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Maslov Index and Quasi-Symplectic Isomorphisms

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Abstract: Maslov index is defined as the number of the intersection of a loop of Lagrangian subspaces with a 1-codimensional cycle in the Lagrangian Grassmannian. It is well-known that linear symplectomorphisms preserve the Maslov index. We show how quasi-symplectic isomorphisms change Maslov index.

***Keywords*:** Maslov index, linear symplectomorphism, Lagrangian Grassmannian

MSC: 53D05, 53D12

1. Introduction

In the process of treating the asymptotic expression of the solution of the quasiclassical question, e.g. the Schrödinger equation, Maslov [1] defined an index by the intersection number of an oriented closed curve in an n-dimensional Lagrangian submanifold M with a two-sided 1-codimensional cycle on M. Arnold [2] proved that the Maslov index coincides with a cohomology class and also with the index for the corresponding loop in the Lagrangian Grassmaniann $\mathcal{L}(n)$ (the manifold consists of all Lagrangian subspaces in $\mathbb{R}^{2n}$), which is defined as an intersection number of this corresponding loop with a singular cycle called Maslov cycle. Arnold's work can be generalized to the case of a path of Lagrangian subspaces with its endpoints lying in the complement of the Maslov cycle. Robbin and Salamon [3] generalized a new definition of Maslov index for any path even if its endpoints lie in the Maslov cycle. They defined an associated form $Q : \mathcal{L}(n) \times \mathcal{L}(n) \to \mathbb{R}$, when a Lagrangian subspace is represented by a Lagrangian frame, the form Q can be expressed explicitly in a matrix form. Robbin and Salamon also defined the relative Maslov index for a pair of loops of Lagrangian subspaces. In [4] Robbin and Salamon also showed that the Maslov index agrees with the spectral flow of an associated matrix family. On the other hand, Cappell, Lee and Miller [5] showed four definitions of Maslov index with respect to a pair of loops of Lagrangian subspaces and proved that they are equivalent to each other. The Maslov index also can be used to other objects, for example, Schrödinger operators [6], loops in a coisotropic submanifold [7] and so on. So it is necessary to develop the properties of the Maslov index.

One important property of Maslov index is that the linear symplectomorphisms, the linear isomorphism of $\mathbb{R}^{2n}$ preserving the symplectic form, preserve Maslov index. In this article, we study how more general isomorphisms, such as quasi-symplectic isomorphisms which change the symplec-tic form based on a fixed coefficient, act on Maslov index. Explicitly, let the vector space $\mathbb{R}^{2n}$ be equipped with the standard symplectic form ω_0, the quasi-symplectic isomorphisms Ψ_λ in $(\mathbb{R}^{2n}, \omega_0)$ are the isomorphisms satisfying $\Psi_\lambda^*\omega_0 = \lambda\omega_0$ with a nonzero constant coefficient λ. As in [3]

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the Maslov index μ for a loop $\Lambda(t)$ of Lagrangian subspaces is defined as the sum of signature of a crossing form $\Gamma(\Lambda, V)$ where V is a fixed Lagrangian subspace (see (12)) and the Maslov index for a pair of loops is defined as the sum of the signature of a relative crossing form (see (14)). Then

Theorem 1.1 For a pair of loops $\Lambda_1(t)$, $\Lambda_2(t)$ in $\mathcal{L}(n)$, quasi-symplectic isomorphisms Ψ_λ change the sign of the Maslov index depended on the sign of the coefficient λ. i.e.,

$$\mu(\Psi_\lambda\Lambda_1(t), \Psi_\lambda\Lambda_2(t)) = \begin{cases} \mu(\Lambda_1(t), \Lambda_2(t)) & \lambda > 0, \\ -\mu(\Lambda_1(t), \Lambda_2(t)) & \lambda < 0. \end{cases} \quad (1)$$

Analogous to $\mathrm{Sp}(2n)$, all the λ-coefficient quasi-symplectic matrices form a manifold denoted by $\mathrm{QSp}_\lambda(2n)$. Let $\Psi_\lambda(t)$ be a loop in $\mathrm{QSp}_\lambda(2n)$, then

Theorem 1.2 In the above setting, we have

$$\mu(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\Lambda_2(t)) = \begin{cases} \mu(\Lambda_1(t), \Lambda_2(t)) & \lambda > 0, \\ -\mu(\Lambda_1(t), \Lambda_2(t)) & \lambda < 0. \end{cases} \quad (2)$$

In particular, if $\Lambda_2(t) \equiv V$ where V is the fixed Lagrangian subspace and let $\Lambda_1(t) = \Lambda(t)$ for simplicity, we have

Remark 1.3

$$\mu(\Psi_\lambda(t)\Lambda(t), \Psi_\lambda(t)V) = \begin{cases} \mu(\Lambda(t), V) & \lambda > 0, \\ -\mu(\Lambda(t), V) & \lambda < 0. \end{cases} \quad (3)$$

All the quasi-symplectic matrices with any nonzero coefficient also form a Lie group denoted by $\mathrm{QSp}(2n)$. For a loop $\tilde{\Psi}_\lambda(t)$ in $\mathrm{QSp}(2n)$, the coefficient also is a smooth function $\lambda(t)$ which is nonzero for any t. We have

Remark 1.4

$$\mu(\tilde{\Psi}_\lambda(t)\Lambda_1(t), \tilde{\Psi}_\lambda(t)\Lambda_2(t)) = \mu(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\Lambda_2(t)) \quad (4)$$

where $\Psi_\lambda(t) = \pi(\tilde{\Psi}_\lambda(t))$ is a loop in some $\mathrm{QSp}_\lambda(2n)$ via a projection $\pi : \mathrm{QSp}(2n) \to \mathrm{QSp}_\lambda(2n)$.

