \vartheta_{m-1}) g_{m-1}(\tau) d\tau \right] a_{m-1}(\vartheta_{m-1}) f_{m-1}\left(\frac{\lambda_m (p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \right.$$

$$+\mu_m \int_0^1 \left[R(\vartheta_{m-1}, \vartheta_m) + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau, \vartheta_m) g_m(\tau) d\tau \right]$$

$$\left. a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) d\vartheta_{m-1}$$

$$\le \frac{\lambda_{m-1}(p-r-1)!}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} + \mu_{m-1}\int_0^1 \left[R(1, \vartheta_{m-1}) + \frac{\eta_{m-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} \right.$$

$$\left. \int_0^1 S(\tau, \vartheta_{m-1}) g_{m-1}(\tau) d\tau \right] a_{m-1}(\vartheta_{m-1}) d\vartheta_{m-1} (f_{m-1,0} + \epsilon) H_1$$

$$\le \frac{H_1}{2} + \frac{H_1}{2} = H_1.$$

Continuing with this bootstrapping argument, we have, for $0 \le x \le 1$,

$$\frac{\lambda_1 (p-r-1)! x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x, \vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau, \vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) f_1\left(\frac{\lambda_2 (p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1, \vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \int_0^1 S(\tau, \vartheta_2) g_2(\tau) d\tau \right] \right.$$

$$a_2(\vartheta_2) \cdots f_{m-1}\left(\frac{\lambda_m (p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \right.$$

$$+\mu_m \int_0^1 \left[R(\vartheta_{m-1}, \vartheta_m) + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau, \vartheta_m) g_m(\tau) d\tau \right]$$

$$\left. \left. a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) \cdots d\vartheta_2 \right) d\vartheta_1 \le H_1,$$

so that for $0 \le x \le 1$,

$$\mathfrak{D}w_1(x) \le H_1.$$

Hence, $\|\mathfrak{D}w_1\| \le H_1 = \|w_1\|$. If we set

$$\Omega_1 = \{w \in \mathfrak{B} : \| w\| < H_1\},$$

then

$$\| \mathfrak{D}w_1 \| \le \| w_1 \|, \text{ for } w_1 \in \rho \cap \partial\Omega_1. \tag{12}$$

Next, from the definition of $f_{i\infty}$, $1 \le i \le m$, there exists $\bar{H}_2 \ge 0$, such that, for each $1 \le i \le m$,

$$f_i(w) \ge (f_{i\infty} - \epsilon)w, \ w \ge \bar{H}_2.$$

Let

$$H_2 = \max\{2H_1, 4^{p-1}\bar{H}_2\}.$$

Choose $w_1 \in \rho$ and $\|w_1\| = H_2$. Then

$$\min_{x\in I} w_1(x) \ge \frac{1}{4^{p-1}}\| w_1 \| \ge \bar{H}_2.$$

From Lemma 2.2 and choice of ϵ, for $\frac{1}{4} \le \vartheta_{m-1} \le \frac{3}{4}$, we have

$$\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} + \mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m) + \frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m$$

$$\ge \mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m) + \frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m$$

$$\ge \frac{\mu_m}{4^{p-1}}\int_{\vartheta_m\in I}\left[R(1,\vartheta_m) + \frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)}\int_{\tau\in I} S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)(f_{m\infty} - \epsilon)w_1(\vartheta_m)d\vartheta_m$$

$$\ge \frac{\mu_m}{4^{2p-2}}\int_{\vartheta_m\in I}\left[R(1,\vartheta_m) + \frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)}\int_{\tau\in I} S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)d\vartheta_m(f_{m\infty} - \epsilon)\| w_1 \| \ge \| w_1 \| = H_2.$$

