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Research Article

$C^1(\mathbb{R}^N)$ Versus $W^{1,p}(\mathbb{R}^N)$ Local Minimizers

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Abstract: This paper deals with the energy functional associated with a quasilinear elliptic equation in $\mathbb{R}^N$ which is driven by the p-Laplacian operator. It is shown for such functional that any $C^1(\mathbb{R}^N)$ local minimizer in an appropriate sense is a $W^{1,p}(\mathbb{R}^N)$ local minimizer. This extends to $\mathbb{R}^N$ the celebrated property of Brezis-Nirenberg type known for bounded domains.

***Keywords*:** quasilinear elliptic equation in $\mathbb{R}^N$, p-Laplacian, energy functional, local minimizer, Sobolev spaces

MSC: 35J20, 35B38, 35J91

1. Introduction

For the variational theory of nonlinear elliptic boundary value problems, the relationship between the local minimizers of the corresponding energy functionals in spaces of smooth functions with respect to Sobolev spaces is fundamental. It permits to pass from local pointwise estimates to global weak formulations regarding the solution set. The study started with [1] in the case of $C_0^1(\overline{\Omega})$ versus $W_0^{1,2}(\Omega)$ local minimizers when Ω is a bounded domain in $\mathbb{R}^N$. The extension to the spaces $W_0^{1,p}(\Omega)$, with $1 < p < \infty$, was done in [2]. This extension underlies the idea of passing from the ordinary Laplacian Δ to the p-Laplacian Δ_p in the corresponding Dirichlet problem. There are various results of this type applicable to different contexts (see, e.g., [3-6]). Major applications of such results on a bounded domain $\Omega \subset \mathbb{R}^N$ concern for instance the enclosure of the solution in the ordered interval formed by a sub-supersolution (see [1]) or the location of a sign-changing solution by means of extremal constant-sign solutions (see [7]).

In all the mentioned references, the domain $\Omega \subset \mathbb{R}^N$ for the formulated boundary value problem is supposed to be bounded. Clearly, it is difficult to drop this condition due to the lack of compactness for the embeddings of the related function spaces. On the other hand, it is expected that finding a way of handling this issue would have a major impact on boundary value problems on unbounded domains. The only available result of Brezis-Nirenberg type for an unbounded domain is the one in [8] that we now briefly describe. Let the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, for $N \geq 3$, which is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\||\nabla u|\|_{L^2(\mathbb{R}^N)}$, and consider its closed subspace

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$$V = \left\{ v \in \mathcal{D}^{1,2}(\mathbb{R}^N) : v \in C(\mathbb{R}^N) \text{ and } \| v \|_V := \sup_{x \in \mathbb{R}^N} (1 + |x|^{N-2}) |v(x)| < \infty \right\}$$

endowed with the norm $\|v\|_V$. Under some hypotheses, it is shown in [8] that for the energy functional associated with a semilinear elliptic equation in $\mathbb{R}^N$, the local minimizers in the V topology are local minimizers in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

The aim of the present paper is to address the natural question of local minimizers for the energy functionals associated with quasilinear elliptic equations in $\mathbb{R}^N$ regarding the Sobolev space $W^{1,p}(\mathbb{R}^N)$ versus the much smaller space $W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. A major difficulty is the lack of compactness in the Sobolev embedding theorem over $\mathbb{R}^N$. In this respect, it is worth mentioning that in addition to the L^p-norm of the gradient we include in the functional a term describing the L^p-norm of the function, which makes a striking difference in regard to what happens for the corresponding problem on a bounded domain. We emphasize the decisive part played by a weight $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ to guarantee for the lower order term the required integrability properties. A relevant part of this study focuses on the regularity of minimizers. More precisely, we establish that an element of $W^{1,p}(\mathbb{R}^N)$ which is a $C^1(\mathbb{R}^N)$ local minimizer in the sense of the subsequent Definition 1 is necessarily an element of $C^1(\mathbb{R}^N)$. Then we prove the main statement that a $C^1(\mathbb{R}^N)$ local minimizer in the sense of Definition 1 is a $W^{1,p}(\mathbb{R}^N)$ local minimizer. Our approach relies on the Lagrange multiplier rule and estimates regarding the interaction between the minimizer and the multiplier. A passing to the limit process is developed on the basis of a sequence of expanding bounded domains covering $\mathbb{R}^N$. It is worth noting that we build a completely different functional setting with respect to the previous results. In particular, compared to [8], here we pass from the equations in $\mathbb{R}^N$ driven by the ordinary Laplacian Δ to the equations in $\mathbb{R}^N$ with p-Laplacian as leading operator.

