Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Maslov Index and Quasi-Symplectic Isomorphisms

Jin Wu

Department of Mathematics, Jinan University, Guangzhou 510632, China
E-mail:wj393745678@163.com

Received: 13 February 2022; **Revised:** 24 April 2022; **Accepted:** 29 April 2022

Abstract: Maslov index is defined as the number of the intersection of a loop of Lagrangian subspaces with a 1-codimensional cycle in the Lagrangian Grassmannian. It is well-known that linear symplectomorphisms preverse the Maslov index. We show how quasi-symplectic isomorphisms change Maslov index.

***Keywords*:** Maslov index, linear symplectomorphism, Lagrangian Grassmannian

MSC: 53D05, 53D12

1. Introduction

In the process of treating the asymptotic expression of the solution of the quasiclassical question, e.g. the Schrödinger equation, Maslov [1] defined an index by the intersection number of an oriented closed curve in an n-dimensional Lagrangian submanifold M with a two-sided 1-codimensional cycle on M. Arnold [2] proved that the Maslov index coincides with a cohomology class and also with the index for the corresponding loop in the Lagrangian Grassmaniann $\mathcal{L}(n)$ (the manifold consists of all Lagrangian subspaces in $\mathbb{R}^{2n}$), which is defined as an intersection number of this corresponding loop with a singular cycle called Maslov cycle. Arnold's work can be generalized to the case of a path of Lagrangian subspaces with its endpoints lying in the complement of the Maslov cycle. Robbin and Salamon [3] generalized a new definition of Maslov index for any path even if its endpoints lie in the Maslov cycle. They defined an associated form $Q : \mathcal{L}(n) \times \mathcal{L}(n) \to \mathbb{R}$, when a Lagrangian subspace is represented by a Lagrangian frame, the form Q can be expressed explicitly in a matrix form. Robbin and Salamon also defined the relative Maslov index for a pair of loops of Lagrangian subspaces. In [4] Robbin and Salamon also showed that the Maslov index agrees with the spectral flow of an associated matrix family. On the other hand, Cappell, Lee and Miller [5] showed four definitions of Maslov index with respect to a pair of loops of Lagrangian subspaces and proved that they are equivalent to each other. The Maslov index also can be used to other objects, for example, Schrödinger operators [6], loops in a coisotropic submanifold [7] and so on. So it is necessary to develop the properties of the Maslov index.

One important property of Maslov index is that the linear symplectomorphisms, the linear isomorphism of $\mathbb{R}^{2n}$ preserving the symplectic form, preserve Maslov index. In this article, we study how more general isomorphisms, such as quasi-symplectic isomorphisms which change the symplec-tic form based on a fixed coefficient, act on Maslov index. Explicitly, let the vector space $\mathbb{R}^{2n}$ be equipped with the standard symplectic form ω_0, the quasi-symplectic isomorphisms Ψ_λ in $(\mathbb{R}^{2n}, \omega_0)$ are the isomorphisms satisfying $\Psi_\lambda^*\omega_0 = \lambda\omega_0$ with a nonzero constant coefficient λ. As in [3]

DOI: https://doi.org/10.37256/cm.3220221363

the Maslov index μ for a loop $\Lambda(t)$ of Lagrangian subspaces is defined as the sum of signature of a crossing form $\Gamma(\Lambda, V)$ where V is a fixed Lagrangian subspace (see (12)) and the Maslov index for a pair of loops is defined as the sum of the signature of a relative crossing form (see (14)). Then

Theorem 1.1 For a pair of loops $\Lambda_1(t)$, $\Lambda_2(t)$ in $\mathcal{L}(n)$, quasi-symplectic isomorphisms Ψ_λ change the sign of the Maslov index depended on the sign of the coefficient λ. i.e.,

$$\mu(\Psi_\lambda\Lambda_1(t), \Psi_\lambda\Lambda_2(t)) = \begin{cases} \mu(\Lambda_1(t), \Lambda_2(t)) & \lambda > 0, \\ -\mu(\Lambda_1(t), \Lambda_2(t)) & \lambda < 0. \end{cases} \tag{1}$$

Analogous to $\mathrm{Sp}(2n)$, all the λ-coefficient quasi-symplectic matrices form a manifold denoted by $\mathrm{QSp}_\lambda(2n)$. Let $\Psi_\lambda(t)$ be a loop in $\mathrm{QSp}_\lambda(2n)$, then

Theorem 1.2 In the above setting, we have

$$\mu(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\Lambda_2(t)) = \begin{cases} \mu(\Lambda_1(t), \Lambda_2(t)) & \lambda > 0, \\ -\mu(\Lambda_1(t), \Lambda_2(t)) & \lambda < 0. \end{cases} \tag{2}$$

In particular, if $\Lambda_2(t) \equiv V$ where V is the fixed Lagrangian subspace and let $\Lambda_1(t) = \Lambda(t)$ for simplicity, we have

Remark 1.3

$$\mu(\Psi_\lambda(t)\Lambda(t), \Psi_\lambda(t)V) = \begin{cases} \mu(\Lambda(t), V) & \lambda > 0, \\ -\mu(\Lambda(t), V) & \lambda < 0. \end{cases} \tag{3}$$

All the quasi-symplectic matrices with any nonzero coefficient also form a Lie group denoted by $\mathrm{QSp}(2n)$. For a loop $\tilde{\Psi}_\lambda(t)$ in $\mathrm{QSp}(2n)$, the coefficient also is a smooth function $\lambda(t)$ which is nonzero for any t. We have

Remark 1.4

$$\mu(\tilde{\Psi}_\lambda(t)\Lambda_1(t), \tilde{\Psi}_\lambda(t)\Lambda_2(t)) = \mu(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\Lambda_2(t)) \tag{4}$$

where $\Psi_\lambda(t) = \pi(\tilde{\Psi}_\lambda(t))$ is a loop in some $\mathrm{QSp}_\lambda(2n)$ via a projection $\pi : \mathrm{QSp}(2n) \to \mathrm{QSp}_\lambda(2n)$.

