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Galerkin Method and Its Residual Correction with Modified Legendre Polynomials

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Abstract: Accuracy and error analysis is one of the significant factors in computational science. This study employs the Galerkin method to solve second order linear or nonlinear Boundary Value Problems (BVPs) of Ordinary Differential Equations (ODEs) with modified Legendre polynomials to seek numerical solutions. The residual function of a differential operator is used as non-homogeneous term information of an error differential equation. The Galerkin approximation is then improved or corrected by solving the error differential equation by the Galerkin method using the same polynomials. Thus we apply the double layer Galerkin method to a variety of instances. We compare approximate solutions with exact ones and results available in the literature, and in every case, we find better accuracy.

***Keywords*:** galerkin method, linear and nonlinear BVP, modified legendre polynomials, residual correction

MSC: 26A33, 65M06, 65M12, 65M55

1. Introduction

Computational research plays an essential role in science, engineering, and industrial applications to get the best solution in our everyday lives. While considering the nonlinear Boundary Value Problems (BVPs), it has various applications in science and engineering, and the problems are widely solved either analytically or numerically. For example, many researchers attempted to solve second-order boundary value problems using numerous methods available in the literature. Falkner-type methods, block methods or even block Falkner-type methods have been used efficiently for solving initial-value problems [1-6]. Ramos et al. [7] have recently proposed a third derivative continuous 2-step block Falkner-type approach for the universal solution of second-order boundary value problems of ordinary differential equations for various boundary conditions. In [8] Sankar, Sreedhar, and Prasad discuss the existence and uniqueness of solutions to the nonlinear differential equations. Taking Chebyshev points and using the Chebyshev differentiation matrix, spectral methods have been used to solve linear and nonlinear boundary value problems in Irin et al. [9]. The Finite Difference Method (FDM) is widely used, but it requires many parameters to achieve high accuracy. Since finite difference methods generate numerical solutions at grid points only, Galerkin's weighted residual method finds approximate results at any point in the domain of a problem. For this, Bhatti and Bracken [10] paid their attention to using the Galerkin method for solving two points BVP using Bernstein polynomials, but it is restricted to second-

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order linear BVP with Dirichlet boundary conditions and to the first order nonlinear BVP. Oliveira [11] applied a residual correction to solve linear BVP using the collocation method for the original differential equation with the FDM to solve the error differential equation. Celik [12] studied the same equation applying the Chebyshev series method. Nevertheless, they have limited their discussions to only linear differential equations.

Twizell and Tirmizi [13] have studied special nonlinear BVP through multi-derivatives with the Pade approximation method. Since piecewise polynomials are easy to differentiate and integrate, they may be used to approximate any function accurately [10]. Spline functions have also been widely employed [14-16]. Amodio and Segura [17] used a high order finite difference scheme to solve second-order boundary value problems with a uniform mesh grid. To solve a nonlinear system of second-order boundary value problems, Lu [18] used the variational iteration approach. Assuming uniqueness of the solution, Nazan and Hikmet [19] introduced and studied the B-spline approach for solving a linear system of second-order boundary value problems. The BS2 schemes [20] are linked to the B-Spline (BS) approaches described in [21, 22] to deal with first-order BVP. Manni et al. [20] demonstrated that utilizing an even number of steps yields solutions with good general behavior. Dehghan and Nikpour [23] used Radial Basis Functions (RBFs) to estimate the derivatives, which required less computing effort than the widely used RBFs collocation approach. Several techniques were compared to the performance of the Haar wavelet-based method in [24].

This paper's main objective and motivation are to solve the second-order nonlinear BVPs using the Galerkin technique with residual correction approach utilizing modified Legendre polynomials since no one has been attempted in the literature as far our knowledge.

The paper is organized as follows: Section 1 contains the background and the importance of the study. We briefly describe modified Legendre polynomials and their properties in Section 2. The method of residual correction and its formulation for linear and nonlinear BVP by the Galerkin weighted residual method is discussed in Section 3. Numerical examples are considered subsequently in Section 4 for nonlinear BVPs (obviously linear as well), and the results are compared with the solutions obtained previously by several methods. All computations have been performed using the Matlab (version 2020a) programming language and a Zeon Processor I7 machine.

