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Research Article

Extending the Traub Theory for Solving Nonlinear Equations

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Abstract: In this paper, we extend the Traub theory for solving nonlinear equation. The extension is based on recurrent functions, the center Lipschitz condition and the notion of the restricted convergence domain. Numerical examples indicate that the new results can be utilized to solve nonlinear equations, but not earlier ones.

***Keywords*:** Traub method, semi-local convergence, Banach space

MSC: 65K05, 49M37, 49M15

1. Introduction

Let X and Y stand for Banach spaces, $\Omega \subset X$ denote a nonempty convex set. By $C^2(\Omega)$ we denote the space of twice differentiable operators defined on the set Ω. Let $F : \Omega \to Y$ be a $C^2(\Omega)$ operator. Let also $\mathcal{L}(X, Y)$ denote the space of bounded linear operators mapping X into Y. A plethora of problems reduces to determining a solution $x^* \in \Omega$ of the nonlinear equation

$$F(x) = 0. \tag{1}$$

This is one of the most challenging tasks in computational sciences. Most solution methods for equation (1) are of iterative nature, since the point x^* is found in closed form only in special cases.

We present the semi-local convergence analysis of the class of methods defined for each $n = 0, 1, 2, \ldots$ by

$$y_n = x_n - F'(x_n)^{-1} F(x_n)$$

$$x_{n+1} = y_n - (I + M_n L_n + L_n^2 A_n) F'(x_n)^{-1} F(y_n), \tag{2}$$

where $F'(x) : \Omega \to \mathcal{L}(X, Y)$, $F'(x)^{-1} : \Omega \to \mathcal{L}(Y, X)$, $M_n := M(x_n)$, $M : \Omega \to \mathcal{L}(X, Y)$, $A_n = A(x_n)$, $A : \Omega \to \mathcal{L}(X, Y)$,

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$L_n = L(x_n)$, $L : \Omega \rightarrow \mathcal{L}(Y, X)$ and $L_n = L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$, where F', F'' stand for the first and second Fréchet derivative of operator F [1].

The second derivative is easy to evaluate in some interesting cases: to mention a few

(1) The operator F'' is constant.

(2) The operator in equilibrium problems depends on the connection between two elements $v_i.v_j$, and the second derivative is constant.

(3) Many integral equations can be solved so that the second derivative is also constant.

Many popular fourth convergence order methods are special cases of (2).

Traub's Method [2, 3]; Choose $M_n = I$ to obtain

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

$$x_{n+1} = y_n - (I + L_n + L_n^2 A_n)F'(x_n)^{-1}F(y_n). \tag{3}$$

Two-Step Method [4, 5]: Choose $M_n = I$ and $A_n \equiv 0$ to have

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

$$x_{n+1} = y_n - (I + L_n)F'(x_n)^{-1}F(y_n). \tag{4}$$

Special Two-Step Method [4]: Choose $A(x) = \frac{1}{2}\left(\frac{5}{2}I - L_{F'}(x)\right)$ to get

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

$$x_{n+1} = y_n - \left(I + L_n + \frac{1}{2}L_n^2\left(\frac{5}{2}I - L_{F'}(x)\right)\right)F'(x_n)^{-1}F(y_n). \tag{5}$$

Many other choices for linear operators M_n and A_n are possible [2, 6-8]. Therefore, it is important to provide a unified convergence analysis for these methods by studying (2).

If $\|x^* - x_{n+1}\| \leq C\|x^* - x_n\|^q$ for some constant $C > 0$ and $q \geq 1$, we say that the convergence order of the method is q.

In this paper we are also motivated by Traub's theorem for scalar equations and optimization considerations. Let $X = \mathbb{R}$.

Theorem 1.1 (Traub [2]) If $A(x)$ is a sufficiently many times differentiable function, then method

$$y_n = Q(x_n)$$

$$x_{n+1} = y_n - (1 + L_n + L_n^2 A_n)F'(x_n)^{-1}F(y_n) \tag{6}$$

has order of convergence $\min\{s + 2, 2s\}$, where s is the order of iterative function Q.

The convergence of the aforementioned four methods has been shown using Lipschitz-type continuity conditions on F'' or higher than order two derivatives which are not on these methods [4, 10-12]. Hence, the utilization of these methods is limited although they may converge. But to solve equations containing operators that are at least three times differentiable or whose F'' is not Lipschitz-type continuous.