Theorem 1.5 In the above setting, we have

$$\mu(\Psi_\lambda(t)\Lambda(t), V) = \begin{cases} \mu(\Lambda(t), V) + \mu(\Psi_\lambda(t)V, V) & \lambda > 0, \\ -\mu(\Lambda(t), V) + \mu(\Psi_\lambda(t)V, V) & \lambda < 0. \end{cases} \tag{5}$$

2. Preliminaries

In this section, we recall some fundamental definitions and results that we will use throughout the article.

The vector space $\mathbb{R}^{2n}$ is called symplectic if it is equipped with a nondegenerate skew-symmetric bilinear 2-form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$, which is called a symplectic form.

In particular, the standard symplectic form ω_0 has the form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ under the coordinate system $\{x_1, \ldots, x_n; y_1, \ldots, y_n\}$ of $\mathbb{R}^{2n}$. For any vector $\xi_k = (u_k, v_k) \in \mathbb{R}^n \times \mathbb{R}^n$ with $k = 1, 2$, ω_0 also can be described as follow

$$\omega_0(\xi_1, \xi_2) = \langle u_1, v_2 \rangle - \langle v_1, u_2 \rangle = u_1^T v_2 - v_1^T u_2 \tag{6}$$

where $< \cdot, \cdot >$ is the standard inner product of $\mathbb{R}^n$.

There exist some special subspaces in a symplectic vector space. In particular, the subspace V of $(\mathbb{R}^{2n}, \omega)$ is called Lagrangian if V is identified with the subspace $V^{\omega} = \{v \in \mathbb{R}^{2n} \mid \omega(v, w) = 0, \forall w \in V\}$. All the Lagrangian subspaces of $\mathbb{R}^{2n}$ form a manifold, which is called Lagrangian Grassmanian and denoted by $\mathcal{L}(n)$. In this article, a loop $\Lambda(t)$ means $\Lambda : [0, 1] \to \mathcal{L}(n)$ is a smooth curve in $\mathcal{L}(n)$ and $\Lambda(0) = \Lambda(1)$.

A linear isomorphism $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is called symplectic if it presverses the symplectic form, explicitly, for any pair of vectors $\xi_1, \xi_2 \in \mathbb{R}^{2n}$

$$\omega_0(\xi_1, \xi_2) = \omega_0(f(\xi_1), f(\xi_2)) \tag{7}$$

and the equation (7) is usually abbreviated as $f^*\omega_0 = \omega_0$. We consider some isomorphisms analogous to symplectic isomorphisms.

Definition 2.1 A linear isomorphism $f : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is called quasi-symplectic if f satisfing $f^*\omega = \lambda\omega$ where λ is a nonzero constant. In particular, f is called anti-symplectic if $\lambda = -1$.

We can identify a linear map with a matrix in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ when we work in $\mathbb{R}^{2n}$ with a fixed cannonical basis. In this article we make no differentiation between the linear map and the corresponding matrix. Moreover, a matrix is called symplectic if the corresponding linear transformation is a symplectomorphism, is called quasi-symplectic if the corresponding linear isomorphism is quasi-symplectic. Note that symplectic matrix Ψ has the following form (one also can see [8, Page 20]).

Lemma 2.2 If Ψ has the form

$$\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are real $n \times n$ matrices, then Ψ is symplectic if and only if the following equations hold

$$A^T C = C^T A,$$

$$B^T D = D^T B,$$

$$A^T D - C^T B = I. \tag{8}$$

Proof. For any two vectors $z_k = (u_k, v_k) \in \mathbb{R}^n \times \mathbb{R}^n$ where $k = 1, 2$, We have

$$\omega(z_1, z_2) = u_1^T v_2 - v_1^T u_2 = \omega(\Psi z_1, \Psi z_2)$$

$$=< Au_1 + Bv_1, Cu_2 + Dv_2 > - < Au_2 + Bv_2, Cu_1 + Dv_1 >$$

$$= u_1^T (A^T C - C^T A)u_2 + v_1^T (B^T D - D^T B)v_2$$

$$+ u_1^T (A^T D - C^T B)v_2 + v_1^T (B^T C - D^T A)u_2.$$

This completes the proof. □

In this article, we denote by Ψ the symplectic matrix and we denote by Ψ_λ the quasi-symplectic matrix when the corresponding quasi-symplectic isomorphism Ψ_λ satisfies $\Psi_\lambda^*\omega = \lambda\omega$ for a nonzero constant λ. Analogous to the proof of Lemma 2.2, it is obvious that the quasi-symplectic matrices have the following form.

Corollary 2.3 If quasi-symplectic matrix Ψ_λ has the form

$$\Psi_\lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are real $n \times n$ matrices, then the following equations hold

$$A^T C = C^T A,$$

$$B^T D = D^T B,$$

$$A^T D - C^T B = \lambda I = diag(\lambda, \lambda, \cdots, \lambda). \tag{9}$$

In this article, we assume that symplectic matrix Ψ has the form $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying condition (8) and quasi-symplectic matrix Ψ_λ has the form $\Psi_\lambda = \begin{bmatrix} \lambda A & B \\ \lambda C & D \end{bmatrix}$ unless otherwise stated. Note that there exists a diffeomorphism $\delta : \Psi \mapsto \Psi_\lambda = \Psi I_\lambda$ with $I_\lambda = \begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix}$ where I is $n \times n$ identity matrix.