2. Modified legendre polynomials

The Legendre polynomial [25, 26] of degree n is defined in the interval $[-1, 1]$ as follows:

$$L_n(x) = \sum_{r=0}^{N} (-1)^r \frac{\underline{|2n-2r}}{2^n \underline{|r} \underline{|n-r} \underline{|n-2r}} x^{n-2r}$$

where

$$N = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even} \\ \frac{n-1}{2}, & \text{when } n \text{ is odd} \end{cases}$$

Legendre polynomials can be defined by the Rodrigues' formula $L_n(x) = \frac{1}{2^n \underline{|n}} \frac{d^n}{dx^n}(x^2-1)^n$

The first few Legendre polynomials are

$$L_0(x) = 1,\ L_1(x) = x,\ L_2(x) = \frac{1}{2}(3x^2-1),\ L_3(x) = \frac{1}{2}(5x^3-3x)\ldots$$

Here we modify the above Legendre polynomial of degree n on the interval [0, 1] as

$$P_n(x) = x(L_n(x)-1) \text{ or } P_n(x) = \frac{x}{2^n \lfloor n} \frac{d^n}{dx^n}\left(x^2-1\right)^n - x$$

For example, first few modified Legendre polynomials are

$$P_0(x) = 0,\ P_1(x) = x^2 - x,\ P_2(x) = \frac{1}{2}\left(3x^3 - 3x\right),\ P_3(x) = \frac{1}{2}\left(5x^4 - 3x^2 - 2x\right), \ldots$$

Observe that Legendre polynomials and Modified Legendre polynomials have the following properties:

1. $L_n(1) = 1$, for $n = 0, 1, 2, \ldots$
2. $L_n(-1) = (-1)^n$, for $n = 0, 1, 2, \ldots$
3. $L_n(-x) = (-1)^n L_n(x)$, for $n = 0, 1, 2, \ldots$
4. $P_n(-1) = 1-(-1)^n$, for $n = 1, 2, 3, \ldots$
5. $P_n(-x) = x\left\{1+(-1)^{n+1} L_n(x)\right\}$, for $n = 1, 2, 3, \ldots$
6. $P_n'(1) = \frac{1}{2} n(n+1)$, for $n = 1, 2, 3, \ldots$

3. Formulation of residual correction method

Consider the general form of a linear second order boundary value problem

$$L[y] = y''(x) + p_1(x)y'(x) + p_0(x)y(x) = f(x),\ a \le x \le b \tag{1}$$

with linearly independent boundary conditions

$$\sum_{k=0}^{1}\left[\alpha_{ik} y^k(a) + \beta_{ik} y^k(b)\right] = \gamma_i,\ i = 1,2 \tag{2}$$

Let $y(x)$ be the exact solution of (1) and (2), and $\tilde{y}(x)$ be a trial solution in Galerkin method. The coefficients of the trial solution are determined to satisfy the standard Galerkin equations obtained by imposing the boundary conditions. The residual function $R(x)$ of the operator equation is

$$R(x) = L\left[\tilde{y}(x)\right] - f(x),\ a \le x \le b \tag{3}$$

Olivera [11] analyzed the error function, $E(x) = y(x) - \tilde{y}(x)$. Since L is a linear operator, we have

$$L\left[E(x)\right] = L\left[y(x)\right] - L\left[\tilde{y}(x)\right] = -R(x) \tag{4}$$

Solving the differential equation (4) along with the boundary conditions same as that of the original equation numerically by finite difference method of order $\rho \ge m$ [11], Olivera showed that $\tilde{y}_1(x) = \tilde{y}(x) + E(x)$ improves accuracy. In this study, $\tilde{y}(x)$ has been obtained by the Galerkin method with modified Legendre polynomials, and the error differential equation (4) has also been solved by the same method.

We derive the matrix formulation for second-order BVP and extend our concept to solve nonlinear BVP to demonstrate the approach. A second-order linear differential equation is considered as

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=r(x),\ a\le x\le b \tag{5}$$

with boundary conditions,

$$\alpha_0 y(a)+\alpha_1 y'(a)=c_1 \tag{6}$$

$$\beta_0 y(b)+\beta_1 y'(b)=c_2 \tag{7}$$

where α_0, α_1, β_0, β_1, c_1, c_2 are constants and $p(x)$, $q(x)$, $r(x)$ are continuous functions. We use the Galerkin technique in [27] to solve the BVP of the form (5-7) using the modified Legendre polynomials as shape functions. The solution of the differential equation (5-7) is approximated as

$$\tilde{y}(x)=\sum_{i=1}^{n} a_i P_i(x),\ n\ge 1 \tag{8}$$

Substituting (8) into equation (5), the Galerkin weighted residual equations are:

$$\int_a^b\left[-\frac{d}{dx}\left(p(x)\frac{d\tilde{y}}{dx}\right)+q(x)y-r(x)\right]P_i(x)dx=0,\ i=1,2,\cdots,n \tag{9}$$

Simplifying, we obtain

$$\sum_{j=1}^{n}\left[\int_a^b\left[p(x)\frac{dP_i}{dx}\frac{dP_j}{dx}+q(x)P_i(x)P_j(x)\right]dx+\frac{\beta_0 p(1)P_i(1)P_j(1)}{\beta_1}-\frac{\alpha_0 p(0)P_i(0)P_j(0)}{\alpha_1}\right]a_j \tag{10}$$

$$=\int_a^b r(x)P_i(x)dx+\frac{c_2 p(1)P_i(1)P_j(1)}{\beta_1}-\frac{c_1 p(0)P_i(0)P_j(0)}{\alpha_1}$$