Theorem 1.5 In the above setting, we have

$$\mu(\Psi_\lambda(t)\Lambda(t), V) = \begin{cases} \mu(\Lambda(t), V) + \mu(\Psi_\lambda(t)V, V) & \lambda > 0, \\ -\mu(\Lambda(t), V) + \mu(\Psi_\lambda(t)V, V) & \lambda < 0. \end{cases} \quad (5)$$

2. Preliminaries

In this section, we recall some fundamental definitions and results that we will use throughout the article.

The vector space $\mathbb{R}^{2n}$ is called symplectic if it is equipped with a nondegenerate skew-symmetric bilinear 2-form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$, which is called a symplectic form.

In particular, the standard symplectic form ω_0 has the form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ under the coordinate system $\{x_1, \ldots, x_n; y_1, \ldots, y_n\}$ of $\mathbb{R}^{2n}$. For any vector $\xi_k = (u_k, v_k) \in \mathbb{R}^n \times \mathbb{R}^n$ with $k = 1, 2$, ω_0 also can be described as follow

$$\omega_0(\xi_1, \xi_2) = < u_1, v_2 > - < v_1, u_2 > = u_1^T v_2 - v_1^T u_2 \quad (6)$$

where $< \cdot, \cdot >$ is the standard inner product of $\mathbb{R}^n$.

There exist some special subspaces in a symplectic vector space. In particular, the subspace V of $(\mathbb{R}^{2n}, \omega)$ is called Lagrangian if V is identified with the subspace $V^\omega = \{v \in \mathbb{R}^{2n} \mid \omega(v, w) = 0, \forall w \in V\}$. All the Lagrangian subspaces of $\mathbb{R}^{2n}$ form a manifold, which is called Lagrangian Grassmanian and denoted by $\mathcal{L}(n)$. In this article, a loop $\Lambda(t)$ means $\Lambda : [0, 1] \to \mathcal{L}(n)$ is a smooth curve in $\mathcal{L}(n)$ and $\Lambda(0) = \Lambda(1)$.

A linear isomorphism $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is called symplectic if it presverses the symplectic form, explicitly, for any pair of vectors $\xi_1, \xi_2 \in \mathbb{R}^{2n}$

$$\omega_0(\xi_1, \xi_2) = \omega_0(f(\xi_1), f(\xi_2)) \tag{7}$$

and the equation (7) is usually abbreviated as $f^*\omega_0 = \omega_0$. We consider some isomorphisms analogous to symplectic isomorphisms.

Definition 2.1 A linear isomorphism $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is called quasi-symplectic if f satisfing $f^*\omega = \lambda\omega$ where λ is a nonzero constant. In particular, f is called anti-symplectic if $\lambda = -1$.

We can identify a linear map with a matrix in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ when we work in $\mathbb{R}^{2n}$ with a fixed cannonical basis. In this article we make no differentiation between the linear map and the corresponding matrix. Moreover, a matrix is called symplectic if the corresponding linear transformation is a symplectomorphism, is called quasi-symplectic if the corresponding linear isomorphism is quasi-symplectic. Note that symplectic matrix Ψ has the following form (one also can see [8, Page 20]).

Lemma 2.2 If Ψ has the form

$$\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are real $n \times n$ matrices, then Ψ is symplectic if and only if the following equations hold

$$A^T C = C^T A,$$

$$B^T D = D^T B,$$

$$A^T D - C^T B = I. \tag{8}$$

Proof. For any two vectors $z_k = (u_k, v_k) \in \mathbb{R}^n \times \mathbb{R}^n$ where $k = 1, 2$, We have

$$\omega(z_1, z_2) = u_1^T v_2 - v_1^T u_2 = \omega(\Psi z_1, \Psi z_2)$$

$$=< Au_1 + Bv_1, Cu_2 + Dv_2 > - < Au_2 + Bv_2, Cu_1 + Dv_1 >$$

$$= u_1^T (A^T C - C^T A) u_2 + v_1^T (B^T D - D^T B) v_2$$

$$+ u_1^T (A^T D - C^T B) v_2 + v_1^T (B^T C - D^T A) u_2.$$

This completes the proof. □

In this article, we denote by Ψ the symplectic matrix and we denote by Ψ_λ the quasi-symplectic matrix when the corresponding quasi-symplectic isomorphism Ψ_λ satisfies $\Psi_\lambda^*\omega = \lambda\omega$ for a nonzero constant λ. Analogous to the proof of Lemma 2.2, it is obvious that the quasi-symplectic matrices have the following form.