The rest of the paper is organized as follows. Section 2 contains the necessary background and the statement of our result. Section 3 consists of the proof of the result.

2. Background and statement of result

Let a real number $p > 1$ and let an integer $N \geq 2$ be such that $1 < p < N$. Throughout this paper, we argue upon the Sobolev space

$$W^{1,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx < \infty \right\},$$

which is a Banach space endowed with the norm $\| u \|_{W^{1,p}(\mathbb{R}^N)}$ given by

$$\| u \|_{W^{1,p}(\mathbb{R}^N)} := (\| u \|^p_{L^p(\mathbb{R}^N)} + \| \nabla u \|^p_{L^p(\mathbb{R}^N)})^{\frac{1}{p}}.$$

Setting the critical exponent

$$p^* = \frac{Np}{N-p},$$

there is the continuous embedding $W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$.

Let $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the growth condition

$$|g(s)| \le c(1+|s|^{r-1}),\ \forall s \in \mathbb{R}, \tag{1}$$

with constants $c > 0$ and $p < r < p^*$. Denoting

$$G(s) = \int_0^s g(t)dt,\ \forall s \in \mathbb{R},$$

it follows that

$$|G(s)| \le c|s| + \frac{c}{r}|s|^r \le c + \frac{c+1}{r}|s|^r,\ \forall s \in \mathbb{R}. \tag{2}$$

Consider the functional $\Phi : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + |u|^p)dx - \int_{\mathbb{R}^N} a(x)G(u)dx,\ \forall u \in W^{1,p}(\mathbb{R}^N). \tag{3}$$

For the functional Φ we introduce the following notion of C^1 local minimizer.

Definition 1 We say that $\hat{u} \in W^{1,p}(\mathbb{R}^N)$ is a $C^1(\mathbb{R}^N)$ local minimizer of the functional Φ if there exists a sequence $\{\Omega_n\}$ of open subsets of $\mathbb{R}^N$, with $\overline{\Omega_n}$ compact, the boundary $\partial\Omega_n$ of class C^2, $\overline{\Omega_n} \subset \Omega_{n+1}$ and $\cup\Omega_n = \mathbb{R}^N$, such that for every sequence $\{h_m\} \subset W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ with $h_m \to 0$ in $C^1(\overline{\Omega_n})$, for all n, as $m \to \infty$, there exists an integer m_0 for which

$$\Phi(\hat{u}) \le \Phi(\hat{u} + h_m),\ \forall m \ge m_0.$$

Our main result reads as follows.

Theorem 2 Assume that condition (1) is satisfied. If $\hat{u} \in W^{1,p}(\mathbb{R}^N)$ is a $C^1(\mathbb{R}^N)$ local minimizer in the sense of Definition 1 for the functional Φ, then $\hat{u} \in C^1(\mathbb{R}^N)$ and $\hat{u}$ is a local minimizer for Φ on $W^{1,p}(\mathbb{R}^N)$, i.e., there exists a constant $\delta > 0$ such that

$$\Phi(\hat{u}) \le \Phi(\hat{u} + u),\ \forall u \in W^{1,p}(\mathbb{R}^N),\ \|u\|_{W^{1,p}(\mathbb{R}^N)} \le \delta.$$

In the proof of Theorem 2 we will need the (negative) p-Laplacian $-\Delta_p : W^{1,p}(\mathbb{R}^N) \to W^{1,p}(\mathbb{R}^N)^*$ which is defined as

$$\langle -\Delta_p u, v\rangle_{W^{1,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx \tag{4}$$

for all $u, v \in W^{1,p}(\mathbb{R}^N)$. Actually, the operator $-\Delta_p : W^{1,p}(\mathbb{R}^N) \to W^{1,p}(\mathbb{R}^N)^*$ is the Fréchet differential of the C^1 functional $J : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx,\ \forall u \in W^{1,p}(\mathbb{R}^N).$$

On the basis of (3) and (4), it is seen that the functional Φ is the energy function associated with the quasilinear equation

$$-\Delta_p u + |u|^{p-2}u = a(x)g(u) \text{ in } \mathbb{R}^N.$$

The critical points $u \in W^{1,p}(\mathbb{R}^N)$ of the functional Φ are exactly the weak solutions of the preceding quasilinear equation provided the nonlinearity $g \in C(\mathbb{R})$ fulfills the growth condition (1).