Or in matrix notations,

$$\sum_{j=1}^{n}K_{ij}a_j=F_i,\ i=1,2,...,n \tag{11}$$

where

$$K_{ij}=\int_a^b\left[p(x)\frac{dP_i}{dx}\frac{dP_j}{dx}+q(x)P_i(x)P_j(x)\right]dx+\frac{\beta_0 p(1)P_i(1)P_j(1)}{\beta_1}-\frac{\alpha_0 p(0)P_i(0)P_j(0)}{\alpha_1} \tag{12}$$

$$F_i=\int_a^b r(x)P_i(x)dx+\frac{c_2 p(1)P_i(1)P_j(1)}{\beta_1}-\frac{c_1 p(0)P_i(0)P_j(0)}{\alpha_1} \tag{13}$$

We obtain the values of the parameters a_i's by solving the system (11) and then substitute into (8) to get the approximate solution $\tilde{y}(x)$ of the desired BVP (5-7). The error differential equation is thus

$$-\frac{d}{dx}\left(p(x)\frac{dE}{dx}\right)+q(x)E=-R(x),\ a\leq x\leq b \tag{14}$$

with boundary conditions,

$$\alpha_0 E(a)+\alpha_1 E'(a)=c_1' \tag{15}$$

$$\beta_0 E(b)+\beta_1 E'(b)=c_2' \tag{16}$$

where $R(x)$ is the residual function obtained from the differential equation (5-7) has an explicit form

$$R(x)=-\frac{d}{dx}\left(p(x)\frac{d\tilde{y}}{dx}\right)+q(x)y-r(x) \tag{17}$$

and c_1', c_2', are some constants defined by $c_1' = c_1 - \alpha_0\tilde{y}(a) - \alpha_1\tilde{y}'(a)$, $c_2' = c_2 - \beta_0\tilde{y}(b) - \beta_1\tilde{y}'(b)$, we solve equation (14-16) by Galerkin method using the same polynomials as above. The solution of the error differential equation is approximated as

$$\tilde{E}(x)=\sum_{i=1}^{n}a_iP_i(x),\ n\geq 1 \tag{18}$$

We obtain the values of the parameters a_i a by solving system (11) and then substitute them into (18) to get the approximate solution $\tilde{E}(x)$ of the desired BVP (14-16). Summing the approximate solutions $\tilde{y}(x)$ and $\tilde{E}(x)$, we get an improved approximation $\tilde{y}_1(x) = \tilde{y}(x) + \tilde{E}(x)$.

4. Convergence and error analysis

Suppose $\mathcal{F} = C^p[a, b]$ is the vector space of p times differentiable functions. L^2 Inner product on F is defined as $\langle f, g\rangle=\int_a^b \alpha(x)f(x)g(x)dx$, for some weight function $r(x)$, and $\| f \|$ is the L^2 norm induced from the inner product giving $\mathcal{F}$ a Hilbert space structure. Assume that $\mathcal{B} = \{\psi_l | l = 1, 2, \cdots\}$ is a Schauder basis of $\mathcal{F}$ formed from the modified Legendre polynomials satisfying the appropriate boundary conditions. We begin with an approximation space $\mathcal{F}^n$ generated by $\{\psi_l | l = 1, 2, \cdots, n\}$. The Galerkin weighted residual equation $\langle R(\tilde{y}(x), x), \psi_l(x)\rangle=\int_a^b \alpha(x)R(\tilde{y}(x), x)$ $\psi_l(x)dx=0$ determines the coefficients in $\tilde{y}(x)=\sum_1^n c_l\psi_l(x)$ so that $R(\tilde{y}(x), x)$ is orthogonal to the subspace $\mathcal{F}^n$. In particular, $\mathcal{R}(\tilde{y}, x)$ is orthogonal to the generating functions $\{\psi_l | l = 1, 2, \cdots, n\}$ of the subspace. Notice that if $n \to \infty$ the residual $\mathcal{R}(\tilde{y}, x)$ is orthogonal to each basis function in $\mathcal{B}$. That is $\mathcal{R}(\tilde{y}, x)$ orthogonal to any function in $\mathcal{F}$. Therefore, $\lim_{n\to\infty} \mathcal{R}(\tilde{y}, x) = \lim_{n\to\infty}(L[y(x)] - L[\tilde{y}(x)]) = 0$. Hence $\lim_{n\to\infty}\tilde{y}(x) = y(x)$. In this article we apply the same approximation scheme on the error function, $E(x) = y(x) - \tilde{y}(x)$ to get the better approximate solution.

Lemma 1 Let $\mathscr{R}(x) \in C[a, b]$ and $g(x) \in C^1[a, b]$ and $\int_a^b \mathscr{R}(x)g(x)dx = 0$ for every $g(x) \in C^1[a, b]$ then it leaves that $\mathscr{R}(x) \equiv 0$ on $[a, b]$.

