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Research Article

Geodesic Triangles in $\mathbb{H}^2$ with Short Sides

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Abstract: We prove that if M is an orientable hyperbolic surface without boundary (possibly compact, possibly with infinitely generated fundamental group) and γ is a closed geodesic in M then any side of any triangle formed by distinct lifts of γ in $\mathbb{H}^2$ is shorter than γ.

***Keywords*:** hyperbolic surface, Poincaré disc, geodesic, covering space, tree, free group

MSC: 51M10, 57M15, 57M10, 53C22, 30F45, 20E05

1. Introduction

Behavior of closed geodesics in hyperbolic surfaces has been a fruitful subject of research for many years. Such geodesics are often studied by looking at their lifts in covering spaces of the surface, cf. [1-7]. In this paper we consider three geodesics in $\mathbb{H}^2$ which are lifts of the same closed geodesic γ in an orientable hyperbolic surface without boundary (possibly compact, possibly with infinitely generated fundamental group), cf. [8] and [3]. We prove that if these three geodesics intersect to form a triangle then each side of that triangle is shorter than γ. Note that two lifts of γ can intersect if and only if γ is self intersecting.

In contrast, a triangle in $\mathbb{H}^2$ formed by three arbitrary geodesics can have sides of any length (see Section 2 of this paper).

Geodesic triangles in hyperbolic plane have been investigated for a long time. The best known result, proved independently by several mathematicians in the nineteenth century, is that the area of a geodesic triangle in the hyperbolic plane is equal to [$\pi -$ (sum of the angles of the triangle)].

The main result of this paper is the following theorem.

Theorem 1. Let M be an orientable hyperbolic surface without boundary (possibly compact, possibly with infinitely generated fundamental group) and let γ be a closed geodesic in M. Any side of any triangle formed by distinct lifts of γ in $\mathbb{H}^2$ is shorter than γ.

Remark. Note that triangles considered in Theorem 1 need not be innermost.

The proof of Theorem 1 utilizes an algebraic lemma proven in Section 5 and geometric results proven in Section 3 and Section 6.

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2. The hyperbolic plane

In this paper we work with the Poincaré disk model of the hyperbolic plane $\mathbb{H}^2$ and the Poincaré metric, given by

$$ds^2 = \frac{4(da^2 + db^2)}{(1 - a^2 - b^2)^2}$$

The Poincaré disc is the open unit disc in the complex plane

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

Any two points a and b in the Poincaré disk are joined by a unique geodesic, which is a part of the circle or the straight line passing through a and b and orthogonal to the boundary of the Poincaré disc.

The hyperbolic distance between a and b is given by

$$d(a, b) = 2tanh^{-1}\frac{|a - b|}{|1 - \bar{b}a|}$$

where $|a - b|$ denotes the Euclidean distance between a and b; $\bar{b}$ denotes the complex conjugate of b and $|1 - \bar{b}a|$ denotes the Euclidean distance between 1 and $\bar{b}a$. In particular, the hyperbolic distance between 0 and any real positive $c \in D$ is given by

$$d(0, c) = 2tanh^{-1}(c)$$

As $\lim_{c\to 1} 2tanh^{-1}(c) = \lim_{c\to 1} \frac{1}{2} ln(\frac{1+c}{1-c}) = \infty$, it follows that any diameter of D, which is a geodesic in D in the hyperbolic metric, has infinite length in the hyperbolic metric.

A polygon in D which has all its vertices on the boundary circle $\{z \in \mathbb{C} : |z| = 1\}$ of D is called an ideal polygon.

Consider an ideal geodesic triangle in D with vertices -1, i, and 1. All sides of this triangle have infinite length, in contrast with Theorem 1.

3. An intermediate cover

Let M be an orientable hyperbolic surface without boundary (possibly compact, possibly with infinitely generated fundamental group) and let γ be a closed geodesic in M. Let l, m, and n be distinct lifts of γ to $\mathbb{H}^2$ which form a triangle. Let g and h be elements of $\pi_1(M)$ such that $m = gl$ and $n = hl$. Let α generate the stabilizer of l in $\pi_1(M)$, so $\alpha(l) = l$.

Let X be the cover of M corresponding to the subgroup of $\pi_1(M)$ generated by α, g, and h. Note that X is an orientable hyperbolic surface with the fundamental group generated by either one, two, or three elements.

If $\pi_1(X)$ has 1 generator then X is a double-punctured sphere.

If $\pi_1(X)$ has 2 generators then X is either a punctured torus or a three-punctured sphere.

