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Explicit Solution for a Family of Hermann Riccati Differential Equations

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Abstract: This paper deals with a parametric family of ordinary differential equations originated by an ancient geometric problem posed in 1719 by the Swiss mathematician Jakob Hermann (1678-1733) and partially solved by Jacopo Vincenzo Riccati (1676-1754). We translate Riccati approach, called by himself dimidiata separazione (splitted, or halved, separation), in modern terms and we give the full solution of the Hermann problem completing the partial solution provided by Riccati.

***Keywords*:** Hermann problem, two steps separation, change of variables, Lambert function

MSC: 34A05, 01A50, 33B99

1. Introduction

The separation of variables for a scalar nonlinear first order ordinary differential equations is, in general, a problem that often results analytically intractable. In some well known and particular situations, separation is obtainable, according to some specific structure of the equation itself, searching for an integrating factor. When the integrating factor depends on both (independent and dependent) variables, say (x, y), some old-school textbooks like for instance [1] pages 50-51, [2] pages 53-55 or [3] chapter 6, use an "inspection method", otherwise "method of grouping" to detect it; this is possible when the given equation presents some recognizable pattern. As a matter of fact, this grouping technique was indeed introduced for the first time ever by the Italian mathematician Jacopo Vincenzo Riccati, in the treatise *Della separazione delle indeterminate nelle equazioni differenziali del primo grado e della riduzione delle equazioni differenziali del secondo e di altri gradi ulteriori* [The separation of indeterminates in differential equations of the first degree and the reduction of differential equations of the second and further degrees] which was originated by his lectures on the differential equations. Although not directly involved in teaching activities, Riccati willingly instructed, in addition to his children, some interested talented students, including Maria Gaetana Agnesi (1718-1799). Riccati, around 1710, communicated to Hermann, who was teaching at the University of Padua at the time, his first attempts to find a general method for the separation of variables. This method was completed in 1714 and was first presented in an essay sent to the scientist who was the main reference for "continental" European mathematicians at that time: Gottfried Wilhelm von Leibniz (1646-1716). The definitive exposition of the separation method was printed in 1757 by his son Giordano in the first of four tomes Opera Omnia [4], dedicated to his lectures on differential equations. Riccati called

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his method “dimidiata separazione” that can be translated as “two steps separation” or “halved separation”. A quick but very interesting scientific biography of Jacopo Riccati, is available (in Italian) from the University of Pisa, Ennio De Giorgi Research Centre [http://mathematica. sns.it/autori/1349/].

In a recent article [5], we gave an account of the two steps separation, applying it to a class of differential equations:

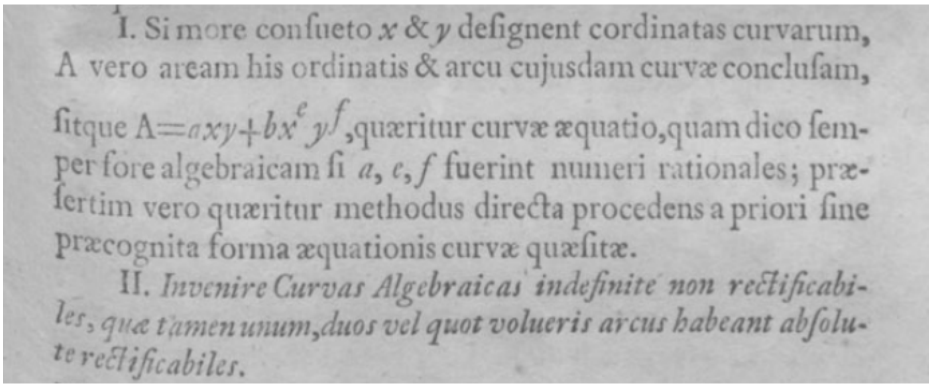
$$y^{m-2}\left(x\mathrm{d}x + y\mathrm{d}y\right) = x^{n}\,\frac{\left(y\mathrm{d}x - x\mathrm{d}y\right)}{y^{2}}.$$

In that paper, we extended Riccati’s results, thanks also to the reinterpretation of the methods used at the time in terms of double change of variables and Lie symmetries, to a more general family of equations. Our letter goes in the same direction considering a parametric family of differential equations, studied by Riccati in order to solve the geometrical problem, posed by Hermann in [6]. In this way, we revisit the method introduced by Riccati in the eighteenth century, in the light of current knowledge about variable transformations in differential equations and the computational power of computer algebra. Clearly, our approach remains symbolic in nature, as a matter of fact, founded on tailor-made procedures that depend on the structure of the problem: in its real essence, it is a problem of finding an integrating factor for a first-order nonlinear ordinary differential equation.