For example: Let $X = Y = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define function Ψ on Ω by

$$\Psi(t)=\begin{cases} t^3\log t^2+t^5-t^4 & if\ t\neq 0\\ 0 & if\ t=0. \end{cases}$$

Then, $t^*=1$, is a zero of equation $\Psi(t)=0$, and

$$\Psi''(t)=6\log t^2+60t^2-24t+22.$$

Obviously $\Psi''(t)$ is not bounded on Ω.

We address these concerns by providing a semi-local convergence analysis for method (2) based on operator F' and F'' that only appear on it. Hence, we extend the usage of these methods and in the more general setting of method (2).

Scalar majorizing sequences are introduced in Section 2. The semi-local convergence appears in Section 3 followed by numerical examples in Section 4 and the conclusions in Section 5.

2. Majorizing sequences

Let K_0, K, K_1, K_2, m, α, b be positive and η be a non-negative number. The numbers are connected to the operators appearing on method (2) in Section 3. Define sequence $\{t_n\}$ for $n=0, 1, \ldots$ by $t_0=0$, $s_0=\eta$,

$$t_{n+1}=s_n+\frac{bK\gamma_n(s_n-t_n)^2}{2(1-bK_0t_n)}$$

$$s_{n+1}=t_{n+1}+\frac{bq_nd_{n+1}(t_{n+1}-s_n)}{1-bK_0t_{n+1}}, \tag{7}$$

where $\ell_n=\frac{bK_2(s_n-t_n)}{1-bK_0t_n}$, $\beta_n=m\ell_n+\alpha\ell_n^2$, $q_n=\frac{1}{1-\beta_n}$, $\gamma_n=1+\beta_n$ and $d_{n+1}=K\left(s_n-t_n+\frac{1}{2}(t_{n+1}-s_n)\right)+\ell_nK_1(m+\alpha\ell_n)$. This sequence is shown to be majorizing for method (2) in Section 3. But first, we present some convergence results for this sequence.

Lemma 2.1 Suppose

$$t_n<\frac{1}{bK_0}. \tag{8}$$

Then, sequence $\{t_n\}$ is such that $t_n\le s_n\le t_{n+1}<\frac{1}{bK_0}$, $\lim_{n\to\infty}t_n=t^*\le\frac{1}{bK_0}$, and t^* is the least upper bound of sequence $\{t_n\}$.

Proof. It follows from (7) and (8) that sequence $\{t_n\}$ is non-decreasing and bounded from above by $\frac{1}{bK_0}$ and as such it converges to t^*.

Next, we present some stronger results for the convergence of sequence $\{t_n\}$ but which are easier to verify than (8).

Define polynomials $f_n^{(1)}$, $f_n^{(2)}$ and $f_n^{(3)}$ on the interval $S=[0, 1)$ by

$$f_n^{(1)}(t)=bK_2t^{n-1}\eta+bK_0(1+t+t^n)\eta-1$$

$$f_n^{(2)}(t)=\frac{bK}{2}(1+mt+\alpha t^2)t^{n-1}\eta$$

$$+bK_0(1+t+\ldots+t^n)\eta-1,$$

$$f_n^{(3)}(t)=2b\left(Kt^n\eta+\frac{K}{2}t^{n+1}\eta+mK_1t+\alpha K_1t^2\right)$$