If $\pi_1(X)$ has 3 generators then X is either a double-punctured torus or a four-punctured sphere.

In any case X is a non-compact surface.

As $\pi_1(X)$ contains α, it follows that γ lifts to a closed geodesic γ_X in X, and γ and γ_X have the same length. Note that γ_X is the quotient of l by the action of α. As X is hyperbolic, γ_X is the unique closed geodesic in its homotopy class. Note also that the stabilizer of m in $\pi_1(M)$ is generated by $g\alpha g^{-1}$, and the stabilizer of n in $\pi_1(M)$ is generated by $h\alpha h^{-1}$.

As $m = gl$ and $g \in \pi_1(X)$, it follows that l and m have the same image in X, namely γ_X. As $n = hl$ and $h \in \pi_1(X)$, it follows that l and n have the same image in X, namely γ_X. Hence the geodesics l, m, and n are lifts of γ_X to $\mathbb{H}^2$.

It follows that in order to prove that Theorem 1 holds for the surface M which might be compact or have infinitely generated fundamental group, it suffices to prove Theorem 1 for a non-compact surface X with finitely generated fundamental group. Moreover, surface X is restricted to have one of the five homeomorphism classes described above. The proof is completed in section 6.

4. The tree T in $\mathbb{H}^2$

Excellent expositions of hyperbolic geometry can be found in [7, 9-12].

Let M be an orientable non-compact hyperbolic surface without boundary which has finitely generated fundamental group. We consider the standard metric on the hyperbolic surface M, given by the covering map from $\mathbb{H}^2$ to M.

The following classical facts are contained, for example, in [11] pp. 415-427 and [12] pp. 122-125.

Let the genus of M be g and let h be the number of holes in M. Let $n = 2g + h - 1$. Note that M is the interior of a compact surface N with h boundary components, and N also has genus g. The surface N can be cut by n properly embedded disjoint arcs into a disc. It follows that there exist infinite simple disjoint geodesics $x_1, \cdots, x_n$ in M such that M cut along the union of x_i, $1 \le i \le n$, is an open two-dimensional polygon P. Also there exist closed geodesics $y_1, \cdots, y_n$ in M such that $x_i \cap y_i$ = point and $x_i \cap y_j = \varnothing$ for $i \neq j$, which generate the fundamental group of M. Note that the fundamental group of M is a free group of rank n. The universal cover of M is the hyperbolic plane $\mathbb{H}^2$, so M is the quotient of $\mathbb{H}^2$ by the action of $\pi_1(M)$. Let $\widetilde{P}$ be a lift of the polygon P to $\mathbb{H}^2$. Note that $\widetilde{P}$ has $2n$ sides.

Recall that an end of a surface without boundary and with finitely generated fundamental group is homeomorphic to a product $S^1 \times [0, \infty)$.

The following fact is contained, for example, in Theorem 9.8.6. on p. 424 of [11]. Hyperbolic surfaces without boundary and with finitely generated fundamental group have two kinds of ends: a cusp end, which has finite area, and a flare end, which has infinite area. If all the ends of M are cusps then $\widetilde{P}$ is an ideal polygon in $\mathbb{H}^2$. The action of $\pi_1(M)$ on $\mathbb{H}^2$ creates a tessellation of $\mathbb{H}^2$ by the translates of the closure of $\widetilde{P}$. Let T be the graph in $\mathbb{H}^2$ dual to that tessellation, i.e. the vertices of T are located one in each translate of $\widetilde{P}$, and each edge of T connects two vertices of T in adjacent copies of $\widetilde{P}$, so each edge of T intersects just one lift of one x_i in one point. As $\mathbb{H}^2$ is simply connected, T is a tree.

The tree T was introduced in [8] pp. 111-112. It can be considered to be the Cayley graph of the group $\pi_1(M)$ which is a free group of rank n generated by the set $y_1, \cdots, y_n$. Define the distance $d_T(u, v)$ between two vertices u and v of T in T to be the number of edges in a shortest path in T connecting u and v. A geodesic between vertices u and v in T is a shortest path in metric d_T joining u and v. An infinite geodesic in T is an infinite path in T such that any of its finite subpaths is a geodesic in metric d_T in T.

5. Algebraic lemma

We use the notation of Section 4.