3. Proof of Theorem 2

Arguing by contradiction, suppose that $\hat{u} \in W^{1,p}(\mathbb{R}^N)$ is not a local minimizer for the functional Φ. Then, for each $\varepsilon > 0$, it holds

$$m_\varepsilon := \inf_{h \in D_r(0,\varepsilon)} \Phi(\hat{u}+h) < \Phi(\hat{u}), \tag{5}$$

where

$$D_r(0,\varepsilon) := \{v \in W^{1,p}(\mathbb{R}^N) : \|v\|_{L^r(\mathbb{R}^N)} \le \varepsilon\}.$$

Here r stands for the number introduced in condition (1) with $p < r < p^*$. Inequality (5) is true due to the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ (see, e.g., [9, Corollary 9.10]), so $D_r(0, \varepsilon)$ contains an arbitrarily small neighborhood of 0 in $W^{1,p}(\mathbb{R}^N)$ provided $\varepsilon > 0$ is sufficiently small.

We claim that any minimizing sequence of (5) is bounded in $W^{1,p}(\mathbb{R}^N)$. Indeed, let $\{h_n\} \subset D_r(0, \varepsilon)$ be a minimizing sequence of (5). We infer that $\{h_n\}$ is bounded in $L^r(\mathbb{R}^N)$ and

$$\Phi(\hat{u}+h_n) < \Phi(\hat{u})$$

provided n is sufficiently large. Then, by (2), (3) and Hölder's inequality, we see that

$$\begin{aligned}
\frac{1}{p}\int_{\mathbb{R}^N} (|\nabla(\hat{u}+h_n)|^p + |\hat{u}+h_n|^p)dx &= \Phi(\hat{u}+h_n) + \int_{\mathbb{R}^N} aG(\hat{u}+h_n)dx \\
&< \Phi(\hat{u}) + \int_{\mathbb{R}^N} |a||G(\hat{u}+h_n)|\,dx \\
&\le \Phi(\hat{u}) + c\|a\|_{L^{\frac{r}{r-1}}(\mathbb{R}^N)} \|\hat{u}+h_n\|_{L^r(\mathbb{R}^N)} + \frac{c}{r}\|a\|_{L^\infty(\mathbb{R}^N)} \|\hat{u}+h_n\|^r_{L^r(\mathbb{R}^N)} \le C,
\end{aligned}$$

with a constant $C > 0$. The estimate above ensures that the claim holds true.

Now we show that m_ε is attained in (5). By the preceding claim we can fix a minimizing sequence $\{h_n\}$ for (5) which is bounded in $W^{1,p}(\mathbb{R}^N)$. Passing to a relabeled subsequence we have the weak convergence $h_n \rightharpoonup h_\varepsilon$ in $W^{1,p}(\mathbb{R}^N)$, with some $h_\varepsilon \in W^{1,p}(\mathbb{R}^N)$. The continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ implies $h_n \rightharpoonup h_\varepsilon$ in $L^r(\mathbb{R}^N)$. The lower semicontinuity of the norm on $L^r(\mathbb{R}^N)$ renders $h_\varepsilon \in D_r(0, \varepsilon)$. For every compact set $K \subset \mathbb{R}^N$, we can find a subsequence of $\{h_n\}$ converging to h_ε in $L^r(K)$. Covering $\mathbb{R}^N$ with a countable family of compact sets and using the diagonal process, we are able to get a relabeled subsequence of $\{h_n\}$ such that $h_n \to h_\varepsilon$ almost everywhere in $\mathbb{R}^N$ (see [9, Theorem 4.9]). It turns out that $G(\hat{u} + h_n) \to G(\hat{u} + h_\varepsilon)$ almost everywhere in $\mathbb{R}^N$ because $G : \mathbb{R} \to \mathbb{R}$ is continuous. Based on this, we show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)G(\hat{u}+h_n)dx = \int_{\mathbb{R}^N} a(x)G(\hat{u}+h_\varepsilon)dx.$$

Indeed, writing $a = a^+ - a^-$ with $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$, we may suppose that $a \geq 0$. Then it results from (2) that the sequence $\{a^{\frac{r}{p^*}} G(\hat{u}+h_n)\}$ is bounded in $L^{\frac{p^*}{r}}(\mathbb{R}^N)$ (note that $\{\hat{u} + h_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, so in $L^{p^*}(\mathbb{R}^N)$, and $a \in L^\infty(\mathbb{R}^N)$), while $a^{\frac{p^*-r}{p^*}} \in L^{\frac{p^*}{p^*-r}}(\mathbb{R}^N)$ (note that $a \in L^1(\mathbb{R}^N)$).