$$-bK_0(1+t+\ldots+t^{n+1})\eta-1,$$

$$g_1(t)=K_0t^2+K_2t-K_2,$$

$$g_2(t)=\frac{K}{2}(1+mt+\alpha t^2)t-\frac{K}{2}(1+mt+\alpha t^2)+K_0t^2,$$

$$g_3(t)=K_0t^2+2Kt-2K$$

and

$$g_4(t)=2bK_1t(1-t)(m+\alpha t)+t+bK_0\eta-1.$$

By these definitions, we have $g_1(0)=-K_2$, $g_1(1)=K_0$, $g_2(0)=-\frac{K}{2}$, $g_2(1)=K_0$, $g_3(0)=-2K$, $g_3(1)=K_0$, $g_4(0)=bK_0\eta-1<0$ (if $bK_0\eta<1$) and $g_4(1)=bK_0\eta$. Then, the intermediate value theorem assures that equation $g_i(t)=0$, $i=1$, 2, 3, 4 have zeros in the interval (0, 1). Denote by ξ_i the smallest such zeros, respectively. Let $a_1=bK_2\eta$, $a_2=\frac{bK\gamma_0\eta}{2}$, $a_3=\frac{bq_0d_1(t_1-s_0)}{\eta 91-bK_0t_1}$ for $n\neq 0$ $a=\max\{a_1, a_2, a_3\}$, $\xi_5=\min\{\xi_1, \xi_2, \xi_3\}$, $\xi=\max\{\xi_1, \xi_2, \xi_3\}$, $\lambda=\min\{1-bK_0\eta, \xi_4, \mu\}$, where $\mu=\frac{1+\sqrt{m^2+2\alpha}}{2\alpha}$.

Notice that these functions and parameters depend on the original constants. Next, we show the second result on majorizing sequences for method (2) based on the developed notation.

Lemma 2.2 Suppose

$$bK_0t_1<1 \tag{9}$$

and

$$0\leq a\leq\xi_5\leq\xi\leq\lambda. \tag{10}$$

Then, the following assertions hold

$$0\leq s_n-t_n\leq\xi(s_{n-1}-t_{n-1})\leq\xi^{2n}\eta\leq\xi^n\eta, \tag{11}$$

$$0\leq t_{n+1}-s_n\leq\xi(t_n-s_{n-1})\leq\xi^{2n+1}\eta\leq\xi^{n+1}\eta, \tag{12}$$

$$t_n\leq s_n\leq t_{n+1}\leq t^{**}:=\frac{\eta}{1-\xi} \tag{13}$$

and

$$\lim_{n\to\infty} t_n = t^* \le t^{**}. \tag{14}$$

Proof. Mathematical induction is employed to show

$$0 \le \frac{bK_2(s_n - t_n)}{1 - bK_0 t_n} \le \xi, \tag{15}$$

$$0 \le \frac{bK(1 + m\xi + \alpha\xi^2)(s_n - t_n)}{2(1 - bK_0 t_n)}, \tag{16}$$

$$2\beta_n \le 1 \tag{17}$$

and

$$0 \le \frac{bq_n d_{n+1}(t_{n+1} - s_n)}{1 - bK_0 t_{n+1}} \le \xi(s_n - t_n). \tag{18}$$

These estimates are true for $n = 0$ by the definition of sequence $\{t_n\}$, (9) and (10).

Then, we have $0 \le s_1 - t_1 \le \xi(s_0 - t_0)$, $0 \le t_1 - s_0 \le \xi(s_0 - t_0)$ and $t_1 \le \eta + \xi_n = \frac{1-\xi^2}{1-\xi}\eta < t^{**}$.

Suppose that

$$0 \le s_n - t_n \le \xi^n \eta,$$

$$0 \le t_{n+1} - s_n \le \xi^{n+1}\eta$$

and

$$t_n \le \frac{1-\xi^{n+1}}{1-\xi}\eta < t^{**}. \tag{19}$$

Evidently, estimate (15) holds if

$$bK_2\xi^n\eta + \xi bK_0(1 + \xi + \ldots + \xi^n)\eta - \xi \le 0 \tag{20}$$

or

$$f_n^{(1)}(t) \le 0 \text{ at } t = \xi_1. \tag{21}$$

We must relate two consecutive polynomials $f_n^{(1)}$

$$f_{n+1}^{(1)}(t) = f_{n+1}^{(1)}(t) - f_n^{(1)}(t) + f_n^{(1)}(t)$$

$$= bK_2 t^n \eta + bK_0 (1 + t + \ldots + t^{n+1})\eta - 1 + f_n^{(1)}(t)$$

$$-bK_2 t^{n-1}\eta - bK_0 (1 + t + \ldots + t^n)\eta + 1$$

$$= f_n^{(1)}(t) + g_1(t) t^{n-1} b\eta,$$

so

$$f_{n+1}^{(1)}(t) = f_n^{(1)}(t) + g_1(t) t^{n-1} b\eta. \tag{22}$$

In particular, by the definition of g_1 and ξ_1, we get

$$f_{n+1}^{(1)}(t) = f_n^{(1)}(t) \text{ at } t = \xi_1. \tag{23}$$

Define function on the interval [0, 1)