Corollary 2.3 If quasi-symplectic matrix Ψ_λ has the form

$$\Psi_\lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are real $n \times n$ matrices, then the following equations hold

$$A^T C = C^T A,$$

$$B^T D = D^T B,$$

$$A^T D - C^T B = \lambda I = diag(\lambda, \lambda, \cdots, \lambda). \tag{9}$$

In this article, we assume that symplectic matrix Ψ has the form $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying condition (8) and quasi-symplectic matrix Ψ_λ has the form $\Psi_\lambda = \begin{bmatrix} \lambda A & B \\ \lambda C & D \end{bmatrix}$ unless otherwise stated. Note that there exists a diffeomorphism $\delta : \Psi \mapsto \Psi_\lambda = \Psi I_\lambda$ with $I_\lambda = \begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix}$ where I is $n \times n$ identity matrix.

Remark 2.4 It is known that all symplectic matrices Ψ form a Lie group Sp($2n$). The diffeomorphism δ shows that the set consisting of all quasi-symplectic matrices Ψ_λ where λ is a nonzero constant, is a smooth manifold denoted by QSp$_\lambda$($2n$). It is easy to verify that QSp$_\lambda$($2n$) is not a group and the set consisting of all quasi-symplectic matrices Ψ_λ with any nonzero λ is a Lie group denoted by QSp($2n$).

In this article, we study how the quasi-symplectic matrices change Maslov index. Here we introduce the fundamental definitions about Maslov index, and give a definition of Maslov index based on [3].

Lemma 2.5 Let X and Y be real $n \times n$ matrices and define $\Lambda \subset \mathbb{R}^{2n}$ by

$$\Lambda = \mathrm{im} Z,$$

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{10}$$

Then $\Lambda \in \mathcal{L}(n)$ if and only if the matrix Z has rank n and

$$X^T Y = Y^T X.$$

Proof. Given two vectors $z_1 = (Xu, Yu)$ and $z_2 = (Xv, Yv)$ in Λ, according to formula (6), we have $\omega(z_1, z_2) = u^T(X^TY - Y^TX)v$. This completes the proof. □

A matrix $Z \in \mathbb{R}^{2n\times n}$ of the form (8) which satisfies $X^TY - Y^TX$ and has rank n is called a Lagrangian frame. If the matrix

$$U = X + iY$$

is unitary, Z is called a unitary Lagrangian frame.

Lemma 2.6 If $\Lambda \in \mathcal{L}(n)$ and $\Psi \in \mathrm{Sp}(2n)$, then $\Psi\Lambda \in \mathcal{L}(n)$. And if $\Psi_\lambda \in \mathrm{QSp}_\lambda(2n)$, then $\Psi_\lambda\Lambda \in \mathcal{L}(n)$.

Proof. Let Ψ be a symplectic matrix as in Lemma 2.2 and Z a Lagrangian frame of Λ. Then

$$\Psi Z = \begin{pmatrix} AX + BY \\ CX + DY \end{pmatrix}$$

is the frame of $\Psi\Lambda$. Given two vectors $z_1 = \Psi Zu$ and $z_2 = \Psi Zv$ in $\Psi\Lambda$, we have

$$\begin{aligned} \omega(z_1, z_2) &= \omega(\Psi Zu, \Psi Zv) \\ &= u^T(X^T A^T CX + X^T A^T DY + Y^T B^T CX + Y^T B^T DY \\ &\quad -X^T C^T AX - X^T C^T BY - Y^T D^T AX - Y^T D^T BY)v \\ &= u^T X^T (A^T D - C^T B)Yv + u^T Y^T (B^T C - D^T A)Xv \\ &= u^T (X^T Y - Y^T X)v \\ &= \omega(Zu, Zv). \end{aligned}$$

If $\Psi_\lambda \in \mathrm{QSp}_\lambda(2n)$, then it follows from condition (9) that

$$\omega(z_1, z_2) = \lambda u^T (X^T Y - Y^T X)v = \lambda\omega(Zu, Zv). \qquad \square$$

The Maslov index can be defined as the intersection number of the loop $\Lambda(t)$ with the Maslov cycle $\Sigma(n)$ of all Lagrangian subspaces which intersect one chosen Lagrangian subspace V nontransversally. This set is a singular hypersurface of $\mathcal{L}(n)$ of codimension 1 which admits a natural coorientation (one can see [2]). $\Sigma(n)$ is stratified by the dimension of the intersection with V. A generic loop will intersect only the highest stratum (where the interction is 1-dimensional) and all the intersections will be transverse.

More explicitly, let $\Lambda(t) : [0, 1] \to \mathcal{L}(n)$ be a path of Lagrangin planes with $\Lambda(0) = \Lambda$ and $\dot{\Lambda}(0) = \dot{\Lambda}$. We define a form

$$\begin{aligned} Q(\Lambda, \dot{\Lambda})(v) &= < X(0)u, \dot{Y}(0)u > - < Y(0)u, \dot{X}(0)u > \\ &= u^T (X(0)^T \dot{Y}(0) - Y(0)^T \dot{X}(0))u \end{aligned} \tag{11}$$

where $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ is a frame of $\Lambda(t)$ and $v = Z(0)u$. A crossing for $\Lambda(t)$ is a number $t \in [0, 1]$ for which $\Lambda(t) \in \Sigma(n)$. At each crossing time $t \in [0, 1]$ we define the crossing form

$$\Gamma(\Lambda, V, t) = Q(\Lambda(t), \dot{\Lambda}(t))|_{\Lambda(t)\cap V}\ . \tag{12}$$

A crossing is called regular if the crossing form $\Gamma(\Lambda, V, t)$ is nonsingular. Then for a loop $\Lambda(t)$:[0, 1] $\rightarrow \mathcal{L}(n)$ with only regular crossings, we define the Maslov index

$$\mu(\Lambda,V)=\sum_{t}\text{sign}\Gamma(\Lambda,V,t) \tag{13}$$

where sign$\Gamma(\Lambda, V, t)$ is the signature (the number of positive minus the number of negative eigenvalues) of the crossing form and the sum runs over all crossings t.