Remark 2.4 It is known that all symplectic matrices Ψ form a Lie group Sp($2n$). The diffeomorphism δ shows that the set consisting of all quasi-symplectic matrices Ψ_λ where λ is a nonzero constant, is a smooth manifold denoted by $\mathrm{QSp}_\lambda(2n)$. It is easy to verify that $\mathrm{QSp}_\lambda(2n)$ is not a group and the set consisting of all quasi-symplectic matrices Ψ_λ with any nonzero λ is a Lie group denoted by QSp($2n$).

In this article, we study how the quasi-symplectic matrices change Maslov index. Here we introduce the fundamental definitions about Maslov index, and give a definition of Maslov index based on [3].

Lemma 2.5 Let X and Y be real $n \times n$ matrices and define $\Lambda \subset \mathbb{R}^{2n}$ by

$$\Lambda = \mathrm{im} Z,$$

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{10}$$

Then $\Lambda \in \mathcal{L}(n)$ if and only if the matrix Z has rank n and

$$X^T Y = Y^T X.$$

Proof. Given two vectors $z_1 = (Xu, Yu)$ and $z_2 = (Xv, Yv)$ in Λ, according to formula (6), we have $\omega(z_1, z_2) = u^T(X^T Y - Y^T X)v$. This completes the proof. □

A matrix $Z \in \mathbb{R}^{2n\times n}$ of the form (8) which satisfies $X^T Y - Y^T X$ and has rank n is called a Lagrangian frame. If the matrix

$$U = X + iY$$

is unitary, Z is called a unitary Lagrangian frame.

Lemma 2.6 If $\Lambda \in \mathcal{L}(n)$ and $\Psi \in \mathrm{Sp}(2n)$, then $\Psi\Lambda \in \mathcal{L}(n)$. And if $\Psi_\lambda \in \mathrm{QSp}_\lambda(2n)$, then $\Psi_\lambda\Lambda \in \mathcal{L}(n)$.

Proof. Let Ψ be a symplectic matrix as in Lemma 2.2 and Z a Lagrangian frame of Λ. Then

$$\Psi Z = \begin{pmatrix} AX + BY \\ CX + DY \end{pmatrix}$$

is the frame of $\Psi\Lambda$. Given two vectors $z_1 = \Psi Zu$ and $z_2 = \Psi Zv$ in $\Psi\Lambda$, we have

$$\begin{aligned}\omega(z_1, z_2) &= \omega(\Psi Zu, \Psi Zv)\\ &= u^T(X^T A^T CX + X^T A^T DY + Y^T B^T CX + Y^T B^T DY\\ &\quad -X^T C^T AX - X^T C^T BY - Y^T D^T AX - Y^T D^T BY)v\\ &= u^T X^T (A^T D - C^T B)Yv + u^T Y^T (B^T C - D^T A)Xv\\ &= u^T(X^T Y - Y^T X)v\\ &= \omega(Zu, Zv).\end{aligned}$$

If $\Psi_\lambda \in \mathrm{QSp}_\lambda(2n)$, then it follows from condition (9) that

$$\omega(z_1, z_2) = \lambda u^T(X^T Y - Y^T X)v = \lambda\omega(Zu, Zv). \qquad \square$$

The Maslov index can be defined as the intersection number of the loop $\Lambda(t)$ with the Maslov cycle $\Sigma(n)$ of all Lagrangian subspaces which intersect one chosen Lagrangian subspace V nontransversally. This set is a singular hypersurface of $\mathcal{L}(n)$ of codimension 1 which admits a natural coorientation (one can see [2]). $\Sigma(n)$ is stratified by the dimension of the intersection with V. A generic loop will intersect only the highest stratum (where the interction is 1-dimensional) and all the intersections will be transverse.

More explicitly, let $\Lambda(t) : [0, 1] \to \mathcal{L}(n)$ be a path of Lagrangin planes with $\Lambda(0) = \Lambda$ and $\dot{\Lambda}(0) = \dot{\Lambda}$. We define a form

$$\begin{aligned}Q(\Lambda, \dot{\Lambda})(v) &= < X(0)u, \dot{Y}(0)u > - < Y(0)u, \dot{X}(0)u >\\ &= u^T(X(0)^T \dot{Y}(0) - Y(0)^T \dot{X}(0))u \qquad (11)\end{aligned}$$

where $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ is a frame of $\Lambda(t)$ and $v = Z(0)u$. A crossing for $\Lambda(t)$ is a number $t \in [0, 1]$ for which $\Lambda(t) \in \Sigma(n)$. At each crossing time $t \in [0, 1]$ we define the crossing form

$$\Gamma(\Lambda, V, t) = Q(\Lambda(t), \dot{\Lambda}(t))|_{\Lambda(t)\cap V}. \qquad (12)$$

A crossing is called regular if the crossing form $\Gamma(\Lambda, V, t)$ is nonsingular. Then for a loop $\Lambda(t)$:$[0, 1] \to \mathcal{L}(n)$ with only regular crossings, we define the Maslov index

$$\mu(\Lambda, V) = \sum_t \mathrm{sign}\Gamma(\Lambda, V, t) \tag{13}$$

where $\mathrm{sign}\Gamma(\Lambda, V, t)$ is the signature (the number of positive minus the number of negative eigenvalues) of the crossing form and the sum runs over all crossings t.