It follows in a similar manner from Lemma 2.2 and the choice of ϵ, for $\frac{1}{4} \leq \vartheta_{m-1} \leq \frac{3}{4}$,

$$\frac{\lambda_{m-1}(p-r-1)!\vartheta_{m-2}^{p-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} + \mu_{m-1}\int_0^1 \left[R(\vartheta_{m-2}, \vartheta_{m-1}) + \frac{\eta_{m-1}\vartheta_{m-2}^{p-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} \right.$$

$$\left. \int_0^1 S(\tau, \vartheta_{m-1}) g_{m-1}(\tau) d\tau \right] a_{m-1}(\vartheta_{m-1}) f_{m-1} \left(\frac{\lambda_m (p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \right.$$

$$+\mu_m \int_0^1 \left[R(\vartheta_{m-1}, \vartheta_m) + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau, \vartheta_m) g_m(\tau) d\tau \right]$$

$$\left. a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) d\vartheta_{m-1}$$

$$\geq \frac{\mu_{m-1}}{4^{p-1}} \int_{\vartheta_{m-1}\in I} \left[R(1, \vartheta_{m-1}) + \frac{\eta_{m-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} \int_{\tau\in I} S(\tau, \vartheta_{m-1}) g_{m-1}(\tau) d\tau \right]$$

$$a_{m-1}(\vartheta_{m-1}) d\vartheta_{m-1} (f_{m-1,\infty} - \epsilon) H_2$$

$$\geq \frac{\mu_{m-1}}{4^{2p-2}} \int_{\vartheta_{m-1}\in I} \left[R(1, \vartheta_{m-1}) + \frac{\eta_{m-1}}{(p-1)!(1-\eta_{m-1}\theta_{m-1})} \int_{\tau\in I} S(\tau, \vartheta_{m-1}) g_{m-1}(\tau) d\tau \right]$$

$$a_{m-1}(\vartheta_{m-1}) d\vartheta_{m-1} (f_{m-1,\infty} - \epsilon) H_2 \geq H_2.$$

Again, using a bootstrapping argument, we have

$$\frac{\lambda_1(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x, \vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau, \vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) f_1 \left(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1, \vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \right. \right.$$

$$\left. \int_0^1 S(\tau, \vartheta_2) g_2(\tau) d\tau \right] a_2(\vartheta_2) \cdots f_{m-1} \left(\frac{\lambda_m (p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \right.$$

$$+\mu_m \int_0^1 \left[R(\vartheta_{m-1}, \vartheta_m) + \frac{\eta_m \vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau, \vartheta_m) g_m(\tau) d\tau \right]$$

$$\left. \left. a_m(\vartheta_m) f_m(w_1(\vartheta_m)) d\vartheta_m \right) \cdots d\vartheta_2 \right) d\vartheta_1 \geq H_2,$$

so that, for $0 \le x \le 1$,

$$\mathfrak{D}w_1(x) \ge H_2 = \| w_1 \|.$$

Hence, $\|\mathfrak{D}w_1\| \ge \|w_1\|$. If we set

$$\Omega_2 = \{ w \in \mathfrak{B} : \| w\| < H_2 \},$$

then

$$\| \mathfrak{D}w_1 \| \ge \| w_1 \|, \text{ for } w_1 \in \rho \cap \partial\Omega_2. \tag{13}$$

Applying Theorem 2.3 to (12) and (13), we obtain that $\mathfrak{D}$ has a fixed point $w_1 \in \rho \cap (\bar{\Omega}_2 \backslash \Omega_1)$. As such setting $w_{m+1} = w_1$, we obtain a positive solution $(w_1, w_2, ..., w_m)$ of (1) and (2), given iteratively by

$$w_i(x) = \frac{\lambda_i (p-r-1)! x^{p-1}}{(p-1)!(1-\eta_i\theta_i)} + \mu_i \int_0^1 \left[R(x,\vartheta) + \frac{\eta_i x^{p-1}}{(p-1)!(1-\eta_i\theta_i)} \int_0^1 S(\tau,\vartheta) g_i(\tau) d\tau \right]$$

$$a_i(\vartheta) f_i(w_{i+1}(\vartheta)) d\vartheta, \; i = m, m-1, \ldots, 1.$$ □

For the next result, we define the positive numbers F_3 and F_4 by

$$F_3 = \max_{1 \le i \le m} \left\{ \left(\frac{f_{i0}}{4^{2p-2}} \int_{\vartheta \in I} \left[R(1,\vartheta) + \frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)} \int_{\tau \in I} S(\tau,\vartheta) g_i(\tau) d\tau \right] a_i(\vartheta) d\vartheta \right)^{-1} \right\} \tag{14}$$

and

$$F_4 = \min_{1 \le i \le m} \left\{ \left(2 f_{i\infty} \int_0^1 \left[R(1,\vartheta) + \frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)} \int_0^1 S(\tau,\vartheta) g_i(\tau) d\tau \right] a_i(\vartheta) d\vartheta \right)^{-1} \right\}. \tag{15}$$