For a pair of loops of Lagrangian subspaces Λ_1, Λ_2 : [0, 1] $\rightarrow \mathcal{L}(n)$, we define the relative crossing form as follow

$$\begin{aligned}\Gamma(\Lambda_1,\Lambda_2,t)&=\Gamma(\Lambda_1,\Lambda_2(t),t)-\Gamma(\Lambda_2,\Lambda_1(t),t)\\ &=Q(\Lambda_1(t),\dot{\Lambda}_1(t))\,|_{\Lambda_1(t)\cap\Lambda_2(t)}-Q(\Lambda_2(t),\dot{\Lambda}_2(t))\,|_{\Lambda_1(t)\cap\Lambda_2(t)}\end{aligned} \tag{14}$$

and called the crossing t regular if the form is nondegenerate. For a pair of loops with only regular crossing we define the relative Maslov index by

$$\mu(\Lambda_1,\Lambda_2)=\sum_{t}\text{sign}\Gamma(\Lambda_1,\Lambda_2,t) \tag{15}$$

where the sum runs over all crossings t. And if $\Lambda_2 \equiv V$, this definition agrees with (13).

3. Proof of the main results

To prove the Theorem 1.1, we first consider the case that $\Lambda_2(t) \equiv V$ where V is the fixed Lagrangian subspace and let $\Lambda_1(t) = \Lambda(t)$ for simplicity.

Proof for one loop case of Theorem 1.1. It is sufficient to show how Ψ_λ acts on the signature of the form $Q(\Lambda, \dot{\Lambda})(v)$. Let the matrix Ψ_λ of Ψ_λ and the frame $Z(t)$ of $\Lambda(t)$ be defined as follow

$$\Psi_\lambda=\begin{bmatrix}\lambda A & B\\ \lambda C & D\end{bmatrix},$$

$$Z(t)=\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}.$$

Then the frame $\Psi_\lambda Z(t)$ of $\Psi_\lambda\Lambda(t)$ has the form

$$\Psi_\lambda Z(t)=\begin{pmatrix}E(t)\\ F(t)\end{pmatrix}=\begin{pmatrix}\lambda AX(t)+BY(t)\\ \lambda CX(t)+DY(t)\end{pmatrix}$$

and

$$\begin{aligned}Q(\Psi_\lambda\Lambda,\Psi_\lambda\dot{\Lambda})(\Psi_\lambda v)&=<E(t)u,\dot{F}(t)u>-<F(t)u,\dot{E}(t)u>\\ &=u^T(E(t)^T\dot{F}(t)-F(t)^T\dot{E}(t))u\end{aligned}$$

where $v = Z(t)u \in \Lambda(t) \cap V$ for some $u \in \mathbb{R}^n$ and

$$E(t)^T \dot{F}(t) - F(t)^T \dot{E}(t) = (\lambda AX(t) + BY(t))^T (\lambda C\dot{X}(t) + D\dot{Y}(t))$$

$$-(\lambda CX(t) + DY(t))^T (\lambda A\dot{X}(t) + B\dot{Y}(t))$$

$$= \lambda^2 X(t)^T (A^T C - C^T A)\dot{X}(t) + \lambda^2 Y(t)^T (B^T D - D^T B)\dot{Y}(t)$$

$$+\lambda X(t)^T (A^T D - C^T B)\dot{Y}(t) + \lambda Y(t)^T (B^T C - D^T A)\dot{X}(t).$$

It follows from Corollary 2.3 that

$$E(t)^T \dot{F}(t) - F(t)^T \dot{E}(t) = \lambda(X(t)^T \dot{Y}(t) - Y(t)^T \dot{X}(t)).$$

Thus

$$Q(\Psi_\lambda \Lambda, \Psi_\lambda \dot{\Lambda})(\Psi_\lambda v) = \lambda u^T (X(t)^T \dot{Y}(t) - Y(t)^T \dot{X}(t))u = \lambda Q(\Lambda, \dot{\Lambda})(v). \tag{16}$$

It is clear that Ψ_λ preserves the signature of the form $Q(\Lambda, \dot{\Lambda})(v)$ if λ is positive. When $\lambda = -1$, it is obvious that the positive eigenvalues of $Q(\Lambda, \dot{\Lambda})(v)$ become the negative eigenvalues of $Q(\Psi_\lambda \Lambda, \Psi_\lambda \dot{\Lambda})(\Psi_\lambda v)$. So if λ is negative, Ψ_λ changes the sign of signature of the form $Q(\Lambda, \dot{\Lambda})(v)$. □

When $\lambda = 1$, this proof also shows that symplectic matrices preserve the Maslov index.

To interpret the change of the Maslov index under quasi-symplectic isomorphisms, we assume the sympelctic vector space to be $\mathbb{R}^2$. Then each straight line crossing zero is a Lagrangian subspace. Choose y-axis as the chosen Lagrangian subspace, then $\Sigma(1)$ only contains y-axis. Then the Maslov index for a loop of Lagrangian subspaces is the intersection number with y-axis and intersecting upper self y-axis counterclockwise counts +1 meanwhile intersecting upper self y-axis clockwise counts −1. Also, intersecting lower self y-axis counterclockwise counts +1 meanwhile intersecting lower self y-axis clockwise counts −1.