$$f_\infty^{(1)}(t) = \lim_{n\to\infty} f_n^{(1)}(t). \tag{24}$$

By the definition of functions $f_n^{(1)}(t)$ and $f_\infty^{(1)}$, we get

$$f_\infty^{(1)}(t) = \frac{bK_0\eta}{1-t} - 1. \tag{25}$$

So, (21) holds if

$$f_\infty^{(1)}(t) \le 0 \text{ at } t = \xi_1, \tag{26}$$

which is true by (10). Next, we need to show

$$0 \le \frac{bK\gamma_n (s_n - t_n)}{2(1 - K_0 t_n)} \le \xi \tag{27}$$

or instead (by (15)) (16). Similarly, (16) holds if

$$\frac{bK}{2}(1 + m\xi + \alpha\xi^2)\xi^n \eta + \xi bK_0 (1 + \xi + \ldots + \xi^n)\eta - \xi \le 0 \tag{28}$$

or

$$f_n^{(2)}(t) \le 0 \text{ at } t = \xi_2. \tag{29}$$

We can write

$$f_{n+1}^{(2)}(t) = f_{n+1}^{(2)}(t) - f_n^{(2)}(t) + f_n^{(2)}(t)$$

$$= \frac{bK}{2}(1 + mt + \alpha t^2)t^n \eta + bK_0(1 + t + \ldots + t^{k+1})\eta - 1$$

$$+ f_n^{(2)}(t) - \frac{bK}{2}(1 + mt + \alpha t^2)t^{n-1}$$

$$- bK_0(1 + t + \ldots + t^n)\eta + 1$$

$$= f_n^{(2)}(t) + g_2 t^{n-1} b\eta.$$

So, we get

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) + g_2 t^{n-1} b\eta. \tag{30}$$

In particular, we have

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) \text{ at } t = \xi_2. \tag{31}$$

Define function $f_\infty^{(2)}$ on the interval [0, 1) by

$$f_\infty^{(2)}(t) = \overset{(2)}{\lim_{n\to\infty}}(t). \tag{32}$$

It follows that

$$f_\infty^{(2)}(t) = f_\infty^{(2)}(t),$$

so (29) holds, if $f_\infty^{(2)}(t) \le 0$ at $t = \xi_2$, which is true by (10). By the definition of β_n, estimate (17) holds if

$$2(m\xi + \alpha\xi^2) \le 1,$$

which is true by the choice of μ and (10). It follows that

$$q_n \le 2. \tag{33}$$

Hence, estimate (18) certainly holds if

$$\frac{2b\left(K(s_n - t_n) + \frac{K}{2}\xi(s_n - t_n) + m\xi K_1 + \alpha\xi^2 K_1\right)\xi}{1 - bK_0 t_{n+1}} \le \xi, \tag{34}$$

or

$$2b\left[K\xi^n\eta + \frac{K}{2}\xi^{n+1}\eta + mK_1\xi + \alpha K_1\xi^2\right]$$

$$+bK_0(1+\xi+\ldots+\xi^{n+1})\eta-1\leq 0, \tag{35}$$

or

$$f_n^{(3)}(t)\leq 0 \text{ at } t=\xi_3.$$

This time we have

$$f_{n+1}^{(3)}(t)=f_{n+1}^{(3)}(t)=f_n^{(3)}(t)+f_n^{(3)}(t)$$

$$=f_n^{(3)}(t)+g_3(t)t^n\eta b, \tag{36}$$

and

$$f_{n+1}^{(3)}(t)=f_n^{(3)}(t) \text{ at } t=\xi_3.$$

Define function $f_\infty^{(3)}(t)=\lim_{n\to\infty} f_n^{(3)}(t)$. Then, we obtain

$$f_\infty^{(3)}(t)=2b(mtK_1+\alpha t^2K_1)+\frac{bK_0\eta}{1-t}-1.$$

So,

$$f_\infty^{(3)}(t)\leq 0 \text{ at } t=\xi_4,$$

if

$$g_4(t)\leq 0 \text{ at } t=\xi_4, \tag{37}$$

which is true by (10). The induction for estimates (15)-(18) is complete. Hence, sequence $\{t_n\}$ is non-decreasing and bounded from above by t^{**}, and as such it converges to $t^*\in[0, t^{**}]$.