Theorem 3.3 Suppose the conditions mentioned in ($C1$), ($C2$), ($C3$) and ($C4$) are fulfilled. Then, for each $\mu_1, \mu_2, ..., \mu_m$ satisfying

$$F_3 < \mu_1 < F_4, \; F_3 < \mu_2 < F_4, \ldots, F_3 < \mu_m < F_4, \tag{16}$$

there exists an m-tuple $(w_1, w_2, ..., w_m)$ satisfying (1) and (2) such that $w_i(x) > 0$ on $(0, 1]$ and $\lambda_i \in (0, \infty)$ is sufficiently small for $1 \le i \le m$.

Proof. Let μ_i, $1 \le i \le m$, be given as in (16). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1 \le i \le m} \left\{ \left(\frac{(f_{i0} - \epsilon)}{4^{2p-2}} \int_{\vartheta \in I} \left[R(1,\vartheta) + \frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)} \int_{\tau \in I} S(\tau,\vartheta) g_i(\tau) d\tau \right] a_i(\vartheta) d\vartheta \right)^{-1} \right\}$$

$$\le \min\{\mu_1, \mu_2, \ldots, \mu_m\}$$

and

$$\max\{\mu_1, \mu_2, \ldots, \mu_m\}$$

$$\leq \min_{1\leq i\leq m}\left\{\left(2(f_{i\infty}+\epsilon)\int_0^1\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta\right)^{-1}\right\}.$$

Now, we seek fixed point of the completely continuous operator $\mathfrak{D}:\rho\to\rho$ defined in (7).

From the definition of f_{i0}, $1\leq i\leq m$, there exists an $\bar{H}_3>0$ such that, for each $1\leq i\leq m$,

$$f_i(w)\geq(f_{i0}-\epsilon)w,\ \ 0<w\leq\bar{H}_3.$$

Also, from the definition of f_{i0}, it follows that $f_{i0}=0$, $1\leq i\leq m$, and so there exists $0<l_m<l_{m-1}<\cdots<l_2<\bar{H}_3$ such that

$$\mu_i f_i(w)\leq\frac{l_{i-1}}{2\int_0^1\left[R(1,\vartheta)+\frac{\eta_i}{(p-1)!(1-\eta_i\theta_i)}\int_0^1 S(\tau,\vartheta)g_i(\tau)d\tau\right]a_i(\vartheta)d\vartheta},\ w\in[0,l_i],$$

$$0<\lambda_i\leq\frac{(p-1)!(1-\eta_i\theta_i)l_{i-1}}{(p-r-1)!2},\ \text{for } 3\leq i\leq m,$$

and

$$\mu_2 f_2(w)\leq\frac{\bar{H}_3}{2\int_0^1\left[R(1,\vartheta)+\frac{\eta_2}{(p-1)!(1-\eta_2\theta_2)}\int_0^1 S(\tau,\vartheta)g_2(\tau)d\tau\right]a_2(\vartheta)d\vartheta},\ w\in[0,l_2],$$

$$0<\lambda_2\leq\frac{(p-1)!(1-\eta_2\theta_2)\bar{H}_3}{(p-r-1)!2}.$$

Choose $w_1\in\rho$ with $\|w_1\|=l_m$. Then, we have

$$\frac{\lambda_m(p-r-1)!\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(\vartheta_{m-1},\vartheta_m)+\frac{\eta_m\vartheta_{m-1}^{p-1}}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m$$

$$\leq\frac{\lambda_m(p-r-1)!}{(p-1)!(1-\eta_m\theta_m)}+\mu_m\int_0^1\left[R(1,\vartheta_m)+\frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)}\int_0^1 S(\tau,\vartheta_m)g_m(\tau)d\tau\right]$$

$$a_m(\vartheta_m)f_m(w_1(\vartheta_m))d\vartheta_m$$

$$\leq \frac{l_{m-1}}{2} + \frac{\int_0^1 \left[R(1,\vartheta_m) + \frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau,\vartheta_m) g_m(\tau) d\tau \right] l_{m-1} a_m(\vartheta_m) d\vartheta_m}{2\int_0^1 \left[R(1,\vartheta_m) + \frac{\eta_m}{(p-1)!(1-\eta_m\theta_m)} \int_0^1 S(\tau,\vartheta_m) g_m(\tau) d\tau \right] a_m(\vartheta_m) d\vartheta_m}$$