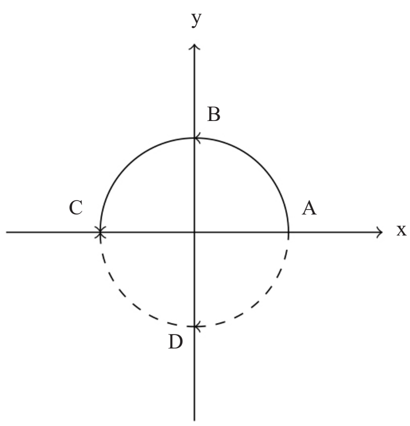

Figure 1. The $\lambda = -1$ case

We consider the $\lambda < 0$ case such as the anti-symplectic isomorphism $\Psi_{-1}(x, y) = (x, -y)$ and a loop $\gamma(t) : [0, 1] \to \mathcal{L}(1)$ such that $\gamma(0) = \gamma(1)$ is the x-axis. It is obvious that Ψ_{-1} maps y-axis to y-axis and does not change $\Sigma(1)$. In order to underline the anti-symplectic isomorphism action, we take a vector A at the starting point of the loop. The movement path of A can be described as the loop $\gamma(t)$. Then A intersects with upper self y-axis at a vector B and intersects with the terminal point, x-axis, at a vector C, see Figure 1. It follows that $\mu(\gamma) = +1$. However, the anti-symplectic isomorphism Ψ_{-1} reverses the loop such that A intersects with lower self y-axis at a vector D and intersects with the terminal point, x-axis, at a vector C. It follows that $\mu(\Psi_{-1}(\gamma)) = -1$. That is because Ψ_{-1} changes the orientation fo the loop but does not change the orientation of $\Sigma(1)$.

Consider the $\lambda > 0$ case such as the quasi-symplectic isomorphism $\Psi_3 = (\sqrt{3}x, \sqrt{3}y)$, Ψ_3 maps each line to itself then the following equations hold

$$\Psi_3(A) = \sqrt{3}A = A'$$

$$\Psi_3(B) = \sqrt{3}B = B'$$

$$\Psi_3(C) = \sqrt{3}C = C'.$$

Then $\Psi_3(A)$ intersects with upper self y-axis at a vector B' and intersects with the terminal point, x-axis, at a vector C', see Figure 2. It follows that $\mu(\Psi_3(\gamma)) = +1$.

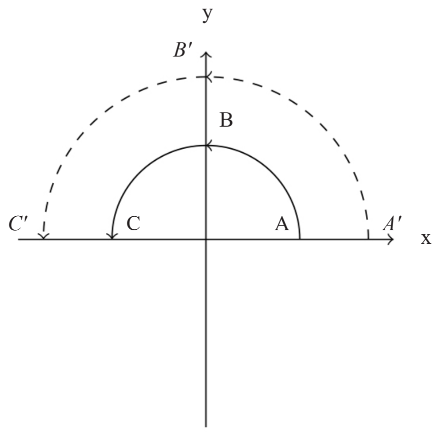

Figure 2. The $\lambda = 3$ case

For a pair of loops $\Lambda_1, \Lambda_2 : [0, 1] \to \mathcal{L}(n)$ with the frame

$$Z_1(t) = \begin{pmatrix} X_1(t) \\ Y_1(t) \end{pmatrix}$$

$$Z_2(t) = \begin{pmatrix} X_2(t) \\ Y_2(t) \end{pmatrix}$$

respectively, the relative crossing form (14) can be expressed as follow

$$\Gamma(\Lambda_1,\Lambda_2,t)=Q(\Lambda_1(t),\dot{\Lambda}_1(t))(v)-Q(\Lambda_2(t),\dot{\Lambda}_2(t))(v)$$

$$=<X_1(t)u,\dot{Y}_1(t)u>-<Y(t)_1u,\dot{X}_1(t)u>$$

$$-(<X_2(t)u,\dot{Y}_2(t)u>-<Y(t)_2u,\dot{X}_2(t)u>)$$

where $v = Z_1(t)u = Z_2(t)u \in \Lambda_1(t) \cap \Lambda_2(t)$ for some $u \in \mathbb{R}^n$. Then analogous to the one loop case, we have

Proof for two loops case of Theorem 1.1. In the above setting, for a quasi-symplectic isomorphism Ψ_λ with the form (9) the relative crossing form has the formula

$$\Gamma(\Psi_\lambda\Lambda_1,\Psi_\lambda\Lambda_2,t)=Q(\Psi_\lambda\Lambda_1(t),\Psi_\lambda\dot{\Lambda}_1(t))(\Psi_\lambda v)-Q(\Psi_\lambda\Lambda_2(t),\Psi_\lambda\dot{\Lambda}_2(t))(\Psi_\lambda v)$$

$$=\lambda(Q(\Lambda_1(t),\dot{\Lambda}_1(t))(v)-Q(\Lambda_2(t),\dot{\Lambda}_2(t))(v))$$

$$=\lambda\Gamma(\Lambda_1,\Lambda_2,t) \tag{17}$$

according to the result (16). Then

$$\mathrm{sign}\Gamma(\Psi_\lambda\Lambda_1,\Psi_\lambda\Lambda_2,t)=\begin{cases}\mathrm{sign}\Gamma(\Lambda_1,\Lambda_2,t) & \lambda>0,\\ -\mathrm{sign}\Gamma(\Lambda_1,\Lambda_2,t) & \lambda<0.\end{cases}$$