2. Materials and methods

2.1 *Revisiting the Riccati contribution*

At page 490 of [4], Riccati presents a problem proposed in 1719 by Jakob Hermann [The article is available at [https://www.beic.it/it/articoli/biblioteca-digitale] the BEIC web site using the research key Acta Eruditorum, Calendis Augusti 1719 page 361], see [6] and Figure 1. Translating in modern terms Hermann required those curves such that their area from (x_0, y_0) to (x, y) could be measured by an algebraic function of both Cartesian coordinates, say: $axy + bx^c y^f$ where a, b, c and f are fixed real numbers.



I. Si more consueto x & y designent cordinatas curvarum, A vero aream his ordinatis & arcu cujusdam curvæ conclusam, sitque A$=axy+bx^c y^f$, quæritur curvæ æquatio, quam dico semper fore algebraicam si a, c, f fuerint numeri rationales; præsertim vero quæritur methodus directa procedens a priori sine præcognita forma æquationis curvæ quæsitæ.

II. *Invenire Curvas Algebraicas indefinite non rectificabiles, quæ tamen unum, duos vel quot volueris arcus habeant absolute rectificabiles.*

Figure 1. The original statement of the problem by Hermann

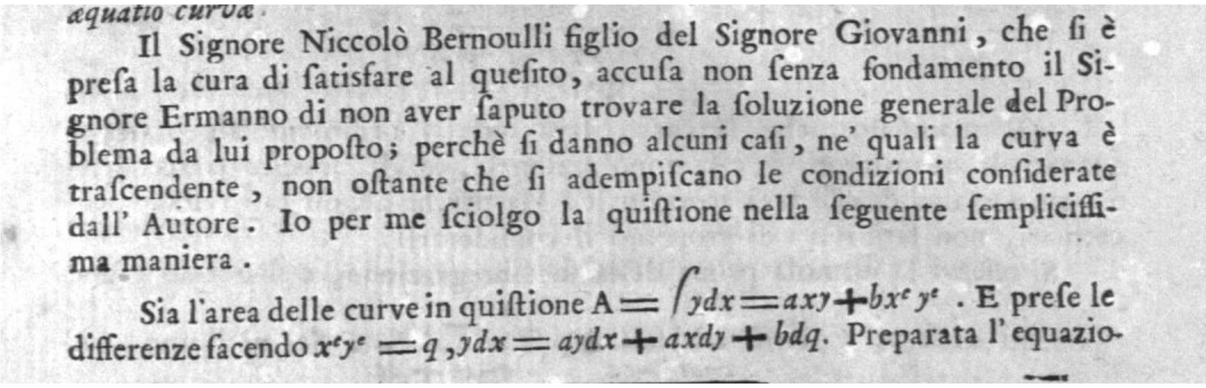
Later Riccati, facing the problem in [4], see Figure 2 assumes the values of the two exponents of the monomial $x^c y^f$ are equal to c. Anyhow, first we treat the problem following Riccati and then we will face the general case, $c \neq f$ as stated by Hermann. The generality is not affected choosing $x_0 = 0$, so we can formulate the Hermann problem as:

Single out a real valued function of one real variable, say $y = y(x)$, of class $\mathcal{C}^{(1)}$ such that:

$$\int_0^x y(u)\mathrm{d}u = axy + bx^c y^f \tag{1}$$

where we assume a, b, c, f given real numbers.

As we said, Riccati faced the problem (1) assuming $c = f$, this as we will see simplifies the relevant treatment. We start exposing Riccati method [Once again from the already addressed BEIC website with the key “Opere del conte Jacopo Riccati, nobile trevigiano, tomo I” one can view the original work on page 490], providing some modern improvements and, in the next section, we solve the original Hermann problem (1).



æquatio curvæ.

Il Signore Niccolò Bernoulli figlio del Signore Giovanni, che ſi è preſa la cura di ſatisfare al queſito, accuſa non ſenza fondamento il Signore Ermanno di non aver ſaputo trovare la ſoluzione generale del Problema da lui propoſto; perchè ſi danno alcuni caſi, ne' quali la curva è traſcendente, non oſtante che ſi adempiſcano le condizioni conſiderate dall' Autore. Io per me ſciolgo la quiſtione nella ſeguente ſempliciſſima maniera.