For a pair of loops of Lagrangian subspaces $\Lambda_1, \Lambda_2 : [0, 1] \to \mathcal{L}(n)$, we define the relative crossing form as follow

$$\begin{aligned}\Gamma(\Lambda_1, \Lambda_2, t) &= \Gamma(\Lambda_1, \Lambda_2(t), t) - \Gamma(\Lambda_2, \Lambda_1(t), t)\\ &= Q(\Lambda_1(t), \dot{\Lambda}_1(t))\,|_{\Lambda_1(t)\cap\Lambda_2(t)} - Q(\Lambda_2(t), \dot{\Lambda}_2(t))\,|_{\Lambda_1(t)\cap\Lambda_2(t)}\end{aligned} \tag{14}$$

and called the crossing t regular if the form is nondegenerate. For a pair of loops with only regular crossing we define the relative Maslov index by

$$\mu(\Lambda_1, \Lambda_2) = \sum_t \mathrm{sign}\Gamma(\Lambda_1, \Lambda_2, t) \tag{15}$$

where the sum runs over all crossings t. And if $\Lambda_2 \equiv V$, this definition agrees with (13).

3. Proof of the main results

To prove the Theorem 1.1, we first consider the case that $\Lambda_2(t) \equiv V$ where V is the fixed Lagrangian subspace and let $\Lambda_1(t) = \Lambda(t)$ for simplicity.

Proof for one loop case of Theorem 1.1. It is sufficient to show how Ψ_λ acts on the signature of the form $Q(\Lambda, \dot{\Lambda})(v)$. Let the matrix Ψ_λ of Ψ_λ and the frame $Z(t)$ of $\Lambda(t)$ be defined as follow

$$\Psi_\lambda = \begin{bmatrix} \lambda A & B \\ \lambda C & D \end{bmatrix},$$

$$Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}.$$

Then the frame $\Psi_\lambda Z(t)$ of $\Psi_\lambda \Lambda(t)$ has the form

$$\Psi_\lambda Z(t) = \begin{pmatrix} E(t) \\ F(t) \end{pmatrix} = \begin{pmatrix} \lambda AX(t) + BY(t) \\ \lambda CX(t) + DY(t) \end{pmatrix}$$

and

$$\begin{aligned}Q(\Psi_\lambda \Lambda, \Psi_\lambda \dot{\Lambda})(\Psi_\lambda v) &= < E(t)u, \dot{F}(t)u > - < F(t)u, \dot{E}(t)u >\\ &= u^T (E(t)^T \dot{F}(t) - F(t)^T \dot{E}(t))u\end{aligned}$$

where $v = Z(t)u \in \Lambda(t) \cap V$ for some $u \in \mathbb{R}^n$ and

$$\begin{aligned} E(t)^T \dot{F}(t) - F(t)^T \dot{E}(t) &= (\lambda AX(t) + BY(t))^T (\lambda C\dot{X}(t) + D\dot{Y}(t)) \\ &\quad -(\lambda CX(t) + DY(t))^T (\lambda A\dot{X}(t) + B\dot{Y}(t)) \\ &= \lambda^2 X(t)^T (A^T C - C^T A)\dot{X}(t) + \lambda^2 Y(t)^T (B^T D - D^T B)\dot{Y}(t) \\ &\quad +\lambda X(t)^T (A^T D - C^T B)\dot{Y}(t) + \lambda Y(t)^T (B^T C - D^T A)\dot{X}(t). \end{aligned}$$

It follows from Corollary 2.3 that

$$E(t)^T \dot{F}(t) - F(t)^T \dot{E}(t) = \lambda(X(t)^T \dot{Y}(t) - Y(t)^T \dot{X}(t)).$$

Thus

$$Q(\Psi_\lambda \Lambda, \Psi_\lambda \dot{\Lambda})(\Psi_\lambda v) = \lambda u^T (X(t)^T \dot{Y}(t) - Y(t)^T \dot{X}(t))u = \lambda Q(\Lambda, \dot{\Lambda})(v). \tag{16}$$

It is clear that Ψ_λ preserves the signature of the form $Q(\Lambda, \dot{\Lambda})(v)$ if λ is positive. When $\lambda = -1$, it is obvious that the positive eigenvalues of $Q(\Lambda, \dot{\Lambda})(v)$ become the negative eigenvalues of $Q(\Psi_\lambda\Lambda, \Psi_\lambda\dot{\Lambda})(\Psi_\lambda v)$. So if λ is negative, Ψ_λ changes the sign of signature of the form $Q(\Lambda, \dot{\Lambda})(v)$. □

When $\lambda = 1$, this proof also shows that symplectic matrices preserve the Maslov index.

To interpret the change of the Maslov index under quasi-symplectic isomorphisms, we assume the sympelctic vector space to be $\mathbb{R}^2$. Then each straight line crossing zero is a Lagrangian subspace. Choose y-axis as the chosen Lagrangian subspace, then $\Sigma(1)$ only contains y-axis. Then the Maslov index for a loop of Lagrangian subspaces is the intersection number with y-axis and intersecting upper self y-axis counterclockwise counts +1 meanwhile intersecting upper self y-axis clockwise counts −1. Also, intersecting lower self y-axis counterclockwise counts +1 meanwhile intersecting lower self y-axis clockwise counts −1.