We are thus led to $a^{\frac{r}{p^*}} G(\hat{u}+h_n) \rightharpoonup a^{\frac{r}{p^*}} G(\hat{u}+h_\varepsilon)$ in $L^{\frac{p^*}{r}}(\mathbb{R}^N)$ and the above claim follows. Hence we derive

$$\begin{aligned}
m_\varepsilon &= \lim_{n\to\infty}\Phi(\hat{u}+h_n)\\
&= \lim_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla(\hat{u}+h_n)|^p + |\hat{u}+h_n|^p)dx - \int_{\mathbb{R}^N} a(x)G(\hat{u}+h_n)dx\right]\\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla(\hat{u}+h_\varepsilon)|^p + |\hat{u}+h_\varepsilon|^p)dx - \lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)G(\hat{u}+h_n)dx\\
&= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla(\hat{u}+h_\varepsilon)|^p + |\hat{u}+h_\varepsilon|^p)dx - \int_{\mathbb{R}^N} a(x)G(\hat{u}+h_\varepsilon)dx = \Phi(\hat{u}+h_\varepsilon).
\end{aligned}$$

In view of (5), the preceding estimate entails

$$\Phi(\hat{u}+h_\varepsilon) = \inf_{h\in D_r(0,\varepsilon)}\Phi(\hat{u}+h) < \Phi(\hat{u}). \tag{6}$$

Obviously, we have that $h_\varepsilon \neq 0$.

At this point we are able to apply the Lagrange multiplier rule to the minimization statement in (6) for the functional $\Phi : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ with the constraint $\| h \|_{L^r(\mathbb{R}^N)} \leq \varepsilon$. Accordingly, there exists $\lambda_\varepsilon \in \mathbb{R}$ such that

$$-\Delta_p(\hat{u}+h_\varepsilon) + |\hat{u}+h_\varepsilon|^{p-2}(\hat{u}+h_\varepsilon) = a(x)g(\hat{u}+h_\varepsilon) + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \text{ in } \mathbb{R}^N. \tag{7}$$

If the strict inequality $\| h_\varepsilon \|_{L^r(\mathbb{R}^N)} < \varepsilon$ occurs, then from (7) we get $\lambda_\varepsilon = 0$ taking into account that $\Phi'(\hat{u} + h_\varepsilon) = 0$. If $\| h_\varepsilon \|_{L^r(\mathbb{R}^N)} = \varepsilon$, we note that the minimization in (6), in conjunction with (7) and (3), yields

$$\begin{aligned}
0 &\geq \lim_{t\downarrow 0}\frac{\Phi(\hat{u}+h_\varepsilon - th_\varepsilon) - \Phi(\hat{u}+h_\varepsilon)}{-t} = \langle \Phi'(\hat{u}+h_\varepsilon), h_\varepsilon\rangle_{W^{1,p}(\mathbb{R}^N)}\\
&= \lambda_\varepsilon \| h_\varepsilon \|^r_{L^r(\mathbb{R}^N)}.
\end{aligned}$$

Consequently, (7) holds true with $\lambda_\varepsilon \leq 0$.

On the other hand, recall that $\hat{u} \in W^{1,p}(\mathbb{R}^N)$ is a $C^1(\mathbb{R}^N)$ local minimizer of the functional $\Phi : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ in the sense of Definition 1. Hence for every $h \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ we have

$$\frac{\Phi(\hat{u}+t_k h)-\Phi(\hat{u})}{t_k} \geq 0,$$

along a sequence of positive numbers $t_k \downarrow 0$ as $k \to \infty$. Letting $k \to \infty$ gives $\langle \Phi'(\hat{u}), h\rangle_{W^{1,p}(\mathbb{R}^N)} \geq 0$. Changing h with $-h$ renders $\langle \Phi'(\hat{u}), h\rangle_{W^{1,p}(\mathbb{R}^N)} = 0$. Notice that in particular this holds for every $h \in C_0^\infty(\mathbb{R}^N)$. The density of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$ implies $\Phi'(\hat{u}) = 0$, which guarantees

$$-\Delta_p \hat{u} + |\hat{u}|^{p-2}\hat{u} = a(x)g(\hat{u}) \text{ in } \mathbb{R}^N. \tag{8}$$