Let a vector space $\mathscr{H} = C^m[a, b]$ be equipped with the inner product $\langle f, g\rangle = \int_a^b r(x)f(x)g(x)dx$, where $r(x)$ is a weight function. Then $\| \cdot \|$ is defined as $\| f \|^2 = \int_a^b r(x)f^2(x)dx$. Now $\mathscr{H} = C^m[a, b]$ is a Hilbert space with a Schauder basis $\mathscr{B} = \{\psi_i, i = 1, 2, 3, \ldots\}$. Let $\mathscr{H}^N$ be an N dimensional approximation subspace of $\mathscr{H}$ generated by $\mathscr{B}^N = \{\psi_i, i = 1, 2, \ldots, N\}$ where ψ_i satisfies the homogeneity criterion of boundary conditions.

Galerkin weighted residual equations become

$$\langle \mathscr{R}(\tilde{y}, x), \psi_i\rangle=\int_a^b r(x)\mathscr{R}(\tilde{y}, x)\psi_i dx=0,\ i=1,2,\ldots,N$$

$r(x) = 1, \tilde{y}(x) = \sum_{i=1}^{N} \alpha_i \psi_i$ and $\mathscr{R}(\tilde{y}, x) = \mathcal{L}[\tilde{y}(x)] - f(x)$

Residual function $\mathscr{R}(\tilde{y}, x)$ is orthogonal to every $\psi_i \in \mathscr{B}^N$ and when $N \to \infty$, Lemma 1 implies $\mathscr{R}(\tilde{y}, x) \equiv 0$ on $[a, b]$. Hence, $\lim_{N\to\infty} e(x) = y - \tilde{y} \equiv 0$ on $[a, b]$.

5. Numerical examples

This section explains two linear and two nonlinear boundary value problems that some researchers have attempted, which are available in the literature. The convergence of each liner BVP is calculated by $E = |\tilde{y}_{n+1}(x) - \tilde{y}_n(x)| < \delta$, $\tilde{y}_n(x)$ denotes the approximate solution. The convergence of nonlinear BVP is assumed when two consecutive iterations are close enough. The convergence of nonlinear BVP is assumed when two consecutive iterations are close enough. That is, when $\delta > 0$ then $|\tilde{y}^{N+1}(x) - \tilde{y}^{N}(x)| < \delta$, where N is the Newton's iteration number and δ is considered as 10^{-8}.

Example 1 We consider a linear second order BVP [14, 15, 28]

$$\frac{d^2y}{dx^2} = \frac{2}{x^2}y - \frac{1}{x}, \; 2 \le x \le 3 \tag{19}$$

$$y(2) = 0, \; y(3) = 0 \tag{20}$$

The exact solution is $y(x) = \frac{1}{38}\left[-5x^2 + 9x - \frac{36}{x}\right]$. The BVP (19-20) is identical to the BVP on the interval [0, 1],

$$\frac{d^2y}{dx^2} = \frac{2}{(x+2)^2}y - \frac{1}{x+2}, \; 0 \le x \le 1 \tag{21}$$

$$y(0) = 0, \; y(1) = 0 \tag{22}$$

The error differential equation is

$$\frac{d^2E}{dx^2} = \frac{2}{(x+2)^2}E - R(x), \; 0 \le x \le 1 \tag{23}$$

$$E(0) = 0, \; E(1) = 0 \tag{24}$$

The residual function with eleven polynomials is then

$$R(x) = 0.012496x^{10} - 0.31688x^9 + 3.5412x^8 - 22.937x^7 + 95.086x^6 - 262.21x^5 + 482.16x^4 - 571.83x^3$$

$$+398.13x^2 - 123.66x - 6.408x^{-1} + 8.0859x^{-2}$$

We can solve equations (21-22) and (23-24) using the formula given in Section 3. Table 1 summarizes the Maximum Absolute Errors (MAE) for various polynomial numbers. We see that only 10 polynomials in residual method give a maximum absolute error of 1.23×10^{-13} which is lower than 8.71×10^{-9} with 10 polynomials used in [28].

Table 1. Maximum Absolute Error (MAE) for Example 1

Number of Polynomials	MAE [26]	Present MAE	Reference results
5	1.24×10^{-6}	1.25×10^{-8}	
7	1.22×10^{-8}	1.26×10^{-10}	8.36×10^{-6} [15]
10	8.71×10^{-9}	1.23×10^{-13}	1.47×10^{-6} [14]
11	--	7.79×10^{-16}	

Observe that high accuracy is obtained for 11 polynomials with an error 7.79×10^{-16}. Parametric cubic spline method with step size $h = \frac{1}{16}$, $\alpha = \frac{1}{14}$, $\beta = \frac{3}{7}$ and cubic spline method with step size $h = \frac{1}{40}$ give errors 8.36×10^{-6} and 1.47×10^{-6}, respectively. As a result, the maximum error acquired with our technique is lower than those obtained with other methods in [15].