Sia l'area delle curve in quiſtione $A = \int ydx = axy + bx^e y^e$. E preſe le differenze facendo $x^e y^e = q$, $ydx = aydx + axdy + bdq$. Preparata l'equazio-

Figure 2. Riccati statement of the problem

Differentiating both sides of (1) we get:

$$y = axy' + ay + bcx^{c-1}y^c + bcx^c y^{c-1}y'. \tag{2}$$

In normal form (2) reads as:

$$y' = \frac{(1-a)y - bcx^{c-1}y^c}{ax + bcx^c y^{c-1}}. \tag{3}$$

Riccati, instead, used Pfaffian form in this way, grouping the terms of the equation as:

$$axy\left(\frac{1-a}{a}\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}y}{y}\right) = bc\left(x^{c-1}y^c\mathrm{d}x + x^c y^{c-1}\mathrm{d}y\right). \tag{4}$$

The structure of (4) suggested by Riccati, here we translate his approach in modern terms, to use the change of variables

$$\begin{cases} Y(x, y) = x^{\frac{1-a}{a}} y^{-1} \\ X(x, y) = x^c y^c \end{cases} \Leftrightarrow \begin{cases} x = Y^a X^{a/c} \\ y = Y^{-a} X^{\frac{1-a}{c}} \end{cases},$$

which indeed separates equation (3), arriving at:

$$Y' := \frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{D_x[Y(x,\ y(x))]}{D_x[X(x,\ y(x))]} = \frac{b}{a}YX^{-1/c}. \tag{5}$$

If $c \neq 1$ using the integration constant k solution to (5) is easily obtainable by separation of variables:

$$Y = k\exp\left(\frac{bc}{a(c-1)}X^{\frac{c-1}{c}}\right).$$

Going back to the original variables in (5) we get solution of (3) assuming $c \neq 1$

$$\frac{x^{\frac{1-a}{a}}}{y} = k\exp\left(\frac{bc}{a(c-1)}x^{c-1}y^{c-1}\right). \tag{6}$$

Observe that, using the Lambert W function, which is indeed the multivalued inverse of $\ell : \mathbb{R} \to [-e^{-1}, +\infty)$, $\ell(u) = ue^u$ (for more informations one can see [7] or [8]), we can solve for y from (6), starting from an equation, in the unknown y of the form

$$H = y\exp(Ay^{\beta}) \Leftrightarrow y^{\beta} = \frac{1}{\beta A}W(\beta AH^{\beta}),$$

thus, specifying the constants, using k to denote an integration constant, we arrive at:

$$y = \frac{1}{x}\left(\frac{bc}{a}\right)^{\frac{1}{1-c}}W\left(\frac{bck}{a}x^{\frac{c-1}{a}}\right)^{\frac{1}{c-1}}. \tag{7}$$

Here it is important to remark that, since the Lambert function is indeed multivalued, when specific values of parameters and the initial value are fixed, it is necessary to individuate which of the branches of the Lambert W function is involved in the process.

It is interesting to note that the solution obtained is validated by symbolic calculation, specifically Mathematica®, as one can check by downloading the file [https://www.dropbox.com/s/uyqo0xh2gw1kaxb/verificationfile.nb?dl=0]. It also seems to us extremely interesting to verify that the function determined here satisfies Hermann's condition (1), in fact, integrating the right hand side of (7), using the change of variable

$$x = \left(\frac{aue^u}{bck}\right)^{\frac{a}{c-1}},$$

we arrive at

$$\left(\frac{bc}{a}\right)^{\frac{1}{1-c}}\int\frac{1}{x}W\left(\frac{bck}{a}x^{\frac{c-1}{a}}\right)^{\frac{1}{c-1}}\mathrm{d}x = \frac{c^{\frac{1}{1-c}}a^{\frac{c}{c-1}}b^{\frac{1}{1-c}}}{c-1}\int(u+1)u^{\frac{1}{c-1}-1}\,\mathrm{d}u.$$

The integral at right hand side is elementary, thus after its computation, going back to the original variable we arrive at

$$\left(\frac{bc}{a}\right)^{\frac{1}{1-c}}\int \frac{1}{x}W\left(\frac{bck}{a}x^{\frac{c-1}{a}}\right)^{\frac{1}{c-1}}\mathrm{d}x=\frac{a\left(\frac{bc}{a}\right)^{\frac{1}{1-c}}}{c}W\left(\frac{bck}{a}x^{\frac{c-1}{a}}\right)^{\frac{1}{c-1}}\left(c+W\left(\frac{bck}{a}x^{\frac{c-1}{a}}\right)\right).$$

At this point (1) follows immediately. This equality is also confirmed symbolically with Mathematica, as can be verified using the file indicated in the previous link.