Combining (7) and (8) ensures

$$-\Delta_p(\hat{u}+h_\varepsilon) + \Delta_p \hat{u} + |\hat{u}+h_\varepsilon|^{p-2}(\hat{u}+h_\varepsilon) - |\hat{u}|^{p-2}\hat{u}$$

$$= a(x)[g(\hat{u}+h_\varepsilon) - g(\hat{u})] + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \text{ in } \mathbb{R}^N.$$

For any $x, y \in \mathbb{R}^N$, let us denote

$$A(x,y) = |\nabla\hat{u}(x)+y|^{p-2}(\nabla\hat{u}(x)+y) - |\nabla\hat{u}(x)|^{p-2}\nabla\hat{u}(x).$$

Notice that

$$-\mathrm{div}A(x,\nabla h_\varepsilon) + |\hat{u}+h_\varepsilon|^{p-2}(\hat{u}+h_\varepsilon) - |\hat{u}|^{p-2}\hat{u}$$

$$= a(x)[g(\hat{u}+h_\varepsilon) - g(\hat{u})] + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \text{ in } \mathbb{R}^N. \tag{9}$$

Taking into account (6), the reasoning to prove the boundedness of a minimizing sequence of (5) entails that $\{|\nabla h_\varepsilon|\}$ is bounded in $L^p(\mathbb{R}^N)$ uniformly with respect to $\varepsilon \in (0, 1)$. By applying the Sobolev-Gagliardo-Nirenberg inequality (see [9, Theorem 9.9]) we can infer that $\{h_\varepsilon\}$ is bounded in $L^{p^*}(\mathbb{R}^N)$ uniformly with respect to $\varepsilon \in (0, 1)$.

The Moser iteration procedure applied to (9) shows that $h_\varepsilon \in L^\infty(\mathbb{R}^N)$ and provides a constant $M > 0$ such that $\| h_\varepsilon \|_{L^\infty(\mathbb{R}^N)} \leq M$ for all $\varepsilon \in (0, 1)$. This follows by proving the assertion separately for the positive part $h_\varepsilon^+ = \max\{h_\varepsilon, 0\}$ and the negative part $h_\varepsilon^- = \max\{-h_\varepsilon, 0\}$ noting that $h_\varepsilon = h_\varepsilon^+ - h_\varepsilon^-$. We focus only on h_ε^+ since the argument proceeds analogously for h_ε^-. For each constant $K > 0$ we set $h_{\varepsilon,K}(x) = \min\{h_\varepsilon^+(x), K\}$, $x \in \mathbb{R}^N$. Acting on (9) with the test function $h_{\varepsilon,K}^{q+1} \in W^{1,p}(\mathbb{R}^N)$ for any $q > 0$ leads to

$$\int_{\mathbb{R}^N} h_{\varepsilon,K}^q A(x,\nabla h_{\varepsilon,K})\nabla h_{\varepsilon,K}\,dx \le (q+1)\int_{\mathbb{R}^N} h_{\varepsilon,K}^q A(x,\nabla h_{\varepsilon,K})\nabla h_{\varepsilon,K}\,dx$$

$$= \int_{\mathbb{R}^N} A(x,\nabla h_{\varepsilon,K})\nabla (h_{\varepsilon,K}^{q+1})dx$$

$$\le \int_{\mathbb{R}^N} a(x)[g(\hat{u}+h_\varepsilon)-g(\hat{u})]h_{\varepsilon,K}^{q+1}dx,$$

where the increasing monotonicity of the function $t \mapsto |t|^{p-2}t$ on $\mathbb{R}$ and the inequality $\lambda_\varepsilon \le 0$ have been used too. From here, by using the continuous embedding $W_0^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$, monotonicity inequalities for the p-Laplacian distinguishing the cases $1 < p < 2$ and $p \ge 2$, the growth condition (1), Hölder's inequality, and the fact that $\lambda_\varepsilon \le 0$, we arrive at the key estimate