Denote the length of the word W in $\pi_1(M)$ by $L(W)$. Note that each oriented edge of T is labeled by one of the generators $\{y_i, 1 \le i \le n\}$ or their inverses $\{y_i^{-1}, 1 \le i \le n\}$, so each oriented path in T is labeled by a word in $\{y_i, 1 \le i \le n\}$ and $\{y_i^{-1}, 1 \le i \le n\}$.

Recall that any element f of $\pi_1(M)$ acts on T leaving invariant a unique infinite geodesic in T, called the axis of f.

The following result is a generalization of Lemma 1 in [3].

Algebraic Lemma. Let f and f' be conjugate elements in $\pi_1(M) = \langle y_1, \cdots, y_n \rangle$, and let A and A' be the axes of f and f' respectively. Let W be a reduced and cyclically reduced conjugate of f. If A and A' intersect in an interval of length $L(W) - 1$ then they coincide.

Proof. Note that W is a shortest word in the conjugacy class of f, and both A and A' are labeled by a bi-infinite word $\cdots WWWWWW \cdots$. Let the intersection of A and A' be labeled by a subword V of $\cdots WWWWWW \cdots$ such that $L(V) = L(W) - 1$. WLOG the word V is the initial subword of W, hence there exists a decomposition $W = Vy$, where y is either a generator or an inverse of a generator in $\pi_1(M) = \langle y_1, \cdots, y_n \rangle$, i.e. $y \in \{y_i, y_i^{-1} | 1 \leq i \leq n\}$. Let W' be a reduced and cyclically reduced conjugate of f' containing V. Then W' is also a shortest word in the conjugacy class of f'. Therefore either $W' = yV$ or $W' = Vy = W$. In either case, the intersection of A and A' contains an interval of length $L(W)$, obtained by adding a single edge with label y to an end of the interval with label V. Hence A and A' coincide.

6. Proof of Theorem 1

In Section 3 the proof of Theorem 1 was reduced to five cases.

Note that a single-punctured hyperbolic sphere has a trivial fundamental group, so it does not have closed geodesics, hence Theorem 1 is vacuously true in this case.

A twice-punctured sphere is homeomorphic to an annulus, so its fundamental group is infinite cyclic. Hence the preimage in $\mathbb{H}^2$ of any closed geodesic in a two-punctured sphere consists of a single geodesic line. It follows that in this case there are no triangles in $\mathbb{H}^2$ which satisfy the hypothesis of Theorem 1.

A proof of Theorem 1 for a hyperbolic single-punctured torus is given in [3].

The remaining two cases can be combined to avoid unnecessary technicalities and they follow from Theorem 2 stated below, completing the proof of Theorem 1.

Theorem 2. Let M be an orientable non-compact hyperbolic surface without boundary which has finitely generated fundamental group and let γ be a closed geodesic in M. Any side of any triangle formed by distinct geodesic lines in the preimage of γ in $\mathbb{H}^2$ is shorter than γ.

Proof. We use the notation from previous sections. Let $x_1, \cdots, x_k$ be infinite simple disjoint geodesics in M which cut M in a polygon P. Consider the tree T in $\mathbb{H}^2$ defined in Section 4. Choose a $\pi_1(M)$-equivariant projection $s : \mathbb{H}^2 \to T$. This means that $g(s(x)) = s(g(x))$ for any $g \in \pi_1(M)$ and $x \in \mathbb{H}^2$. It can be arranged that the restriction of s to each component of the lift of γ in $\mathbb{H}^2$ is monotone, so s maps each component of the lift of γ onto a geodesic in T.

The exposition below generalizes a proof of a special case of Theorem 2 which appeared in [3].

Assume to the contrary that there exists a triangle in $\mathbb{H}^2$ formed by geodesic lines l, m, and n, which are distinct lifts of the geodesic γ, such that the length of the side lying in l is longer than γ. Note that l is stabilized by some element f in $\pi_1(M)$ which acts as a hyperbolic isometry of $\mathbb{H}^2$.

Let P be the intersection of l and n, and let X be the intersection of l and m. The length of γ is equal to the length of the segment $Pf(P)$ which is equal to the length of the segment $f(P)f^2(P)$.

Consider two cases. The graphic illustrations of the cases can be found in [3].

Case 1. The side PX of the triangle formed by lines l, m, and n is shorter than the segment $Pf^2(P)$.

By assumption, the side PX is longer than γ, so the segment $Xf^2(P)$ is shorter than the segment PX. Consider the geodesics $f(n)$ and $f^2(n)$. As f is an isometry, the geodesics n, $f(n)$, and $f^2(n)$ make the same angle with l. Then as $Xf^2(P)$ is shorter than PX, the angle between n and l is equal to the angle between $f^2(n)$ and l, and the opposite angles between m and l are equal, it follows that m and $f^2(n)$ intersect.