Example 2 Consider a linear second order BVP [7, 29]

$$\frac{d^2y}{dx^2} = y + x^2 - 2, \ 0 \le x \le 1 \tag{25}$$

$$y(0) = 0, \ y(1) = 1 \tag{26}$$

The exact solution is $y(x) = \dfrac{\exp(2)x^2 - x^2 + 2\exp(1-x) - 2\exp(x+1)}{1-\exp(2)}$

The error differential equation is

$$\frac{d^2E}{dx^2} = E - R(x), \ 0 \le x \le 1 \tag{27}$$

$$E(0) = 0, \ E(1) = 0 \tag{28}$$

where $R(x)$ is obtained with 10 polynomials is then

$$R(x) = 9.43\times10^{-8}x^{11} - 2.7398\times10^{-7}x^{10} - 5.0445\times10^{-6}x^9 + 0.000024x^8 - 0.000045x^7 + 0.00005x^6 - 0.000029x^5$$

$$+0.00001x^4 - 2.43\times10^{-6}x^3 + 3.00\times10^{-7}x^2 - 1.7410^{-8}x$$

The maximum absolute errors for various polynomial numbers are summarized in the Table 2. A 2-step block Falkner-type method in Ramos [7] was used to solve the problem for different values of step size. They had to solve very large system with the increase of value of h and got a maximum error 1.03×10^{-14} for $h = \frac{1}{32}$. A method based on finite difference technique was developed in Usmani [29] and obtainedmaximum error 4.22×10^{-10} for same h. Table 2 shows that only 10 polynomials in residual method give an excellent maximum absolute error, 5.55×10^{-17} which is lower than 4.22×10^{-10} and 1.03×10^{-14}. Figure 1 depicts the absolute error for 10 polynomials. It is seen that the maximum error is too small and almost negligible compared with other techniques.

Table 2. The Maximum Absolute Error (MAE) for Example 2

h	MAE [7]	MAE [29]	Number of Polynomials	Present MAE
1/2	1.52×10^{-7}	2.64×10^{-5}	3	9.20×10^{-7}
1/4	2.12×10^{-9}	1.66×10^{-6}	4	6.69×10^{-8}
1/8	3.78×10^{-11}	1.07×10^{-7}	6	5.87×10^{-11}
1/16	6.27×10^{-13}	6.72×10^{-9}	8	2.98×10^{-14}
1/32	1.03×10^{-14}	4.22×10^{-10}	10	5.55×10^{-17}

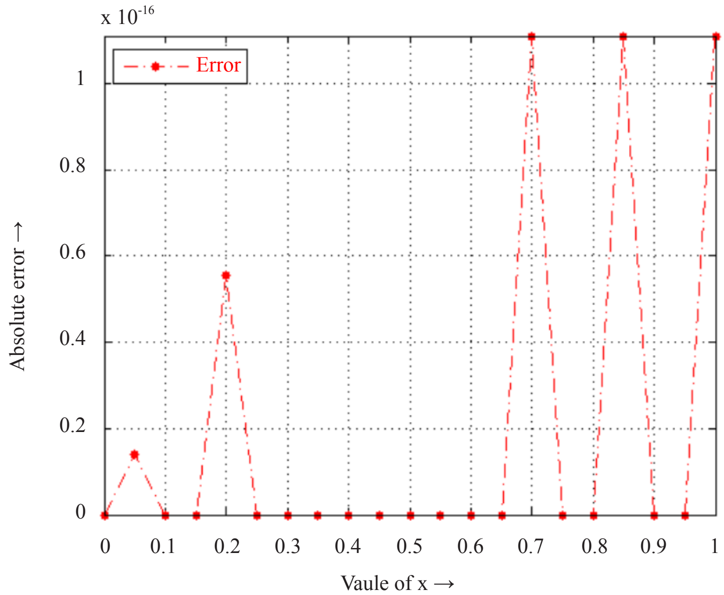

Figure 1. Absolute error for Example 2

Example 3 We consider a second order nonlinear BVP [28, 30] with Robin boundary conditions,

$$\frac{d^2 y}{dx^2} = \frac{1}{2}(1 + x + u)^3, \ 0 \le x \le 1 \tag{29}$$

$$y'(0) - y(0) = -\frac{1}{2}, \ y'(1) + y(1) = 1 \tag{30}$$

The exact solution is given by $y(x) = \frac{2}{2-x} - x - 1$

We assume that the approximate solution

$$\tilde{y}(x) = \sum_{i=1}^{n} a_i P_i(x), \ n \ge 1 \tag{31}$$

where, $P_i(0) = P_i(1) = 0$, $i = 1, 2, 3, \ldots, n$. The error differential equation is

$$\frac{d^2E}{dx^2} - \left\{\frac{3}{2}(1+x)^2 + 3(1+x)\tilde{y} + \frac{3}{2}\tilde{y}^2\right\}E - \left\{\frac{3}{2}(1+x) + \frac{3}{2}\tilde{y}\right\}E^2 - \frac{1}{2}E^3 = -R(x) \tag{32}$$

$$E'(0) - E(0) = -\frac{1}{2} - \tilde{y}'(0) - \tilde{y}(0),\ E'(1) + E(1) = 1 - \tilde{y}'(1) - \tilde{y}(1) \tag{33}$$