3. Semi-local analysis

Let $U(x_0, s)$, $U[x_0, s]$ stand for the open and closed balls, respectively, centered at the point $x_0\in X$ and of radius $s>0$. The following conditions (H) are used. Suppose

(H1) There exist $x_0\in\Omega$, $b>0$, $\eta\geq 0$, $\alpha_n\geq 0$, $\alpha\geq 0$, such that $m_n\geq 0$ and $m\geq 0$ such that

$$F'(x_0)^{-1}\in\mathcal{L}(Y,X),\ \|F'(x_0)^{-1}\|\leq b,$$

$$\|F'(x_0)^{-1}F(x_0)\|\leq\eta.$$

$$\|A_n\|\leq\alpha_n\leq\alpha \text{ and } \|M_n\|\leq m_n\leq m.$$

(H2)

$$\| F'(x) - F'(x_0) \| \le K_0 \| x - x_0 \| \tag{38}$$

for each $x \in \Omega$. Set $\Omega_0 = U(x_0, \frac{1}{bK_0}) \cap \Omega$.

(H3)

$$\| F'(x) \| \le K_1, \tag{39}$$

$$\| F''(x) \| \le K_2 \tag{40}$$

and

$$\| F'(y) - F'(x) \| \le \tilde{K} \| y - x \| \tag{41}$$

for all $x \in \Omega_0$ and $y = x - F'(x)^{-1}F(x) \in \Omega$, or $x, y \in \Omega_0$. Denote by $\bar{K}$ the constant in the first case and by K the one in the second case. Notice that $\bar{K} \le K$. We use K in the results that follow although $\bar{K}$ can also be used.

(H4) Condition (8) of Lemma 2.1 or conditions (8) and (9) of Lemma 2.2 hold.

and

(H5) $U[x_0, t^*] \in \Omega$.

Next, we need an auxiliary result connecting the iterates of method (2).

Lemma 3.1 Suppose that iterates $\{x_n\}$, $\{y_n\}$ exist for each $n = 0, 1, 2, ...$

Set

$$\begin{aligned} D_{n+1} = &\int_0^1 \left(F'\left(y_n + \theta(x_{n+1} - y_n) \right) - F'(x_n) \right) d\theta \\ &+ \int_0^1 M_n L_n F'\left(y_n + \theta(x_{n+1} - y_n) \right) d\theta \\ &\int_0^1 L_n^2 A_n F'\left(y_n + \theta(x_{n+1} - y_n) \right) d\theta \end{aligned} \tag{42}$$

and

$$C_{n+1} = B_n^{-1} \left(B_n F'\left(y_n + \theta(x_{n+1} - y_n) \right) - F'(x_n) \right). \tag{43}$$

Then, the following assertions hold

$$C_{n+1} = B_n^{-1} D_{n+1} \tag{44}$$

and

$$F(x_{n+1}) = C_{n+1}(x_{n+1} - y_n). \tag{45}$$

Proof. In view of (43), we have

$$
\begin{aligned}
C_{n+1} &= B_n^{-1}\left(B_n F'\left(y_n+\theta(x_{n+1}-y_n)\right)-F'(x_n)\right)\\
&= B_n^{-1}\left((I+M_nL_n+L_n^2A_n)F'\left(y_n+\theta(x_{n+1}-y_n)\right)-F'(x_n)\right)\\
&= B_n^{-1}D_{n+1},
\end{aligned}
$$

by the definition (42). Moreover, we can write by the second substep of method (2)

$$
\begin{aligned}
F(x_{n+1}) &= F(x_{n+1})-F(y_n)+F(y_n)\\
&= F(x_{n+1})-F(y_n)-B_n^{-1}F'(x_n)(x_{n+1}-y_n)\\
&= \int_0^1\left(F'\left(y_n+\theta(x_{n+1}-y_n)\right)d\theta-B_n^{-1}F'(x_n)\right)(x_{n+1}-y_n)\\
&= C_{n+1}(x_{n+1}-y_n).
\end{aligned}
$$

□

The semi-local convergence analysis of method (2) follows.