Eventually for $c = 1$ integration of (3) is trivial leading to:

$$y=kx^{\frac{1-a-b}{a+b}}.$$

3. Results and discussion

3.1 *General treatment of Hermann problem*

Hermann problem (1) in the general case leads to the ordinary differential equation

$$y'=\frac{(1-a)y-bcx^{c-1}y^{f}}{ax+bfx^{c}y^{f-1}}. \tag{8}$$

To solve (8) we slightly modify the Riccati's change of variable

$$\begin{cases} Y(x,\,y)=x^{\frac{1-a}{a}}y^{-1} \\ X(x,\,y)=x^{c}y^{f} \end{cases} \Leftrightarrow \begin{cases} x=\left(XY^{f}\right)^{\frac{a}{ac-af+f}} \\ y=\left(X^{1-a}Y^{-ac}\right)^{\frac{1}{ac-af+f}} \end{cases}, \tag{9}$$

obtaining the transformed separable differential equation

$$Y':=\frac{\mathrm{d}Y}{\mathrm{d}X}=\frac{b}{a}Y^{\frac{2ac-2af+f}{ac-af+f}}X^{-\frac{1}{ac-af+f}}. \tag{10}$$

So that separating the variables in (10) we arrive at

$$\frac{(a(c-f)+f)}{a(f-c)}Y^{\frac{a(f-c)}{a(c-f)+f}}=\frac{b(ac-af+f)}{c(ac-af+f-1)}X^{\frac{ac-af+f-1}{ac-af+f}}+k,$$

being k an integration constant. Going back to the original variables using (9), we can express implicitly the solution to (8):

$$\frac{b}{ac-af+f-1}\left(x^{c}y^{f}\right)^{\frac{ac-af+f-1}{ac-af+f}}+\frac{1}{c-f}\left(x^{a-1}y^{a}\right)^{\frac{c-f}{ac-af+f}}=k. \tag{11}$$

Observe that having, assumed $c \neq f$ it is generally not possible in this situation to explicitly derive the dependent variable y from (11).

We present an example, fixed the values of the parameters, $a = 2$, $b = 3$, $c = 4$, $f = 2$ and the initial conditions $x_0 = 1$, $y_0 = 1$ in which we superimpose the implicit solution expressed by the formula (11) with the numerical solution of (8) obtained with Mathematica®, see Figure 3: the implicit solution obtained using Riccati's method drawn in red, while the numerical solution is represented in black, with a slightly thickened line.

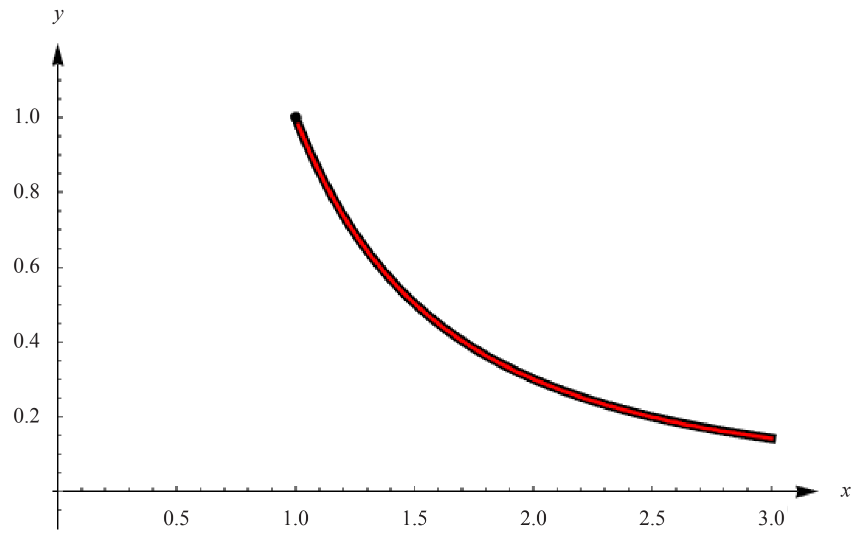


Figure 3. Superposition between the implicit and numerical solution to (8)

This numerical simulation is available in the verification file accompanying the article. To conclude notice that in (11) some occurrences of the parameters are not admitted, namely:

(i) $ac - af + f - 1 = 0$

(ii) $ac - af + f = 0$

while the case $c - f = 0$ was discussed formerly. The particular situations (i) and (ii) result into elementary solutions, which we provide in detail below for the sake of completeness.