The residual function with 10 polynomials

$$R(x) = -5.98\times10^{-6}x^{33} + 0.000072x^{32} - 0.000435x^{31} + 0.0017x^{30} - 0.0047x^{29} + 0.0103x^{28} - 0.017x^{27}$$

$$+0.025x^{26} - 0.029x^{25} + 0.028x^{24} - 0.023659x^{23} + 0.016x^{22} - 0.008x^{21} - 0.004x^{20} + 0.028x^{19} - 0.063x^{18}$$

$$+0.093x^{17} - 0.105x^{16} + 0.085x^{15} - 0.062x^{14} + 0.023x^{13} - 0.022x^{12} - 0.0106x^{11} - 0.017x^{10} + 2.376x^{9}$$

$$-8.26x^{8} + 12.68x^{7} - 10.91x^{6} + 5.678x^{5} - 1.81x^{4} + 0.34x^{3} - 0.036x^{2} + 0.001x - 0.000028$$

Table 3. The Maximum Absolute Error (MAE) for Example 3

Number of Polynomials	MAE [28]	Present MAE
8	5.52×10^{-7}	2.65×10^{-9}
10	6.99×10^{-9}	7.77×10^{-11}
11	--	8.15×10^{-12}
13	--	3.39×10^{-13}

This nonlinear BVP with Robin boundary conditions was solved in [28] by the Galerkin method with Bernoulli polynomials. They used only the Galerkin method. The maximum absolute errors for various polynomials, using thepresent method described in section 3, are shown in Table 3. In this case, we can quickly determine the difference between the two processes. More than ten polynomials cannot be used in [28], but we can use more polynomials and get better results. On the other hand, we observe that eight polynomials in [28] give a maximum error 5.52×10^{-7}, but the residual method gives 2.65×10^{-9} for the same number of polynomials. With the same number of polynomials, we achieve better accuracy.

Example 4 we consider a nonlinear second order BVP [7, 31]

$$\frac{d^2y}{dx^2} = y^3 - yy',\ 1 \le x \le 2 \tag{34}$$

$$y(1) = \frac{1}{2},\ y(2) = \frac{1}{3} \tag{35}$$

The exact solution is $y(x) = \dfrac{1}{x+1}$

The error differential equation is

$$\frac{d^2E}{dx^2} = E^3 - EE' - \tilde{y}E' - \left(\tilde{y}' - 3\tilde{y}^2\right)E + 3\tilde{y}E^2 - R(x),\ 0 \le x \le 1 \tag{36}$$

$$E(0) = 0,\ E(1) = 0 \tag{37}$$

Where, $R(x)$ is the residual function and for 9 polynomials we have

$$R(x) = -2.88\times10^{-8}x^{26} + 2.70\times10^{-8}x^{25} - 2.04\times10^{-6}x^{24} + 1.0\times10^{-5}x^{23} - 6.9\times10^{-5}x^{22} + 0.00031x^{21}$$

$$-0.0012x^{20} + 0.0045x^{19} - 0.014x^{18} + 0.041x^{17} - 0.106x^{16} + 0.25x^{15} - 0.53x^{14} + 1.06x^{13} - 1.96x^{12} + 3.37x^{11}$$

$$-5.41x^{10} + 8.11x^{9} - 11.28x^{8} + 14.45x^{7} - 16.7x^{6} + 17.02x^{5} - 14.35x^{4} + 9.39x^{3} - 4.38x^{2} + 1.28x - 0.17$$

The maximum absolute errors for various polynomials, using the present method described in section 3, are shown in Table 4.

Table 4. Observed Maximum Absolute Error (MAE) for Example 4

Number of Polynomials	Present MAE	Reference results
7	1.23×10^{-11}	
8	1.34×10^{-12}	3.27×10^{-12} [7]
9	1.28×10^{-13}	1.18×10^{-4} [31]
11	1.03×10^{-15}	