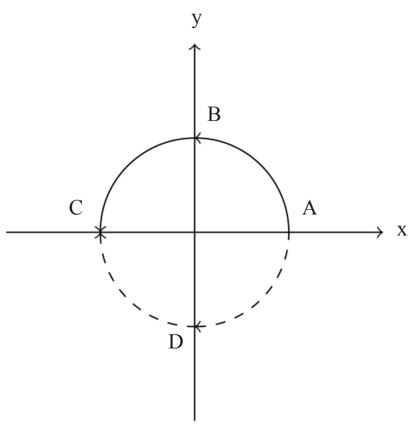

Figure 1. The $\lambda = -1$ case

We consider the $\lambda < 0$ case such as the anti-symplectic isomorphism $\Psi_{-1}(x, y) = (x, -y)$ and a loop $\gamma(t) : [0, 1] \to \mathcal{L}(1)$ such that $\gamma(0) = \gamma(1)$ is the x-axis. It is obvious that Ψ_{-1} maps y-axis to y-axis and does not change $\Sigma(1)$. In order to underline the anti-symplectic isomorphism action, we take a vector A at the starting point of the loop. The movement path of A can be described as the loop $\gamma(t)$. Then A intersects with upper self y-axis at a vector B and intersects with the terminal point, x-axis, at a vector C, see Figure 1. It follows that $\mu(\gamma) = +1$. However, the anti-symplectic isomorphism Ψ_{-1} reverses the loop such that A intersects with lower self y-axis at a vector D and intersects with the terminal point, x-axis, at a vector C. It follows that $\mu(\Psi_{-1}(\gamma)) = -1$. That is because Ψ_{-1} changes the orientation fo the loop but does not change the orientation of $\Sigma(1)$.

Consider the $\lambda > 0$ case such as the quasi-symplectic isomorphism $\Psi_3 = (\sqrt{3}x, \sqrt{3}y)$, Ψ_3 maps each line to itself then the following equations hold

$$\Psi_3(A) = \sqrt{3}A = A'$$

$$\Psi_3(B) = \sqrt{3}B = B'$$

$$\Psi_3(C) = \sqrt{3}C = C'.$$

Then $\Psi_3(A)$ intersects with upper self y-axis at a vector B' and intersects with the terminal point, x-axis, at a vector C', see Figure 2. It follows that $\mu(\Psi_3(\gamma)) = +1$.

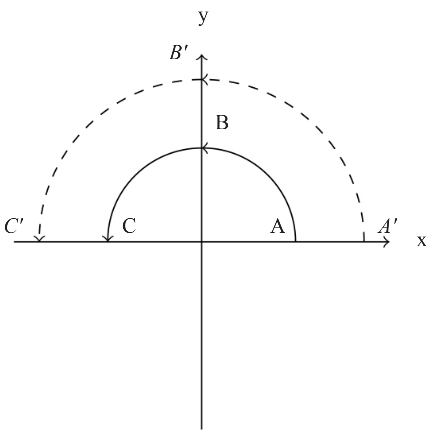

Figure 2. The $\lambda = 3$ case

For a pair of loops $\Lambda_1, \Lambda_2 : [0, 1] \to \mathcal{L}(n)$ with the frame

$$Z_1(t) = \begin{pmatrix} X_1(t) \\ Y_1(t) \end{pmatrix}$$

$$Z_2(t) = \begin{pmatrix} X_2(t) \\ Y_2(t) \end{pmatrix}$$

respectively, the relative crossing form (14) can be expressed as follow

$$\Gamma(\Lambda_1,\Lambda_2,t)=Q(\Lambda_1(t),\dot{\Lambda}_1(t))(v)-Q(\Lambda_2(t),\dot{\Lambda}_2(t))(v)$$

$$=<X_1(t)u,\dot{Y}_1(t)u>-<Y(t)_1u,\dot{X}_1(t)u>$$

$$-(<X_2(t)u,\dot{Y}_2(t)u>-<Y(t)_2u,\dot{X}_2(t)u>)$$

where $v = Z_1(t)u = Z_2(t)u \in \Lambda_1(t) \cap \Lambda_2(t)$ for some $u \in \mathbb{R}^n$. Then analogous to the one loop case, we have

Proof for two loops case of Theorem 1.1. In the above setting, for a quasi-symplectic isomorphism Ψ_λ with the form (9) the relative crossing form has the formula

$$\Gamma(\Psi_\lambda\Lambda_1,\Psi_\lambda\Lambda_2,t)=Q(\Psi_\lambda\Lambda_1(t),\Psi_\lambda\dot{\Lambda}_1(t))(\Psi_\lambda v)-Q(\Psi_\lambda\Lambda_2(t),\Psi_\lambda\dot{\Lambda}_2(t))(\Psi_\lambda v)$$

$$=\lambda(Q(\Lambda_1(t),\dot{\Lambda}_1(t))(v)-Q(\Lambda_2(t),\dot{\Lambda}_2(t))(v))$$

$$=\lambda\Gamma(\Lambda_1,\Lambda_2,t) \tag{17}$$

according to the result (16). Then

$$\mathrm{sign}\Gamma(\Psi_\lambda\Lambda_1,\Psi_\lambda\Lambda_2,t)=\begin{cases}\mathrm{sign}\Gamma(\Lambda_1,\Lambda_2,t) & \lambda>0,\\ -\mathrm{sign}\Gamma(\Lambda_1,\Lambda_2,t) & \lambda<0.\end{cases}$$

This completes the proof of this case and hence the proof of Theorem 1.1. □

A loop $\Psi_\lambda(t)$ in $\mathrm{QSp}_\lambda(2n)$ acting on a fixed Lagrangian subspace V forms a loop $\Psi_\lambda(t)V$ in $\mathcal{L}(n)$ naturally. Then we can define the crossing form $\Gamma(\Psi_\lambda(t)V, V, t)$ of $\Psi_\lambda(t)V$ as in (12) and the Maslov index $\mu(\Psi_\lambda(t)V, V)$ as in (13). The Maslov index for the case that $\mathrm{QSp}_\lambda(2n) = \mathrm{Sp}(2n)$. When $\lambda = 1$ is the definition of Maslov index for a loop of symplectic matrices in [3]. To prove Theorem 1.2, we show that $\mu(\Psi_\lambda(t)V, V)$ is nondependent on the choice of V.