Theorem 3.2 Suppose that the conditions in H hold. Then, iteration $\{x_n\}$ generated by method (2) is well defined in $U(x_0, t^*)$, remains in $U(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$ and converges to solution $x^* \in U[x_0, t^*]$ of equation $F(x) = 0$. Moreover, the following bounds hold

$$\| y_n - x_n \| \le s_n - t_n, \tag{46}$$

$$\| x_{n+1} - y_n \| \le t_{n+1} - s_n, \tag{47}$$

and

$$\| x^* - x_n \| \le t^* - t_n. \tag{48}$$

Proof. It follows from (H1) and (7) that

$$\| y_0 - x_0 \| = \| F'(x_0)^{-1} F(x_0) \| \le \eta = s_0 - t_0,$$

so (46) holds for $n = 0$ and $y_0 \in U(x_0, t^*)$. Let $u \in U(x_0, t^*)$. Then, by (H2), we get that

$$\| F'(x_0)^{-1} \| \ \| F'(u) - F'(x_0) \| \le bK_0 \| u - x_0 \| \le K_0 t^* < 1,$$

so $F'(u)^{-1} \in \mathcal{L}(Y, X)$ by the Banach lemma on linear invertible operators [1, 5, 12] and

$$\| F'(u)^{-1} F'(x_0) \| \le \frac{b}{1 - bK_0 \| u - x_0 \|}. \tag{49}$$

Some estimates are needed:

$$F(y_n) = F(y_n) - F(x_n) - F'(x_n)(y_n - x_n), \tag{50}$$

so by (H3)

$$\| F'(x_0)^{-1} F(y_n) \| \le \frac{K}{2} \| y_n - x_n \|^2 \le \frac{K}{2}(s_n - t_n)^2, \tag{51}$$

$$\begin{aligned} \| L_n \| &= \| F'(x_n)^{-1} F''(x_n) F'(x_n)^{-1} F(x_n)(y_n - x_n) \| \\ &\le \| F'(x_n)^{-1} F'(x_0) \| \; \| F'(x_0)^{-1} F''(x_n) \| \; \| y_n - x_n \| \\ &\le \frac{bK_2 \| y_n - x_n \|}{1 - bK_0 \| x_n - x_0 \|} \le \ell_n, \end{aligned} \tag{52}$$

and

$$\begin{aligned} \| x_{n+1} - y_n \| &\le \left[\| I \| + \| M_n \| \; \| L_n \| + \| L_n \|^2 \| A_n \| \right] \\ &\times \| F'(x_n)^{-1} F'(x_0) \| \; \| F'(x_0)^{-1} F(y_n) \| \\ &\le \left[1 + m_n \frac{bK_2 \| y_n - x_n \|}{1 - bK_0 \| x_n - x_0 \|} + \alpha_n \left(\frac{bK_2 \| y_n - x_n \|}{1 - bK_0 \| x_n - x_0 \|} \right)^2 \right] \\ &\times \frac{b}{1 - b_0 K_0 \| x_n - x_0 \|} \frac{K}{2} \| y_n - x_n \|^2 \end{aligned} \tag{53}$$

$$\le t_{n+1} - s_n, \tag{54}$$

showing (47), where we also used $\| y_n - x_n \| \le s_n - t_n$ and $\| x_n - x_0 \| \le t_n$.

Next, we show the invertability of B_n. We get in turn by the definition of B_n that

$$\begin{aligned} \| B_n - I \| &\le \| M_n \| \; \| L_n \| + \| A_n \| \; \| L_n \|^2 \\ &\le m_n \frac{bK_2 \| y_n - x_n \|}{1 - bK_0 \| x_n - x_0 \|} + \alpha_n \left(\frac{bK_2 \| y_n - x_n \|}{1 - bK_0 \| x_n - x_0 \|} \right)^2 \\ &\le \frac{mbK_2 (s_n - t_n)}{1 - bK_0 t_n} + \alpha \frac{bK_2 (s_n - t_n)}{1 - bK_0 t_n} \\ &= \beta_n < 1, \end{aligned}$$

so

$$\| B_n^{-1} \| \leq q_n. \tag{55}$$

Moreover, we obtain by the definition of D_{n+1} and (H3) that

$$\begin{aligned}
\| D_{n+1} \| &\leq K\left[\| y_n - x_n \| + \frac{1}{2} \| x_{n+1} - y_n \| \right] \\
&+ m\ell_n K_1 + \alpha \ell_n^2 K_1 \\
&\leq K\left(s_n - t_n + \frac{1}{2}(t_{n+1} - s_n) \right) + m\ell_n K_1 + \alpha \ell_n^2 K_1 \\
&= d_{n+1}.
\end{aligned} \tag{56}$$