$$= \frac{l_{m-1}}{2} + \frac{l_{m-1}}{2} = l_{m-1}.$$

Continuing with this bootstrapping argument, it follows that

$$\frac{\lambda_2 (p-r-1)! \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \int_0^1 S(\tau,\vartheta_2) g_2(\tau) d\tau \right] a_2(\vartheta_2)$$

$$f_2 \left(\frac{\lambda_3 (p-r-1)! \vartheta_2^{p-1}}{(p-1)!(1-\eta_3\theta_3)} + \mu_3 \int_0^1 \left[R(\vartheta_2,\vartheta_3) + \frac{\eta_3 \vartheta_2^{p-1}}{(p-1)!(1-\eta_3\theta_3)} \int_0^1 S(\tau,\vartheta_3) g_3(\tau) d\tau \right] \right.$$

$$\left. a_3(\vartheta_3) \cdots f_m(w_1(\vartheta_m)) d\vartheta_m \cdots d\vartheta_3 \right) d\vartheta_2 \leq \bar{H}_3.$$

Then

$$\mathfrak{D} w_1(x) = \frac{\lambda_1 (p-r-1)! x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1 \int_0^1 \left[R(x,\vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) f_1 \left(\frac{\lambda_2 (p-r-1)! \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2 \int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2 \vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \int_0^1 S(\tau,\vartheta_2) g_2(\tau) d\tau \right] \right.$$

$$\left. a_2(\vartheta_2) \cdots f_m(w_1(\vartheta_m)) d\vartheta_m \cdots d\vartheta_2 \right) d\vartheta_1$$

$$\geq \frac{\mu_1}{4^{2p-2}} \int_{\vartheta_1 \in I} \left[R(1,\vartheta_1) + \frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)} \int_{\tau \in I} S(\tau,\vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) d\vartheta_1 (f_{10} - \epsilon) \| w_1 \|$$

$$\geq \| w_1 \|.$$

So, $\|\mathfrak{D} w_1\| \geq \|w_1\|$. If we put

$$\Omega_3 = \{ w \in \mathfrak{B} : \| w \| < l_m \},$$

then

$$\| \mathfrak{D}w_1 \| \geq \| w_1 \|, \text{ for } w_1 \in \rho \cap \partial\Omega_3. \tag{17}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leq i \leq m$, is unbounded at ∞. For each $1 \leq i \leq m$, set

$$f_i^*(w) = \sup_{0 \leq s \leq w} f_i(s).$$

Then, it is straightforward that, for each $1 \leq i \leq m$, $f_i^*(w)$ is a non-decreasing real-valued function, $f_i \leq f_i^*$ and

$$\lim_{w\to\infty} \frac{f_i^*(w)}{w} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leq i \leq m$, there exists $\bar{H}_4$ such that, for each $1 \leq i \leq m$,

$$f_i^*(w) \leq (f_{i\infty} + \epsilon)w, \; w \geq \bar{H}_4.$$

It follows that there exists $H_4 > \max\{2\bar{H}_3, \bar{H}_4\}$ such that, for each $1 \leq i \leq m$,

$$f_i^*(w) \leq f_i^*(H_4), \; 0 < w \leq H_4.$$

Let λ_i, $1 \leq i \leq m$, satisfy

$$0 < \lambda_i \leq \frac{(p-1)!(1-\eta_i\theta_i)H_4}{(p-r-1)!2}.$$

Choose $w_1 \in \rho$ with $\|w_1\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\mathfrak{D}w_1(x) = \frac{\lambda_1(p-r-1)!x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} + \mu_1\int_0^1 \left[R(x,\vartheta_1) + \frac{\eta_1 x^{p-1}}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1)g_1(\tau)d\tau \right]$$

$$a_1(\vartheta_1)f_1\left(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2\int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau \right] \right.$$

$$\left. a_2(\vartheta_2)\cdots f_m(w_1(\vartheta_m))d\vartheta_m \cdots d\vartheta_2 \right) d\vartheta_1$$

$$\leq \frac{\lambda_1(p-r-1)!}{(p-1)!(1-\eta_1\theta_1)} + \mu_1\int_0^1 \left[R(1,\vartheta_1) + \frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau,\vartheta_1)g_1(\tau)d\tau \right]$$