$$\| h_{\varepsilon,K} \|^{p+q}_{L^{\frac{p^*(p+q)}{p}}(\mathbb{R}^N)} \le C_0 \| h_{\varepsilon,K}^{\frac{p+q}{p}} \|^p_{W^{1,p}(\mathbb{R}^N)}$$

$$\le C\left(\frac{p+q}{p}\right)^p \max\{1, \| h_\varepsilon^+ \|^{q+r}_{L^{q+r}(\mathbb{R}^N)}\}, \tag{10}$$

with positive constants C_0 and C, provided $h_\varepsilon^+ \in L^{q+r}(\mathbb{R}^N)$. An essential term in getting the preceding estimate is

$$\int_{\mathbb{R}^N} h_{\varepsilon,K}^q \,|\nabla h_{\varepsilon,K}|^p\, dx = \int_{\mathbb{R}^N} |h_{\varepsilon,K}^{\frac{q}{p}} \nabla h_{\varepsilon,K}|^p\, dx = \left(\frac{p}{p+q}\right)^p \int_{\mathbb{R}^N} |\nabla (h_{\varepsilon,K}^{\frac{p+q}{p}})|^p\, dx.$$

Such calculations on a bounded domain can be found in [5, Theorem C]. At this point we are able to implement a Moser iteration scheme by choosing appropriately q. The first step is to choose $q_0 = p^* - r$, which is possible because $h_\varepsilon \in L^{p^*}(\mathbb{R}^N)$. Then (10) and Fatou's lemma letting $K \to +\infty$ ensure that $h_\varepsilon \in L^{\frac{p^*(p+q_0)}{p}}(\mathbb{R}^N)$. Inductively, we pose $p(q_n + r) = p^*(p + q_{n-1})$ for all $n \ge 1$, and through (10) we are able to deduce that $h_\varepsilon^+ \in L^\tau(\mathbb{R}^N)$ for every $\tau \ge 1$ since $q_n \to +\infty$ as $n \to \infty$. The conclusion about the uniform boundedness of h_ε follows. Furthermore, combining the iterations on the pattern in the proof of [5, Theorem C], the existence of the claimed bound $M > 0$ independent of $\varepsilon \in (0, 1)$ is achieved.

The term $\lambda_\varepsilon|h_\varepsilon|^{r-2}h_\varepsilon$ in (9) describes the interaction between the multiplier λ_ε and the minimizer h_ε on $D_r(0, \varepsilon)$. In order to handle the right-hand side of (9) we must estimate this term. We claim that there exists a constant $d > 0$ such that

$$|\lambda_\varepsilon|\,|h_\varepsilon(x)|^{r-1} \le d,\ x \in \mathbb{R}^N,\ \forall \varepsilon \in (0,1). \tag{11}$$

Denote

$$f(x,s) := a(x)[g(\hat{u}(x)+s) - g(\hat{u}(x))],\ \forall x \in \mathbb{R}^N,\ \forall s \in \mathbb{R}.$$

The growth condition (1) for the function g entails

$$|f(x,s)| \le \|a\|_{L^\infty(\mathbb{R}^N)} [c(1+(\|\hat{u}\|_{L^\infty(\mathbb{R}^N)} + |s|)^{r-1}) + c(1+\|\hat{u}\|^{r-1}_{L^\infty(\mathbb{R}^N)})]$$

$$\le d_0(1+|s|)^{r-1}, \ \forall x \in \mathbb{R}^N, \ \forall s \in \mathbb{R},$$

with a constant $d_0 > 0$. Here we have used that the weak solution $\hat{u}$ to (8) belongs to $L^\infty(\mathbb{R}^N)$, which can be shown as above on the basis of (8) through the Moser iteration technique knowing that g has a subcritical growth (see assumption (1)).