Furthermore, by the first substep of method (2), Lemma 3.1, (49) (for $u = x_{n+1}$), (7) and (56), we have that

$$\begin{aligned}
\| y_{n+1} - x_{n+1} \| &\leq \| F'(x_{n+1})^{-1} F'(x_0) \| \ \| F'(x_0)^{-1} F(x_{n+1}) \| \\
&\leq \frac{b \| C_{n+1} \| \ \| x_{n+1} - y_n \|}{1 - bK_0 \| x_{n+1} - x_0 \|} \\
&\leq \frac{b q_n d_{n+1} \| x_{n+1} - y_n \|}{1 - bK_0 \| x_{n+1} - x_0 \|} \\
&\leq s_{n+1} - t_{n+1}
\end{aligned} \tag{57}$$

showing (46) for n replacing $n + 1$, where we also used

$$\begin{aligned}
\| x_{n+1} - x_0 \| &\leq \| x_{n+1} - y_n \| + \| y_n - x_0 \| \\
&\leq t_{n+1} - s_n + s_n - t_0 \\
&= t_{n+1} < t^*
\end{aligned}$$

and

$$\begin{aligned}
\| y_{n+1} - x_0 \| &\leq \| y_{n+1} - x_{n+1} \| + \| x_{n+1} - x_0 \| \\
&\leq s_{n+1} - t_{n+1} + t_{n+1} - t_0 \\
&= s_{n+1} < t^*.
\end{aligned}$$

That is x_n, $y_n \in U(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$. The induction for estimates (46) and (47) is complete. Hence, sequence $\{x_n\}$ is fundamental (since $\{t_n\}$ is convergent) in a Banach space X and as such it converges to some $x^* \in U[x_0,$

t^*]. By letting $n \to \infty$ in (51), and using the continuity of F we get $F(x^*) = 0$. Finally, let $j \geq 0$ be an integer. Then, we can write

$$\begin{aligned} \| x_{n+j} - x_n \| &\leq \| x_{n+j} - y_{n+j-1} \| + \| y_{n+j-1} - x_{n+j-1} \| + \ldots + \| y_n - x_n \| \\ &\leq t_{n+j} - s_{n+j-1} + s_{n+j-1} - t_{n+j-1} + \ldots \\ &+ s_n - t_n = t_{n+j} - t_n \end{aligned} \tag{58}$$

and if $j \to \infty$ we obtain (48). □

A uniqueness of the solution result is presented next.

Proposition 3.3 Suppose: there exists a simple solution $x^* \in U(x_0, \rho_0)$ of equation $F(x) = 0$ for some $\rho_0 > 0$; condition (H2) holds and there exists $\rho \geq \rho_0$ such that

$$\frac{K_0}{2}(\rho_0 + \rho) < 1. \tag{59}$$

Set $\Omega_1 = U[x_0, \rho] \cap \Omega$. Then, the point x^* is the only solution of equation $F(x) = 0$ in the set Ω_1.

Proof. Let $z^* \in \Omega_1$ with $F(z^*) = 0$. Define $T = \int_0^1 F'(x^* + \theta(z^* - x^*))d\theta$. Then, using (H2) and (59) we obtain

$$\begin{aligned} \| F'(x_0)^{-1}(T - F'(x_0)) \| &\leq K_0 \int_0^1 ((1-\theta) \| x^* - x \| + \theta \| y^* - x_0 \|) d\theta \\ &\leq \frac{K_0}{2}(\rho_0 + \rho) < 1. \end{aligned}$$

Hence, $z^* = x^*$ is implied by the identity $T(z^* - x^*) = F(z^*) - F(x^*) = 0$, and the invertability of linear operator T. □

Remark 3.4 (i) Condition (H5) can be replaced by $U(x_0, \frac{1}{bK_0}) \subset \Omega$ in the case of Lemma 2.1 and by $U(x_0, \frac{\eta}{1-\xi}) \subset \Omega$ in the case of Lemma 2.2.