$$a_1(\vartheta_1)f_1^*\left(\frac{\lambda_2(p-r-1)!\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} + \mu_2\int_0^1 \left[R(\vartheta_1,\vartheta_2) + \frac{\eta_2\vartheta_1^{p-1}}{(p-1)!(1-\eta_2\theta_2)} \int_0^1 S(\tau,\vartheta_2)g_2(\tau)d\tau \right] \right.$$

$$\left. a_2(\vartheta_2)\cdots f_m(w_1(\vartheta_m))d\vartheta_m \cdots d\vartheta_2 \right) d\vartheta_1$$

$$\leq \frac{H_4}{2} + \mu_1 \int_0^1 \left[R(1, \vartheta_1) + \frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau, \vartheta_1) g_1(\tau) d\tau \right] a_1(\vartheta_1) f_1^*(H_4) d\vartheta_1$$

$$\leq \frac{H_4}{2} + \mu_1 \int_0^1 \left[R(1, \vartheta_1) + \frac{\eta_1}{(p-1)!(1-\eta_1\theta_1)} \int_0^1 S(\tau, \vartheta_1) g_1(\tau) d\tau \right]$$

$$a_1(\vartheta_1) d\vartheta_1 (f_{1\infty} + \epsilon) H_4$$

$$\leq \frac{H_4}{2} + \frac{H_4}{2} = H_4.$$

Hence, $\|\mathfrak{D} w_1\| \leq \|w_1\|$. So, if we set

$$\Omega_4 = \{ w \in \mathfrak{B} : \|w\| < H_4 \},$$

then

$$\|Tw_1\| \leq \|w_1\|, \text{ for } w_1 \in \rho \cap \partial\Omega_4. \tag{18}$$

Applying Theorem 2.3 to (17) and (18), we obtain that $\mathfrak{D}$ has a fixed point $w_1 \in \rho \cap (\bar{\Omega}_4 \backslash \Omega_3)$, which is turn with $w_{m+1} = w_1$, yields an m-tuple $(w_1, w_2, ..., w_m)$ satisfying the boundary value problem (1) and (2). □

4. Examples

Here we consider the examples to illustrate our results.

Example 4.1 Let $p = 3$ and $r = 1$. Consider the problem

$$\left.\begin{aligned} w_1'''(x) + \mu_1 a_1(x) f_1(w_2(x)) = 0, \quad x \in [0,1], \\ w_2'''(x) + \mu_2 a_2(x) f_2(w_3(x)) = 0, \quad x \in [0,1], \\ w_3'''(x) + \mu_3 a_3(x) f_3(w_1(x)) = 0, \quad x \in [0,1], \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} w_1(0) = 0, \quad w_1'(0) = 0, \quad w_1'(1) - \eta_1 \int_0^1 g_1(\tau) w_1'(\tau) d\tau = \lambda_1, \\ w_2(0) = 0, \quad w_2'(0) = 0, \quad w_2'(1) - \eta_2 \int_0^1 g_2(\tau) w_2'(\tau) d\tau = \lambda_2, \\ w_3(0) = 0, \quad w_3'(0) = 0, \quad w_3'(1) - \eta_3 \int_0^1 g_3(\tau) w_3'(\tau) d\tau = \lambda_3, \end{aligned}\right\} \tag{20}$$

where

$$f_1(w_2) = (480.6 - 472.8e^{-w_2})(200 - 194.5e^{-2w_2}) w_2,$$

$$f_2(w_3) = (862.5 - 856.4e^{-3w_3})(168 - 152.6e^{-4w_3}) w_3,$$

$$f_3(w_1) = (376.8 - 368.6e^{-5w_1})(260 - 248.2e^{-2w_1}) w_1,$$

$$a_1(x) = a_2(x) = a_3(x) = 1,$$

$$g_1(\tau) = 1,\ g_2(\tau) = \tau,\ g_3(\tau) = \tau^2,$$

and

$$\eta_1 = 1,\ \eta_2 = \frac{1}{2},\ \eta_3 = \frac{1}{3}.$$

The kernels $R(x, \vartheta)$ and $S(\tau, \vartheta)$ are given by

$$R(x, \vartheta) = \frac{1}{2!}\begin{cases} \left[x^2(1-\vartheta) - (x-\vartheta)^2\right], & 0 \le \vartheta \le x \le 1, \\ x^2(1-\vartheta), & 0 \le x \le \vartheta \le 1, \end{cases}$$

and

$$S(\tau, \vartheta) = \begin{cases} \vartheta(1-\tau), & 0 \le \vartheta \le \tau \le 1, \\ \tau(1-\vartheta), & 0 \le \tau \le \vartheta \le 1. \end{cases}$$