If (i) holds true, then (8) becomes:

$$y' = -\frac{y\left(c-1-bc(c-f)x^{c-1}y^{f-1}\right)}{x\left(f-1-bf(c-f)x^{c-1}y^{f-1}\right)}. \tag{12}$$

With the change of variable $y = x^{\alpha}u$ and then selecting α properly we obtain the solution of (12) using elementary methods:

$$x^{\alpha}u' + \alpha u x^{\alpha-1} = -\frac{x^{\alpha-1}u\left(c-1-bc(c-f)x^{c+\alpha(f-1)-1}u^{f-1}\right)}{f-1-bf(c-f)x^{c+\alpha(f-1)-1}u^{f-1}}. \tag{13}$$

Assume first $f-1 \neq 0$. In (13) we can choose α so that $c+\alpha(f-1)-1=0$, then (13) becomes

$$x^{\frac{1-c}{f-1}}u' + \frac{(1-c)x^{\frac{1-c}{f-1}-1}}{f-1}u = -\frac{x^{\frac{1-c}{f-1}-1}u\left(c-1-bc(c-f)u^{f-1}\right)}{f-1-bf(c-f)u^{f-1}}. \tag{14}$$

Writing (14) in normal form, we arrive at the (separable) equation

$$u' = \frac{b(f-c)^2}{(1-f)x}\frac{u^{f+1}}{bf(f-c)u^f-(1-f)u},$$

whose integral is

$$\frac{b(c-f)^2}{1-f}\ln x = bf(f-c)\ln u - u^{1-f} + k. \tag{15}$$

From (15) going back to the original variables, using (i), with $f \neq 1$, solution of (8) is found, in implicit form, using k to represent the relevant integration constant, as:

$$x^{1-c}y^{1-f} = b(f-c)(c\ln x + f\ln y) + k. \tag{16}$$

It is indeed possible, using again the Lambert W, to obtain y from (16), taking full advantage of the symbolic calculus provided by Mathematica®:

$$y = \left(\frac{bf(c-f)}{1-f}\right)^{\frac{1}{1-f}} x^{\frac{c-1}{1-f}}\left(W\left(\frac{(1-f)}{bf(c-f)}e^{-\frac{(f-1)k}{bf(c-f)}}x^{\frac{f-c}{f}}\right)\right)^{\frac{1}{1-f}}.$$

When $f = 1$ equation (12) is separable, it reduces at:

$$y' = y\left(\frac{1}{bx^c}-\frac{c}{x}\right) \text{ with solution } y = \frac{k}{x^c}\exp\left[\frac{x^{1-c}}{b(1-c)}\right].$$

Eventually when (ii) holds, solution is by far easier. In fact, if we assume $a = f/(f-c)$ equation (8) is elementary:

$$y' = -\frac{c}{f}\frac{y}{x} \text{ with solution } y = kx^{-\frac{c}{f}}.$$

4. Conclusions

Starting from a differential equation, arising from the solution of a geometric problem proposed in 1719 by Hermann and partially solved by Jacopo Vincenzo Riccati, we have produced the complete integration of a four parameters family of first order nonlinear differential equations, highlighting the particular cases where the solutions are already known, adding to these the solutions founded by generalizing the Riccati "splitted separation" method: solution (11) of the equation (8), for the best of our knowledge, does not appear in the main repertoires, like [9], [10] or [11]. Often, going back to the original contributions of the great mathematicians of the past, allows, and this is indeed the case, a posthumous revisiting, which in the light of the knowledge subsequently acquired, improves the results then obtained.

Conflicts of interest statement

Authors declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Riccati's lectures on differential equations have been highlighted to me by G. Mingari Scarpello: I take the opportunity to thank him warmly. To access Riccati's book [4] and Hermann's paper [6] we took full advantage of the on-line resources provided by European library of information and culture [https://www.beic.it/it/articoli/biblioteca-digitale].

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