(ii) In Proposition 3.3 we did not assume all conditions (H) except (H2).

(iii) We can use smallest Lipschitz constants if instead of Ω_0 if we consider the set $\Omega_2 = U(x_1, \frac{1}{bK_0} - \eta)$ and suppose $\Omega_2 \subset \Omega$ and $bK_0\eta < 1$. Then, iterates lie in Ω_2 and $\Omega_2 \subset \Omega_0$. Hence, the constants corresponding to K, K_1 and K_2 are at least as small.

(iv) We have chosen Newton's method in the first substep of (2). But if we consider $Q(x) = x - F'(x)^{-1}G(x)F(x)$ in the Banach space version of (6), then we get

$$\begin{aligned} L_n &= -F'(x_n)^{-1} F''(x_n) F'(x_n)^{-1} F'(x_n) F(x_n)^{-1}(y_n - x_n) \\ &= -F'(x_n)^{-1} F''(x_n) G'(x_n)^{-1}(y_n - x_n), \end{aligned}$$

where $G : \Omega \to \mathcal{L}(X, X)$. Moreover, suppose $G^{-1} \in \mathcal{L}(X, X)$ and $\|G(x)^{-1}\| \leq \tau$ for some $\tau > 0$. Then, the conclusions of Theorem 3.2 hold for this method too if we simply multiply ℓ_n by τ.

4. Numerical example

We test the convergence criteria.

Example 4.1 Define the real function $f: \Omega \to \mathbb{R}$ on $\Omega = B[x_0, 1-\delta]$, $x_0 = 1$, $\delta \in (0, 1)$ by

$$f(t) = t^3 - \delta.$$

Then $\| f'(x_0)^{-1} \| \leq \frac{1}{3}$, $\| f'(y) - f'(x) \| \leq 6(2-p) \| y - x \|$ and hence for all $x \in D$, we have

$$\| f'(x)^{-1} \| \leq \frac{\| f'(x_0)^{-1} \|}{1 - \| f'(x_0)^{-1} \| \ \| f'(x) - f'(x_0) \|} \leq \frac{1}{3\left(1 - 4(2-p)(1-p)\right)}.$$

Hence,

$$\left\| f'(x)^{-1}\left(f'(y) - f'(x)\right)\right\| \leq \| f'(x)^{-1} \| \, L \, \| y - x \| \leq \frac{2(2-p)}{1 - 4(2-p)(1-p)} \| y - x \|.$$

Then, the definitions are satisfied for $\eta = \frac{1-\delta}{3}$, $b = \frac{1}{3}$, $m = 1$, $K_0 = 3(3-\delta)$, $K = K_2 = 6\left(1 + \frac{1}{3-\delta}\right)^2$, $K_1 = \frac{K}{2}$, and $\alpha = \frac{K_2}{3}\left(\frac{2-\delta}{1-\delta}\right)^2$. Then, for $\delta = 0.98$, we have $t_1 = 0.0070$, $t_2 = 0.0063$, $t_3 = 0.0063$, $t_4 = 0.0063$, $t_5 = 0.0063$, $t_6 = 0.0063$, and $\frac{1}{bK_0} = 0.4950$. So, condition (8) satisfied, and hence $\lim_{n \to \infty} t_n = t^*$.

Example 4.2 For the motivational example in the introduction, we have for $x_0 = 0.5$; $\eta = 0.2813$, $b = \frac{1}{3}$, $K_0 = 32.3333$, $K_1 = 33$, $K_2 = K = 97/3$, $M_n = I$, $m = 1$, $\alpha = 0.6532$. We obtain $t_1 = 4.5585$, $t_2 = 5.7840$, $t_3 = 5.8081$, $t_4 = 5.8081$, $t_5 = 5.8081$, $t_6 = 5.8081$ and $\frac{1}{bK_0} = 37$. So, condition (8) satisfied. That is $\lim_{n \to \infty} t_n = t^*$.

5. Conclusion

A semi-local convergence of method (2) is presented under weaker than before conditions using majorizing sequences.

Conflict of interest

The authors declare no conflict of interest.

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