This completes the proof of this case and hence the proof of Theorem 1.1. □

A loop $\Psi_\lambda(t)$ in $\mathrm{QSp}_\lambda(2n)$ acting on a fixed Lagrangian subspace V forms a loop $\Psi_\lambda(t)V$ in $\mathcal{L}(n)$ naturally. Then we can define the crossing form $\Gamma(\Psi_\lambda(t)V, V, t)$ of $\Psi_\lambda(t)V$ as in (12) and the Maslov index $\mu(\Psi_\lambda(t)V, V)$ as in (13). The Maslov index for the case that $\mathrm{QSp}_\lambda(2n) = \mathrm{Sp}(2n)$. When $\lambda = 1$ is the definition of Maslov index for a loop of symplectic matrices in [3]. To prove Theorem 1.2, we show that $\mu(\Psi_\lambda(t)V, V)$ is nondependent on the choice of V.

Note that the crossing form $\Gamma(\Psi_\lambda(t)V, V, t)$ is a quadratic form. Explicitly, let $\Psi_\lambda(t)$ and the frame of V be expressed as follows

$$\Psi_\lambda(t)=\begin{bmatrix}A(t) & B(t)\\ C(t) & D(t)\end{bmatrix}$$

$$Z=\begin{pmatrix}X\\ Y\end{pmatrix}$$

where $\Psi_\lambda(t)$ satisfying the condition (9) for each t. Then the crossing form has the formula where $v = ((A(t)X + B(t)Y)u, (C(t)X + D(t)Y)u) = (Xu, Yu) \in \Psi_\lambda(t)V \cap V$ for some $u \in \mathbb{R}^n$

$$\Gamma(\Psi_\lambda(t)V, V, t) = Q(\Psi_\lambda(t)V, \dot{\Psi}_\lambda(t)V)(v)$$

$$= < (A(t)X + B(t)Y)u, (\dot{C}(t)X + \dot{D}(t)Y)u >$$

$$- < (C(t)X + D(t)Y)u, (\dot{A}(t)X + \dot{B}(t)Y)u >$$

$$= < Xu, (A(t)^T \dot{C}(t) - C(t)^T \dot{A}(t))Xu > + < Yu, (B(t)^T \dot{D}(t) - D(t)^T \dot{B}(t))Yu >$$

$$+ < Xu, 2(A^T(t)\dot{D}(t) - C^T(t)\dot{B}(t))Yu > \quad (18)$$

since the following equations hold according to Corollary (8)

$$\dot{A}^T(t)C(t) + A(t)^T \dot{C}(t) = \dot{C}^T(t)A(t) + C(t)^T \dot{A}(t),$$

$$\dot{B}^T(t)D(t) + B(t)^T \dot{D}(t) = \dot{D}^T(t)B(t) + D(t)^T \dot{B}(t),$$

$$\dot{A}^T(t)D(t) + A(t)^T \dot{D}(t) - \dot{C}^T(t)B(t) - C(t)^T \dot{B}(t) = 0.$$

Formula (18) implies that the signature of this crossing form is independent on the choice of $V \in \mathcal{L}(n)$.

On the other hand, for any V, $V' \in \mathcal{L}(n)$, suppose $\Psi' \in \mathrm{Sp}(2n)$ such that $V = \Psi'V'$, then

$$\mu(\Psi_\lambda(t)V, V) = \mu(\Psi_\lambda(t)\Psi'V', \Psi'V') = \mu(\Psi'^{-1}\Psi_\lambda(t)\Psi'V', V')$$

where $\Psi'^{-1}\Psi_\lambda(t)\Psi'$ can be identified with $\Psi_\lambda(t)$. Hence

Lemma 3.1 For any two Lagrangian subspace V, $V' \in \mathcal{L}(n)$, we have

$$\mu(\Psi_\lambda(t)V, V) = \mu(\Psi_\lambda(t)V', V'). \quad (19)$$

Based on Lemma 3.1 and formula (17), the relative crossing form at a crossing t has the analogous result to formula (17) as follow.

$$\Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2, t) = \Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2(t), t) - \Gamma(\Psi_\lambda\Lambda_2, \Psi_\lambda\Lambda_1(t), t)$$

$$= Q(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\dot{\Lambda}_1(t))|_{\Lambda_1(t)\cap\Lambda_2(t)} - Q(\Psi_\lambda(t)\Lambda_2(t), \Psi_\lambda(t)\dot{\Lambda}_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

$$+ Q(\Psi_\lambda(t)\Lambda_1(t), \dot{\Psi}_\lambda(t)\Lambda_1(t))|_{\Lambda_1(t)\cap\Lambda_2(t)} - Q(\Psi_\lambda(t)\Lambda_2(t), \dot{\Psi}_\lambda(t)\Lambda_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

$$= \lambda\Gamma(\Lambda_1, \Lambda_2, t). \quad (20)$$

This completes the proof of Theorem 1.2. Consider the case that $\Lambda_2(t) \equiv V$ and $\Lambda_1(t) \equiv \Lambda(t)$, then the following term

$$Q(\Psi_\lambda(t)\Lambda_2(t), \Psi_\lambda(t)\dot{\Lambda}_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

vanishes. Then

$$\Gamma(\Psi_\lambda\Lambda, \Psi_\lambda V, t) = \lambda\Gamma(\Lambda, V, t). \tag{21}$$

Hence Remark 1.3 holds.