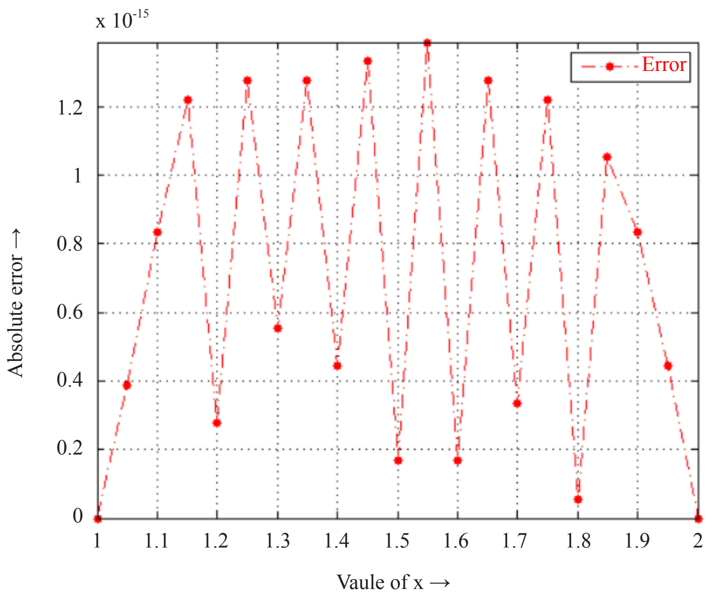

Figure 2. Absolute error for Example 4

The largest absolute error as much as 3.27×10^{-12}. Ha [31] used a shooting method that requires two initial-value problems at each iteration, along with selecting an appropriate relaxation parameter and a starting velocity estimation. We have included the best results in [7] and [31], and Table 4 demonstrates that the residual technique outperforms those results significantly. Figure 2 describes the absolute errors over the interval [1, 2] for 11 polynomials. The results are excellent compared with the existing literatures.

Example 5 The Bratu's problem arises in a wide range of applications, including the radioactive heat transfer, the Chandrasekhar model of universe expansion, thermal reaction, thermal combustion fuel ignition model, nanotechnology, chemical reactor theory etc. We consider the Bratu's BVP [32-36]

$$\frac{d^2 y}{dx^2} + \lambda e^{y} = 0, \; 0 \le x \le 1 \tag{38}$$

$$y(0) = 0, \; y(1) = 0 \tag{39}$$

The exact solution is $y(x) = -2\ln\left[\frac{\cosh\left(\frac{c(x-1/2)}{2}\right)}{\cosh\left(\frac{c}{4}\right)}\right]$ and c is the solution of the equation $c = \sqrt{2\lambda}\cosh\left(\frac{c}{4}\right)$.

The error differential equation is

$$\frac{d^2 E}{dx^2} = \lambda e^{\tilde{y}+E} = -\tilde{y}'', \; 0 \le x \le 1 \tag{40}$$

$$E(0) = 0, \; E(1) = 0 \tag{41}$$

Table 5. Comparison of absolute errors for Example 5 with $\lambda = 1$

x	Present absolute errors for 13 polynomials	Absolute errors in [32]	Absolute errors in [33]	Absolute errors in [34]	MAEs in [35, 36]
0.1	6.76×10^{-14}	4.8×10^{-7}	3.0×10^{-8}	1.24×10^{-9}	6.41×10^{-13} [35]
0.2	4.61×10^{-14}	9.4×10^{-7}	5.9×10^{-8}	3.64×10^{-10}	2.24×10^{-13} [36]
0.3	4.89×10^{-14}	1.3×10^{-7}	8.1×10^{-8}	3.99×10^{-11}	
0.4	7.72×10^{-14}	1.6×10^{-7}	9.4×10^{-8}	1.29×10^{-9}	
0.5	1.19×10^{-13}	1.7×10^{-7}	9.9×10^{-8}	2.04×10^{-10}	
0.6	1.28×10^{-13}	1.6×10^{-7}	9.4×10^{-8}	1.29×10^{-9}	
0.7	1.41×10^{-13}	1.3×10^{-7}	8.1×10^{-8}	1.60×10^{-10}	
0.8	1.27×10^{-13}	9.4×10^{-7}	5.9×10^{-8}	3.64×10^{-10}	
0.9	1.45×10^{-13}	4.8×10^{-7}	3.0×10^{-8}	1.24×10^{-9}	

We have solved this problem for $\lambda = 1$ and had MAE 1.45×10^{-13} for 13 polynomials. Table 5 shows the comparison among various types of methods. MAE 9.40×10^{-7} was obtained in Al-Mazmamy et al. [32] using restarted adomain decomposition method with Taylor. In [33] quintic B-spline was employed and had an MAE 9.90×10^{-8}. Chebyshev-Legendre spectral collocation was used to solve this equation in [34] and obtained maximum absolute error 1.29×10^{-9}. A high order quartic B-spline [35] and quintic B-spline collocation method [36] with step size $h = \frac{1}{256}$ gave MAEs 6.41×10^{-13} and 2.24×10^{-13}, respectively. The absolute errors for 13 polynomials are shown in Figure 3 (a, b). For $\lambda = 2$, we have summarized absolute errors obtained from various types of methods in Table 6 and Figure 3 (b) describes the graphical representation of the errors for 13 polynomials in this case.