Note that the crossing form $\Gamma(\Psi_\lambda(t)V, V, t)$ is a quadratic form. Explicitly, let $\Psi_\lambda(t)$ and the frame of V be expressed as follows

$$\Psi_\lambda(t)=\begin{bmatrix}A(t) & B(t)\\ C(t) & D(t)\end{bmatrix}$$

$$Z=\begin{pmatrix}X\\ Y\end{pmatrix}$$

where $\Psi_\lambda(t)$ satisfying the condition (9) for each t. Then the crossing form has the formula where $v = ((A(t)X + B(t)Y)u, (C(t)X + D(t)Y)u) = (Xu, Yu) \in \Psi_\lambda(t)V \cap V$ for some $u \in \mathbb{R}^n$

$$\Gamma(\Psi_\lambda(t)V, V, t) = Q(\Psi_\lambda(t)V, \dot{\Psi}_\lambda(t)V)(v)$$

$$=< (A(t)X + B(t)Y)u, (\dot{C}(t)X + \dot{D}(t)Y)u >$$

$$- < (C(t)X + D(t)Y)u, (\dot{A}(t)X + \dot{B}(t)Y)u >$$

$$=< Xu, (A(t)^T \dot{C}(t) - C(t)^T \dot{A}(t))Xu > + < Yu, (B(t)^T \dot{D}(t) - D(t)^T \dot{B}(t))Yu >$$

$$+ < Xu, 2(A^T(t)\dot{D}(t) - C^T(t)\dot{B}(t))Yu > \tag{18}$$

since the following equations hold according to Corollary (8)

$$\dot{A}^T(t)C(t) + A(t)^T \dot{C}(t) = \dot{C}^T(t)A(t) + C(t)^T \dot{A}(t),$$

$$\dot{B}^T(t)D(t) + B(t)^T \dot{D}(t) = \dot{D}^T(t)B(t) + D(t)^T \dot{B}(t),$$

$$\dot{A}^T(t)D(t) + A(t)^T \dot{D}(t) - \dot{C}^T(t)B(t) - C(t)^T \dot{B}(t) = 0.$$

Formula (18) implies that the signature of this crossing form is independent on the choice of $V \in \mathcal{L}(n)$.

On the other hand, for any V, $V' \in \mathcal{L}(n)$, suppose $\Psi' \in \mathrm{Sp}(2n)$ such that $V = \Psi'V'$, then

$$\mu(\Psi_\lambda(t)V, V) = \mu(\Psi_\lambda(t)\Psi'V', \Psi'V') = \mu(\Psi'^{-1}\Psi_\lambda(t)\Psi'V', V')$$

where $\Psi'^{-1}\Psi_\lambda(t)\Psi'$ can be identified with $\Psi_\lambda(t)$. Hence

Lemma 3.1 For any two Lagrangian subspace V, $V' \in \mathcal{L}(n)$, we have

$$\mu(\Psi_\lambda(t)V, V) = \mu(\Psi_\lambda(t)V', V'). \tag{19}$$

Based on Lemma 3.1 and formula (17), the relative crossing form at a crossing t has the analogous result to formula (17) as follow.

$$\Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2, t) = \Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2(t), t) - \Gamma(\Psi_\lambda\Lambda_2, \Psi_\lambda\Lambda_1(t), t)$$

$$= Q(\Psi_\lambda(t)\Lambda_1(t), \Psi_\lambda(t)\dot{\Lambda}_1(t))|_{\Lambda_1(t)\cap\Lambda_2(t)} - Q(\Psi_\lambda(t)\Lambda_2(t), \Psi_\lambda(t)\dot{\Lambda}_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

$$+ Q(\Psi_\lambda(t)\Lambda_1(t), \dot{\Psi}_\lambda(t)\Lambda_1(t))|_{\Lambda_1(t)\cap\Lambda_2(t)} - Q(\Psi_\lambda(t)\Lambda_2(t), \dot{\Psi}_\lambda(t)\Lambda_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

$$= \lambda\Gamma(\Lambda_1, \Lambda_2, t). \tag{20}$$

This completes the proof of Theorem 1.2. Consider the case that $\Lambda_2(t) \equiv V$ and $\Lambda_1(t) \equiv \Lambda(t)$, then the following term

$$Q(\Psi_\lambda(t)\Lambda_2(t), \Psi_\lambda(t)\dot{\Lambda}_2(t))|_{\Lambda_1(t)\cap\Lambda_2(t)}$$

vanishes. Then

$$\Gamma(\Psi_\lambda\Lambda, \Psi_\lambda V, t) = \lambda\Gamma(\Lambda, V, t). \tag{21}$$

Hence Remark 1.3 holds.