By direct calculation, we found that

$$\theta_1 = \frac{1}{2},\ \theta_2 = \frac{1}{3},\ \theta_3 = \frac{1}{4},$$

$$f_{10} = 42.8,\ f_{20} = 93.94,\ f_{30} = 96.76,$$

$$f_{1\infty} = 96120,\ f_{2\infty} = 144900,\ f_{3\infty} = 97968,$$

$$F_1 = \max\{0.030080047, 0.02271517303, 0.03128607314\} = 0.03128607314$$

and

$$F_2 = \min\{0.07009345794, 0.05553961363, 0.00588017277\} = 0.05553961363.$$

Applying Theorem 3.2, we get an eigenvalue interval $0.03128607314 < \mu_i < 0.05553961363$, $i = 1, 2, 3$ for which the boundary value problem (19) and (20) have at least one positive solution by choosing λ_1, λ_2 and λ_3 are sufficiently small.

Example 4.2 Let $p = 3$ and $r = 1$. Consider the problem

$$\left.\begin{aligned} w_1'''(x) + \mu_1 a_1(x) f_1(w_2(x)) = 0, \quad x \in [0,1], \\ w_2'''(x) + \mu_2 a_2(x) f_2(w_3(x)) = 0, \quad x \in [0,1], \\ w_3'''(x) + \mu_3 a_3(x) f_3(w_1(x)) = 0, \quad x \in [0,1], \end{aligned}\right\} \tag{21}$$

$$
\left.\begin{aligned}
w_1(0)=0,\quad w_1'(0)=0,\quad w_1'(1)-\eta_1\int_0^1 g_1(\tau)w_1'(\tau)d\tau=\lambda_1,\\
w_2(0)=0,\quad w_2'(0)=0,\quad w_2'(1)-\eta_2\int_0^1 g_2(\tau)w_2'(\tau)d\tau=\lambda_2,\\
w_3(0)=0,\quad w_3'(0)=0,\quad w_3'(1)-\eta_3\int_0^1 g_3(\tau)w_3'(\tau)d\tau=\lambda_3,
\end{aligned}\right\} \tag{22}
$$

where

$$f_1(w_2)=150000\left|sin\left(\frac{5w_2}{2}\right)\right|+35.4w_2e^{-\frac{1}{w_2^2}},$$

$$f_2(w_3)=560000\left|sin\left(\frac{5w_3}{2}\right)\right|+58.2w_3e^{-\frac{1}{w_3}},$$

$$f_3(w_1)=100000\left|sin\left(\frac{5w_1}{2}\right)\right|+65.2w_1e^{-\frac{1}{w_1^2}},$$

$$a_1(x)=a_2(x)=a_3(x)=1,$$

$$g_1(\tau)=1,\ g_2(\tau)=\tau,\ g_3(\tau)=\tau^2,$$

and

$$\eta_1=1,\ \eta_2=\frac{1}{2},\ \eta_3=\frac{1}{3}.$$

By direct calculation, we found that

$$\theta_1=\frac{1}{2},\ \theta_2=\frac{1}{3},\ \theta_3=\frac{1}{4},$$

$$f_{10}=375000,\ f_{20}=560000,\ f_{30}=650000,$$

$$f_{1\infty}=35.4,\ f_{2\infty}=58.2,\ f_{3\infty}=65.2,$$

$$F_3=\max\{0.007710117, 0.007836734, 0.006287249\}=0.007836734$$

and

$$F_4=\min\{0.084745762, 0.089645898, 0.087264649\}=0.084745762.$$

Applying Theorem 3.3, we get an eigenvalue interval $0.007836734<\mu_i<0.084745762$, $i=1, 2, 3$ for which the boundary value problem (21) and (22) have at least one positive solution by choosing λ_1, λ_2 and λ_3 are sufficiently small.

5. Conclusion

We obtained the eigenvalue intervals of the parameters that provide the existence of positive solutions to the boundary value problem with non-homogeneous integral boundary conditions by an application of the Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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