Consider a loop $\tilde{\Psi}_\lambda(t)$ in QSp($2n$) with the form

$$\tilde{\Psi}_\lambda(t) = \begin{bmatrix} \lambda(t)A(t) & B(t) \\ \lambda(t)C(t) & D(t) \end{bmatrix} \tag{22}$$

where $\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$ is a symplectic matrix satisfying condition (8) and $\lambda(t)$ is a smooth nonzero function satisfying the condition

$$\lambda(0) = \lambda(1),$$

$$\dot{\lambda}(0) = \dot{\lambda}(1). \tag{23}$$

Then for each crossing t the relative crossing form is analogous to formula (20) as follow

$$\Gamma(\tilde{\Psi}_\lambda\Lambda_1, \tilde{\Psi}_\lambda\Lambda_2, t) = \lambda(t)\Gamma(\Lambda_1, \Lambda_2, t). \tag{24}$$

When $\lambda(t)$ is nonnegative, the signature of $\Gamma(\tilde{\Psi}_\lambda\Lambda_1, \tilde{\Psi}_\lambda\Lambda_2, t)$ is same as the signature of $\Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2, t)$ where $\Psi_\lambda(t)$ has the form $\Psi_\lambda(t) = \begin{bmatrix} \lambda A(t) & B(t) \\ \lambda C(t) & D(t) \end{bmatrix}$ with $\lambda > 0$. Moreover, $\Psi_\lambda(t)$ can be viewed as the image of $\tilde{\Psi}_\lambda(t)$ under the projection

$$\pi : \mathrm{QSp}(2n) \to \mathrm{QSp}_\lambda(2n)$$

$$\begin{bmatrix} \lambda(t)A & B \\ \lambda(t)C & D \end{bmatrix} \mapsto \begin{bmatrix} \lambda A & B \\ \lambda C & D \end{bmatrix}. \tag{25}$$

Hence Remark 1.4 holds.

Let $\Psi_\lambda(t)$ be a loop in QSp$_\lambda$($2n$), $\Lambda(t)$ a loop in $\mathcal{L}(n)$ and V a fixed Lagrangian subspace, it is hard to find out the relationship between the signature of $\Gamma(\Psi_\lambda\Lambda, V, t)$ and the one of $\Gamma(\Lambda, V, t)$. Hence we apply the analogous way in [3] and [8], firstly we review some results.

Lemma 3.2 If $\Psi = \Psi^T \in \mathrm{Sp}(2n)$ is a symmetric, positive definite and symplectic matrix, then $\Psi^s \in \mathrm{Sp}(2n)$ for any $s \geq 0$.

Proof. Let λ_i be the eigenvalues of Ψ and E_i the corresponding eigenvector spaces for $i = 1, \ldots, k$. It is known that all the eigenvalues are positive and Ψ determines a orthogonal decomposition

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{k} E_i.$$

For any two nonzero vectors

$$\xi = \sum_{i=1}^{k} a_i u_i,\ \eta = \sum_{i=1}^{k} b_i v_i$$

in $\mathbb{R}^{2n}$ where u_i, $v_i \in E_i$ for $i = 1, \ldots, k$. Note that

$$\omega_0(u_i, v_j) = \omega_0(\Psi u_i, \Psi v_j) = \lambda_i \lambda_j \omega_0(u_i, v_j)$$

for all i, j and then either $\lambda_i \lambda_j = 0$ or $\omega_0(u_i, v_j) = 0$ holds. This implies that

$$\omega_0(\Psi^s \xi, \Psi^s \eta) = \sum_{i,j=1}^{k} a_i b_j (\lambda_i \lambda_j)^s \omega_0(u_i, v_j) = \sum_{i,j=1}^{k} a_i b_j \omega_0(u_i, v_j) = \omega_0(\xi, \eta)$$

and hence $\Psi^s \in \mathrm{Sp}(2n)$ for any $s \geq 0$. □

Define a map $g : [0, 1] \times \mathrm{Sp}(2n) \to \mathrm{Sp}(2n)$ by

$$g(t, \Psi) = g_t(\Psi) = \Psi(\Psi^T \Psi)^{-t/2}. \tag{26}$$

Since $\Psi^T\Psi$ is symmetric, positive definite symplectic matirx, then $(\Psi^T\Psi)^{-t/2} \in \mathrm{Sp}(2n)$ according to Lemma 3.2 and hence $g_t(\Psi) \in \mathrm{Sp}(2n)$ for any $t \geq 0$ and any $\Psi \in \mathrm{Sp}(2n)$. Moreover, g is continuous, and

$$g_0 = id,\ g_t|_{\mathrm{U}(n)} = id \text{ for any } t,\ g_1(\mathrm{Sp}(2n)) = \mathrm{U}(n)$$

since $g_1(\Psi)$ is also orthogonal. Hence

Lemma 3.3 $\mathrm{Sp}(2n)$ is homotopy equivalent to $\mathrm{U}(n)$.