Consider a loop $\tilde{\Psi}_\lambda(t)$ in QSp($2n$) with the form

$$\tilde{\Psi}_\lambda(t) = \begin{bmatrix} \lambda(t)A(t) & B(t) \\ \lambda(t)C(t) & D(t) \end{bmatrix} \tag{22}$$

where $\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$ is a symplectic matrix satisfying condition (8) and $\lambda(t)$ is a smooth nonzero function satisfying the condition

$$\lambda(0) = \lambda(1),$$

$$\dot{\lambda}(0) = \dot{\lambda}(1). \tag{23}$$

Then for each crossing t the relative crossing form is analogous to formula (20) as follow

$$\Gamma(\tilde{\Psi}_\lambda\Lambda_1, \tilde{\Psi}_\lambda\Lambda_2, t) = \lambda(t)\Gamma(\Lambda_1, \Lambda_2, t). \tag{24}$$

When $\lambda(t)$ is nonnegative, the signature of $\Gamma(\tilde{\Psi}_\lambda\Lambda_1, \tilde{\Psi}_\lambda\Lambda_2, t)$ is same as the signature of $\Gamma(\Psi_\lambda\Lambda_1, \Psi_\lambda\Lambda_2, t)$ where $\Psi_\lambda(t)$ has the form $\Psi_\lambda(t) = \begin{bmatrix} \lambda A(t) & B(t) \\ \lambda C(t) & D(t) \end{bmatrix}$ with $\lambda > 0$. Moreover, $\Psi_\lambda(t)$ can be viewed as the image of $\tilde{\Psi}_\lambda(t)$ under the projection

$$\pi : \mathrm{QSp}(2n) \to \mathrm{QSp}_\lambda(2n)$$

$$\begin{bmatrix} \lambda(t)A & B \\ \lambda(t)C & D \end{bmatrix} \mapsto \begin{bmatrix} \lambda A & B \\ \lambda C & D \end{bmatrix}. \tag{25}$$

Hence Remark 1.4 holds.

Let $\Psi_\lambda(t)$ be a loop in QSp$_\lambda$($2n$), $\Lambda(t)$ a loop in $\mathcal{L}(n)$ and V a fixed Lagrangian subspace, it is hard to find out the relationship between the signature of $\Gamma(\Psi_\lambda\Lambda, V, t)$ and the one of $\Gamma(\Lambda, V, t)$. Hence we apply the analogous way in [3] and [8], firstly we review some results.

Lemma 3.2 If $\Psi = \Psi^T \in \mathrm{Sp}(2n)$ is a symmetric, positive definite and symplectic matrix, then $\Psi^s \in \mathrm{Sp}(2n)$ for any $s \geq 0$.

Proof. Let λ_i be the eigenvalues of Ψ and E_i the corresponding eigenvector spaces for $i = 1, \ldots, k$. It is known that all the eigenvalues are positive and Ψ determines a orthogonal decomposition

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{k} E_i.$$

For any two nonzero vectors

$$\xi = \sum_{i=1}^{k} a_i u_i, \ \eta = \sum_{i=1}^{k} b_i v_i$$

in $\mathbb{R}^{2n}$ where $u_i, v_i \in E_i$ for $i = 1, \ldots, k$. Note that

$$\omega_0(u_i, v_j) = \omega_0(\Psi u_i, \Psi v_j) = \lambda_i \lambda_j \omega_0(u_i, v_j)$$

for all i, j and then either $\lambda_i \lambda_j = 0$ or $\omega_0(u_i, v_j) = 0$ holds. This implies that

$$\omega_0(\Psi^s \xi, \Psi^s \eta) = \sum_{i,j=1}^{k} a_i b_j (\lambda_i \lambda_j)^s \omega_0(u_i, v_j) = \sum_{i,j=1}^{k} a_i b_j \omega_0(u_i, v_j) = \omega_0(\xi, \eta)$$

and hence $\Psi^s \in \mathrm{Sp}(2n)$ for any $s \geq 0$. □

Define a map $g : [0, 1] \times \mathrm{Sp}(2n) \to \mathrm{Sp}(2n)$ by

$$g(t, \Psi) = g_t(\Psi) = \Psi(\Psi^T \Psi)^{-t/2}. \tag{26}$$

Since $\Psi^T\Psi$ is symmetric, positive definite symplectic matirx, then $(\Psi^T\Psi)^{-t/2} \in \mathrm{Sp}(2n)$ according to Lemma 3.2 and hence $g_t(\Psi) \in \mathrm{Sp}(2n)$ for any $t \geq 0$ and any $\Psi \in \mathrm{Sp}(2n)$. Moreover, g is continuous, and

$$g_0 = id, \ g_t|_{\mathrm{U}(n)} = id \ \text{for any } t, \ g_1(\mathrm{Sp}(2n)) = \mathrm{U}(n)$$

since $g_1(\Psi)$ is also orthogonal. Hence

Lemma 3.3 $\mathrm{Sp}(2n)$ is homotopy equivalent to $\mathrm{U}(n)$.

Robbin and Salamon showed in [3] that the Maslov index is a homotopy invariant and has the equivalent definition as follow.

Lemma 3.4 Let $\Lambda(t)$ be a loop in $\mathcal{L}(n)$ and $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ a lift of unitary frames. Then for any $V \in \mathcal{L}(n)$

$$\mu(\Lambda, V) = \frac{\alpha(1) - \alpha(0)}{\pi}, \ \det_{\mathbb{C}}(X(t) + iY(t)) = e^{i\alpha(t)}. \tag{27}$$

McDuff and Salamon generalized this definition in [8] to the Maslov index $\mu(\Psi(t)) = \mu(\Psi(t)V, V)$ for a loop $\Psi(t)$ in $\mathrm{Sp}(2n)$ and any fixed Lagangrian subspace V.