For each $\rho > 0$, we act on (9) with the test function $(h_\varepsilon - \rho)^+ := \max\{h_\varepsilon - \rho, 0\}$. Through the monotonicity of $A(x, \cdot)$ and since $\lambda_\varepsilon \le 0$, this implies

$$0 \le \int_{\{h_\varepsilon > \rho\}} A(x, \nabla h_\varepsilon) \nabla h_\varepsilon dx$$

$$= \int_{\mathbb{R}^N} A(x, \nabla h_\varepsilon) \nabla (h_\varepsilon - \rho)^+ dx$$

$$\le \int_{\mathbb{R}^N} f(x, h_\varepsilon(x))(h_\varepsilon - \rho)^+ dx - |\lambda_\varepsilon| \int_{\mathbb{R}^N} h_\varepsilon^{r-1}(h_\varepsilon - \rho)^+ dx$$

$$\le d_0(1+\rho^{1-r}) \int_{\mathbb{R}^N} h_\varepsilon^{r-1}(h_\varepsilon - \rho)^+ dx - |\lambda_\varepsilon| \int_{\mathbb{R}^N} h_\varepsilon^{r-1}(h_\varepsilon - \rho)^+ dx.$$

We are led to

$$|\lambda_\varepsilon| \le d_0(1+\rho^{1-r}) \text{ provided } (h_\varepsilon - \rho)^+ \not\equiv 0.$$

Similarly, denoting $(h_\varepsilon + \rho)^- := \max\{-(h_\varepsilon + \rho), 0\}$, we obtain

$$|\lambda_\varepsilon| \le d_0(1+\rho^{1-r}) \text{ provided } (h_\varepsilon + \rho)^- \not\equiv 0.$$

Choose $\rho = \frac{1}{2}\|h_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0$. The previous discussion ensures the estimate

$$|\lambda_\varepsilon| |h_\varepsilon(x)|^{r-1} \le |\lambda_\varepsilon| \|h_\varepsilon\|^{r-1}_{L^\infty(\mathbb{R}^N)}$$

$$\le d_0(1+2^{r-1}\|h_\varepsilon\|^{1-r}_{L^\infty(\mathbb{R}^N)}) \|h_\varepsilon\|^{r-1}_{L^\infty(\mathbb{R}^N)}$$

$$= d_0(2^{r-1} + \|h_\varepsilon\|^{r-1}_{L^\infty(\mathbb{R}^N)})$$

$$\le d_0(2^{r-1} + M^{r-1}),$$

which proves (11).

In view of (11) and recalling the function $f(x, s)$ defined before, it appears that the function $f_\varepsilon : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f_\varepsilon(x,s) := f(x,s) + \lambda_\varepsilon \,|\, h_\varepsilon(x) \,|^{r-2}\, h_\varepsilon(x)$$

is uniformly bounded with respect to $\varepsilon \in (0, 1)$ on $\bar{\Omega} \times [-K, K]$ for each open bounded set Ω in $\mathbb{R}^N$ and constant $K > 0$. Therefore we are able to apply the regularity result up to the boundary in [10, Theorem 1] to equation (9) on any such Ω supposing in addition that $\partial\Omega$ is of class C^2. We deduce that there exist constants $\theta = \theta(\Omega) \in (0, 1)$ and $C = C(\Omega) > 0$ such that $\hat{u} + h_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ and

$$\| \hat{u} + h_\varepsilon \|_{C^{1,\theta}(\bar{\Omega})} \le C, \ \forall \varepsilon \in (0,1).$$

Since $C^{1,\theta}(\bar{\Omega})$ is compactly embedded in $C^1(\bar{\Omega})$ and it holds $\| h_\varepsilon \|_{L^r(\mathbb{R}^N)} \le \varepsilon$, we can assume that

$$\hat{u} + h_\varepsilon \to \hat{u} \text{ in } C^1(\bar{\Omega}) \text{ as } \varepsilon \to 0$$

In particular, it enables us to conclude that $\hat{u} \in C^1(\mathbb{R}^N)$ because of $\hat{u} \in C^1(\Omega)$ with Ω as above belonging to a covering of $\mathbb{R}^N$.

Consider now the sequence of sets $\{\Omega_n\}$ postulated in Definition 1 for $\hat{u}$. Complying with what was said before, we can use a diagonal process to obtain a subsequence of the sequence $\{h_\varepsilon\}$, still denoted $\{h_\varepsilon\}$, for which there holds

$$\hat{u} + h_\varepsilon \to \hat{u} \text{ in } C^1(\overline{\Omega_n}), \text{ for all } n, \text{ as } \varepsilon \to 0.$$

We are in a position to invoke Definition 1 which provides an $\varepsilon_0 > 0$ such that

$$\Phi(\hat{u}) \le \Phi(\hat{u} + h_\varepsilon), \ \forall \varepsilon \le \varepsilon_0.$$

We have thus reached a contradiction with inequality (6). This proves completely Theorem 2.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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