Robbin and Salamon showed in [3] that the Maslov index is a homotopy invariant and has the equivalent definition as follow.

Lemma 3.4 Let $\Lambda(t)$ be a loop in $\mathcal{L}(n)$ and $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ a lift of unitary frames. Then for any $V \in \mathcal{L}(n)$

$$\mu(\Lambda, V) = \frac{\alpha(1) - \alpha(0)}{\pi},\ \det_{\mathbb{C}}(X(t) + iY(t)) = e^{i\alpha(t)}. \tag{27}$$

McDuff and Salamon generalized this definition in [8] to the Maslov index $\mu(\Psi(t)) = \mu(\Psi(t)V, V)$ for a loop $\Psi(t)$ in $\mathrm{Sp}(2n)$ and any fixed Lagangrian subspace V.

Lemma 3.5 Let $\Psi(t)$ be a loop in $\mathrm{Sp}(2n)$ and $U(t) = A(t) + iC(t) = \Psi(t)(\Psi(t)^T\Psi(t))^{-1/2}$ a lift of unitary matrices. Then

$$\mu(\Psi(t)) = \frac{\beta(1) - \beta(0)}{\pi},\ \det_{\mathbb{C}}(A(t) + iC(t)) = e^{2i\beta(t)}. \tag{28}$$

Remark 3.6 In Lemma 3.4 we consider the unoriented Lagrangian subspaces and in Lemma 3.5 the loop $\Psi(t)$ endows the Lagrangian subspaces with an orientation. In other words, it may occur that $\Psi(t_1)V = \Psi(t_2)V$ when $\Psi(t_1) \neq \Psi(t_2)$. Hence the differences between equations (27) and (28) emerge.

It is sufficient to suppose that $\Psi(t) = U(t)$ according to Lemma 3.3, then the frames of $\Psi(t)\Lambda(t)$ have the form $Z(t) = \begin{pmatrix} A(t)X(t) - C(t)Y(t) \\ C(t)X(t) + A(t)Y(t) \end{pmatrix}$, and hence

$$\det_{\mathbb{C}}((A(t)X(t) - C(t)Y(t)) + i(C(t)X(t) + A(t)Y(t))) = e^{i(\alpha(t)+2\beta(t))}$$

$$\mu(\Psi\Lambda, V) = \frac{\alpha(1) + 2\beta(1) - \alpha(0) - 2\beta(0)}{\pi} = \mu(\Lambda, V) + 2\mu(\Psi). \tag{29}$$

Let $\Lambda(t) \equiv V$, equation (29) also shows that $\mu(\Psi V, V) = 2\mu(\Psi)$. Hence

$$\mu(\Psi\Lambda, V) = \mu(\Lambda, V) + \mu(\Psi V, V). \tag{30}$$

Consider the quasi-symplectic case, note that

$$\Psi_\lambda = \Psi I_\lambda$$

where $I_\lambda = \begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix}$ and $\Psi = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathrm{Sp}(2n)$.

Then according to (30), we have

$$\mu(\Psi_\lambda\Lambda, V) = \mu(\Psi(I_\lambda\Lambda), V) = \mu(I_\lambda\Lambda, V) + \mu(\Psi V, V). \tag{31}$$

Let the Lagrangian frame of Λ be $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$, then $Z_\lambda(t) = \begin{pmatrix} \lambda X(t) \\ Y(t) \end{pmatrix}$ is a frame of $I_\lambda\Lambda$, which is not a Lagrangian frame. We transform $Z_\lambda(t)$ to a Lagrangian frame

$$Z'_\lambda(t) = \begin{pmatrix} \frac{\lambda}{|\lambda|} X(t) \\ Y(t) \end{pmatrix}. \tag{32}$$

When $\lambda > 0$, $Z'_\lambda(t) = Z(t)$ and $\Lambda = I_\lambda\Lambda$, then equation (31) yields

$$\mu(\Psi_\lambda\Lambda, V) = \mu(I_\lambda\Lambda, V) + \mu(\Psi V, V) = \mu(\Lambda, V) + \mu(\Psi V, V). \tag{33}$$

When $\lambda > 0$, $Z'_\lambda(t) = \begin{pmatrix} -X(t) \\ Y(t) \end{pmatrix}$, according to the crossing form (12) and this Lagrangian frame, we have

$$\Gamma(I_\lambda\Lambda, V, t) = -\Gamma(\Lambda, V, t). \tag{34}$$

Equation (30) shows that $\Gamma(I_\lambda\Lambda, V, t)$ has the same crossing time as $\Gamma(\Lambda, V, t)$ and hence

$$\mu(I_\lambda \Lambda, V) = -\mu(\Lambda, V) \tag{35}$$

$$\mu(\Psi_\lambda \Lambda, V) = \mu(I_\lambda \Lambda, V) + \mu(\Psi V, V) = -\mu(\Lambda, V) + \mu(\Psi V, V). \tag{36}$$

Moreover, let $\Lambda(t) \equiv V$, eqution (29) also shows

$$\mu(\Psi_\lambda V, V) = \mu(\Psi V, V). \tag{37}$$

Hence Theorem 1.5 holds according to equations (33), (36) and (37).

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Conflict of interest

The author declare that there is no personal or organizational conflict of interest with this work.

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