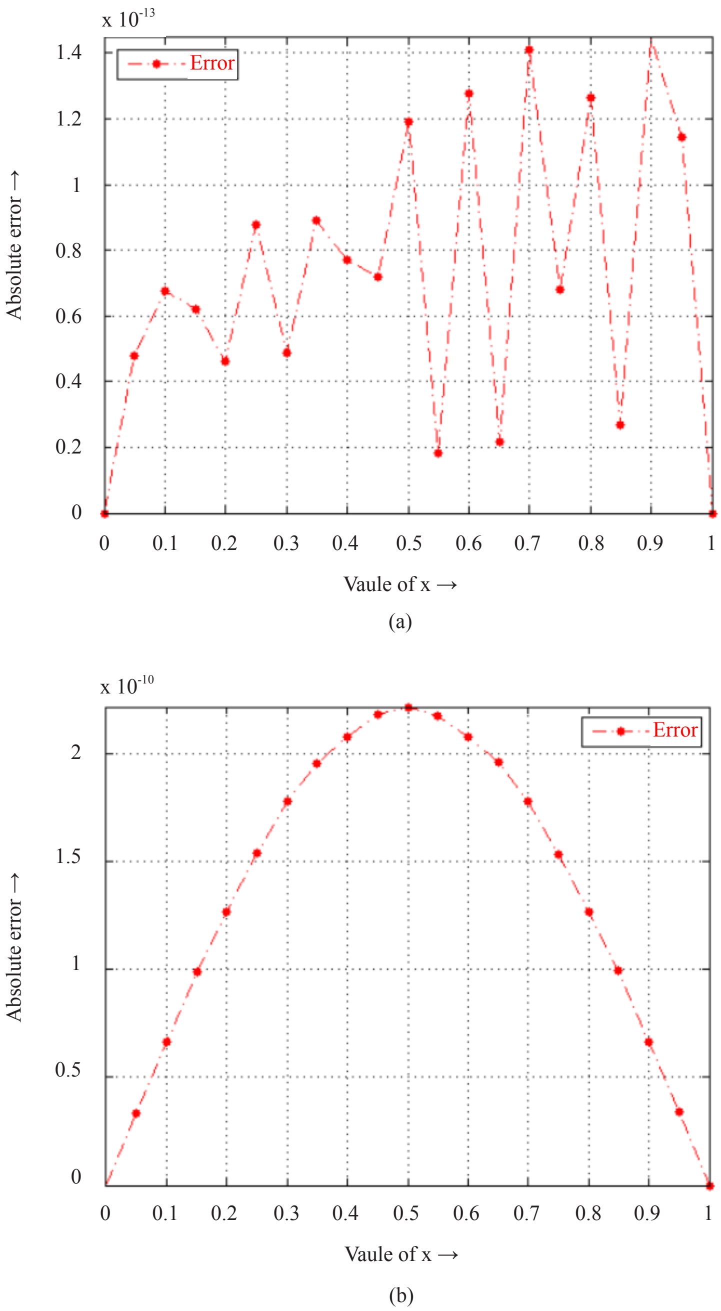

Figure 3. Absolute errors for Example 5 with (a) $\lambda = 1$, and (b) $\lambda = 2$

Table 6. Comparison of absolute errors for Example 5 with $\lambda = 2$

x	Present absolute errors for 13 polynomials	Absolute errors in [32]	Absolute errors in [33]	Absolute errors in [34]
0.1	6.66×10^{-11}	6.5×10^{-5}	9.7×10^{-7}	1.44×10^{-9}
0.2	1.27×10^{-10}	1.3×10^{-4}	5.2×10^{-8}	1.98×10^{-9}
0.3	1.78×10^{-10}	1.8×10^{-4}	8.7×10^{-7}	1.18×10^{-9}
0.4	2.09×10^{-10}	2.1×10^{-4}	1.4×10^{-6}	9.84×10^{-11}
0.5	2.22×10^{-10}	2.3×10^{-4}	1.6×10^{-6}	6.03×10^{-10}
0.6	2.09×10^{-10}	2.1×10^{-4}	1.4×10^{-6}	9.84×10^{-11}
0.7	1.78×10^{-10}	1.8×10^{-4}	8.7×10^{-7}	1.18×10^{-9}
0.8	1.27×10^{-10}	1.3×10^{-4}	5.2×10^{-8}	1.98×10^{-9}
0.9	6.61×10^{-11}	6.5×10^{-5}	9.7×10^{-7}	1.44×10^{-9}

6. Conclusion

In this paper, we have used the residual Galerkin technique to solve linear and nonlinear BVPs using Modified Legendre polynomials focused on the performance of the procedure. We have applied the formulation on secondorder BVP, compared approximate solutions with the analytical/numerical solutions available in the references, and found they are in excellent agreement. The results indicate that the residual correction procedure can obtain accurate numerical solutions to linear and nonlinear boundary value problems with less computational effort. This method may be applied for higher-order nonlinear BVP with ODEs and PDEs.

Conflict of interest

The authors declare no conflict of interest.

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