Lemma 3.5 Let $\Psi(t)$ be a loop in $\mathrm{Sp}(2n)$ and $U(t) = A(t) + iC(t) = \Psi(t)(\Psi(t)^T\Psi(t))^{-1/2}$ a lift of unitary matrices. Then

$$\mu(\Psi(t)) = \frac{\beta(1) - \beta(0)}{\pi}, \ \det_{\mathbb{C}}(A(t) + iC(t)) = e^{2i\beta(t)}. \tag{28}$$

Remark 3.6 In Lemma 3.4 we consider the unoriented Lagrangian subspaces and in Lemma 3.5 the loop $\Psi(t)$ endows the Lagrangian subspaces with an orientation. In other words, it may occur that $\Psi(t_1)V = \Psi(t_2)V$ when $\Psi(t_1) \neq \Psi(t_2)$. Hence the differences between equations (27) and (28) emerge.

It is sufficient to suppose that $\Psi(t) = U(t)$ according to Lemma 3.3, then the frames of $\Psi(t)\Lambda(t)$ have the form $Z(t) = \begin{pmatrix} A(t)X(t) - C(t)Y(t) \\ C(t)X(t) + A(t)Y(t) \end{pmatrix}$, and hence

$$\det_{\mathbb{C}}((A(t)X(t) - C(t)Y(t)) + i(C(t)X(t) + A(t)Y(t))) = e^{i(\alpha(t)+2\beta(t))}$$

$$\mu(\Psi\Lambda, V) = \frac{\alpha(1) + 2\beta(1) - \alpha(0) - 2\beta(0)}{\pi} = \mu(\Lambda, V) + 2\mu(\Psi). \tag{29}$$

Let $\Lambda(t) \equiv V$, equation (29) also shows that $\mu(\Psi V, V) = 2\mu(\Psi)$. Hence

$$\mu(\Psi\Lambda, V) = \mu(\Lambda, V) + \mu(\Psi V, V). \tag{30}$$

Consider the quasi-symplectic case, note that

$$\Psi_\lambda = \Psi I_\lambda$$

where $I_\lambda = \begin{bmatrix} \lambda I & 0 \\ 0 & I \end{bmatrix}$ and $\Psi = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathrm{Sp}(2n)$.

Then according to (30), we have

$$\mu(\Psi_\lambda\Lambda, V) = \mu(\Psi(I_\lambda\Lambda), V) = \mu(I_\lambda\Lambda, V) + \mu(\Psi V, V). \tag{31}$$

Let the Lagrangian frame of Λ be $Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$, then $Z_\lambda(t) = \begin{pmatrix} \lambda X(t) \\ Y(t) \end{pmatrix}$ is a frame of $I_\lambda\Lambda$, which is not a Lagrangian frame. We transform $Z_\lambda(t)$ to a Lagrangian frame

$$Z'_\lambda(t) = \begin{pmatrix} \frac{\lambda}{|\lambda|} X(t) \\ Y(t) \end{pmatrix}. \tag{32}$$

When $\lambda > 0$, $Z'_\lambda(t) = Z(t)$ and $\Lambda = I_\lambda\Lambda$, then equation (31) yields

$$\mu(\Psi_\lambda\Lambda, V) = \mu(I_\lambda\Lambda, V) + \mu(\Psi V, V) = \mu(\Lambda, V) + \mu(\Psi V, V). \tag{33}$$

When $\lambda > 0$, $Z'_\lambda(t) = \begin{pmatrix} -X(t) \\ Y(t) \end{pmatrix}$, according to the crossing form (12) and this Lagrangian frame, we have

$$\Gamma(I_\lambda\Lambda, V, t) = -\Gamma(\Lambda, V, t). \tag{34}$$

Equation (30) shows that $\Gamma(I_\lambda\Lambda, V, t)$ has the same crossing time as $\Gamma(\Lambda, V, t)$ and hence

$$\mu(I_\lambda \Lambda, V) = -\mu(\Lambda, V) \tag{35}$$

$$\mu(\Psi_\lambda \Lambda, V) = \mu(I_\lambda \Lambda, V) + \mu(\Psi V, V) = -\mu(\Lambda, V) + \mu(\Psi V, V). \tag{36}$$

Moreover, let $\Lambda(t) \equiv V$, eqution (29) also shows

$$\mu(\Psi_\lambda V, V) = \mu(\Psi V, V). \tag{37}$$

Hence Theorem 1.5 holds according to equations (33), (36) and (37).

Acknowledgments

This work is partially supported by the project No. 11671209 of NSFC and the project No. 2021A1515010379 of Guangdong Basic and Applied Basic Research Foundation, China. Moreover, the author would like to thank Professor Hailong He for his guidance and suggestions.

Conflict of interest

The author declare that there is no personal or organizational conflict of interest with this work.

References

[1] Maslov VP. Théorie des perturbations et méthodes asymptotiques [Theory of perturbations and asymptotic methods]. *French translation*. Paris: Dunod; 1972.
[2] Arnold VI. Characteristic class entering in quantization conditions. *Functional Analysis and Its Applications*. 1967; 1(1): 1-13.
[3] Robbin J, Salamon D. The Maslov index for paths. *Topology*. 1993; 32(4): 827-844.
[4] Robbin J, Salamon D. The Spectral flow and the Maslov index. *Bulletin of the London Mathematical Society*. 1995; 27(124): 1-33.
[5] Cappell SE, Lee R, Miller EY. On the Maslov index. *Communications on Pure and Applied Mathematics*. 1994; 47(2): 121-186.
[6] Howard P, Latushkin Y, Sukhtayev A. The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$. *Journal of Mathematical Analysis and Applications*. 2017; 451(2): 794-821.
[7] Ginzburg VL. On the Maslov class rigidity for coisotropic submanifolds. *Pacific Journal of Mathematics*. 2011; 250(1): 139-161.
[8] McDuff D, Salamon D. *Introduction to symplectic topology*. New York: Oxford University Press; 2017.