Let T be the tree in $\mathbb{H}^2$ defined above and let W be a reduced and cyclically reduced word conjugate to f in $\pi_1(M)$. The geodesic lines l, m, and n are transversal to the lifts of the geodesics $x_1, \cdots, x_k$ in $\mathbb{H}^2$. Consider the intersections of the lifts of the geodesics $x_1, \cdots, x_k$ with lines l, m, and n.

Let b lifts of $x_1, \cdots, x_k$ intersect both l and n to the left of the point P and let a lifts of $x_1, \cdots, x_k$ intersect both l and n to the right of the point P. Then there are $a + b$ lifts of $x_1, \cdots, x_k$ crossing l and n, hence the length of the intersection $s(l) \cap s(n)$ is $a + b$. The Algebraic Lemma implies that $a + b < L(W) - 1$. By a similar argument, the number c of the lifts of $x_1, \cdots, x_k$ intersecting both l and m is also less than $L(W) - 1$. As f is an isometry, there are b lifts of $x_1, \cdots, x_k$ crossing l and $f^2(n)$ to the left of $f^2(P)$. Then the total number of the lifts of $x_1, \cdots, x_k$ crossing l between the points P and $f^2(P)$ is at most $a + b + c$, which is strictly less than $2L(W)$. However by construction, the number of the lifts of $x_1, \cdots, x_k$ crossing l between the points P and $f^2(P)$ should be equal to $2L(W)$.

This contradiction completes the proof of Theorem 2 in Case 1.

Case 2. The side PX of the triangle formed by lines l, m, and n is longer or equal than the segment $Pf^2(P)$.

Let a lifts of $x_1, \cdots, x_k$ intersect both l and n to the right of the point P. Then the length of the intersection $s(l) \cap s(n)$ is not shorter than a, hence the Algebraic Lemma implies that $a < L(W) - 1$. Let c be the number of the lifts of $x_1, \cdots, x_k$ intersecting both l and m to the left of the point X. Then the length of the intersection $s(l) \cap s(m)$ is not shorter than c, hence the Algebraic Lemma implies that $c < L(W) - 1$. Therefore the total number of the lifts of $x_1, \cdots, x_k$ crossing l between the points P and $f^2(P)$ is at most $a + c$, which is strictly less than $2L(W)$. However by construction, the number of the lifts of $x_1, \cdots, x_k$ crossing l between the points P and $f^2(P)$ should be equal to $2L(W)$.

This contradiction completes the proof of Theorem 2 in Case 2.

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Conflict of interest

The author declares that there is no personal or organizational conflict of interests with this work.

References

[1] Basmajian A. The stable neighborhood theorem and lengths of closed geodesics. *Proceedings of the American Mathematical Society*. 1993; 119(1): 217-224.
[2] Freedman M, Hass J, Scott P. Closed geodesics on surfaces. *Bulletin of the London Mathematical Society*. 1982; 14(5): 385-391.
[3] Gitik R. Conjugate words and intersections of geodesics in $\mathbb{H}^2$. *Journal of Topology and Analysis*. 2021; 13(1): 175-185.
[4] Hass J, Scott P. Curve flows on surfaces and intersections of curves. *Proceedings of Symposia in Pure Mathematics*. 1993; 54(3): 415-421.
[5] Hass J, Scott P. Shortening curves on surfaces. *Topology*. 1994; 33(1): 25-43.
[6] Neumann-Coto M. A characterization of shortest geodesics on surfaces. *Algebraic and Geometric Topology*. 2001; 1(1): 349-368.
[7] Rivin I. Geodesics with one self-intersection and other stories. *Advances in Mathematics*. 2021; 231(5): 2391-2412.
[8] Hass J, Scott P. Intersections of curves on surfaces. *Israel Journal of Mathematics*. 1985; 51: 90-120.
[9] Beardon AF. The geometry of discrete groups. *Graduate Texts in Mathematics*. 1st ed. Springer, New York; 1983.
[10] Lehner J. *A short course in automorphic functions*. Holt, Rinchart, and Wiston, New York; 1966.
[11] Scott P. The Geometries of 3-manifolds. *Bulletin of the London Mathematical Society*. 1983; 15(15): 401-487.
[12] Stillwell J. *Geometry of surfaces*. Springer, New York; 1992.