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Research Article

Spectral Collocation Algorithm for Solving Fractional Volterra-Fredholm Integro-Differential Equations via Generalized Fibonacci Polynomials

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Abstract: In this research article, we build and implement an efficient spectral algorithm for handling linear/nonlinear mixed Volterra-Fredholm integro-differential equations. First, we expand the exact solution as a truncated series of the generalized Fibonacci polynomials, and then we discretize the equation via Simpson's quadrature formula. Finally, we collocate the resulted residual at the roots of the shifted first-kind Chebyshev polynomials. Also, the rate of convergence is studied and the truncation error estimate is reported. Some numerical examples are exhibited to prove the applicability and accuracy of the algorithm.

***Keywords*:** Volterra integral equation, Fredholm integral equation, generalized Fibonacci polynomials, collocation method

MSC: 45B05, 45D05, 65M70, 11B39

1. Introduction

Fractional calculus is a very essential mathematical field that is generalizing the ordinary derivatives and integrals to any non-integer order. It has recently gained popularity in a variety of fields since a lot of physical phenomena are explained by fractional differential equations such as biology, engineering, fluid mechanics, and many other various fields. Many types of research studied fractional differential equations numerically for example: [1] using the variational iteration method to solve fractional optimal control problems, [2] using the Taylor collocation method to solve fractional differential equations, [3] using the ultraspherical wavelets method for solving fractional Riccati differential equations, [4] using a spectral element method for working with nonlinear fractional evolution equation, [5] using Legendre spectral element system for solving time fractional modified anomalous sub-diffusion equation. It is widely known that obtaining theoretical solutions for fractional differential equations is difficult. Accordingly, it is essential to use numerical methods to achieve effective and accurate solutions for the fractional differential equations such as: differential transform method, spectral methods, and finite element methods. The spectral methods are the most important method for solving ordinary and fractional differential equations. This is due to many felicities such as: spectral techniques can give exponential convergence of the solutions of the differential equations and it also gives accurate results which are efficient and simple

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application. The spectral method process is based on finding the approximate solution for a differential equation by a finite sum of specific basis sets that are usually orthogonal then evaluating the expansion coefficient in the sum to satisfy the differential equation and its conditions. There are three common types for evaluating the expansion coefficients for the spectral methods: Galerkin, collocation, and tau techniques. The first technique, Galerkin, includes finding a suitable orthogonal polynomial as a basis function that would satisfy the initial and boundary conditions of the differential equation and then impose the residual to be orthogonal with the basic functions. For examples: [6-7] using the Galerkin technique for finding a direct solution of high even-order differential equations, and solving the time-fractional telegraph equation. The second technique, collocation, ensures that the residual of the differential equation vanishes at a specified set of points. It is a suitable method for dealing with non-linear equations. For example: [8-12] using the collocation method for studying nonlinear FDEs (subject to initial/boundary conditions), studying the second-order multipoint boundary value problems, solving nonlinear FDEs subject to initial/boundary conditions, solving multi-term fractional differential equations, and solving one-dimensional time-fractional convection equation. The last technique, tau, works by decreasing the residual of the differential equation and then applying the initial and boundary conditions. It is considered a particular type of the Petrov-Galerkin method and it is usually applied to solve the differential equation that has complicated boundary conditions. For example: [13-14] using tau technique for solving a coupled system of fractional differential equations through generalized Fibonacci polynomial sequence, and solving a class of fractional optimal control via Jacobi polynomials. The Fibonacci polynomial is a polynomial sequence that could be considered as a circular generalization of the Fibonacci numbers. It is useful for many fields such as: physics, biology, statics, and computer science, see [15]. Many types of research discussed these polynomials and their generalized technically for example: [13] using Spectral tau Algorithm for a certain coupled system of fractional differential e quations through generalized Fibonacci polynomial sequence, [16] studied generalized Fibonacci sequences on an integral domain, [17] studying A novel operational matrix of Caputo fractional derivatives of Fibonacci polynomials, and [18] solving one-dimensional variable-order space-fractional diffusion equations using Fibonacci collocation m ethod. The integral equations have an important role in many branches of applied mathematics. It appears in many useful formulas such as Fredholm and Volterra integral equations. The Volterra-Fredholm integral equations evolved from parabolic boundary value problems and From spatio-temporal statistical modeling The Epidemic Growth. Many researchers obtained several methods for solving linear/nonlinear Volterra-Fredholm integral equations for example: [19] using the Bernstein's approximation method to solve the numerical solution of nonlinear Fredholm and Volterra integral equations, and [20] studied an hp-version collocation method for solving nonlinear Volterra integral equations of the first kind.

This paper is concerned with the numerical solution of the following general mixed Volterra-Fredholm integro-differential equation [21].

$$AD^{a}u(x)+Bu(x)=Cf(x)+\lambda\int_0^x \kappa_1(x,\ t,\ u(t))dt+\mu\int_0^1 \kappa_2(x,\ t,\ u(t))dt. \tag{1}$$

Where A, B, C, λ and μ are known constants, $0 < a \leq 1$, $0 \leq x \leq 1$, If $A \neq 0$, we have the initial condition $u(0) = u_0$ such that: $f(x)$, $\lambda(x, t)$ and $\mu(x, t)$ are analytic known functions, a and b are constants and $u(x)$, should be a continuous function, is the unknown function that we will try to obtain it.

2. An overview on generalized Fibonacci polynomials

In the beginning, we know that we can get the Fibonacci polynomials from the following recurrence relation

$$E_i(y) = yE_{i-1}(y) + E_{i-2}(y);\ i \geq 2, \tag{2}$$

with the initial values: $E_0(y) = 0$ and $E_1(y) = 1$.

The sequence of Fibonacci polynomials $\{E_1(y); i \geq 0\}$ can be extended to produce the sequence $\{\varphi_i^{m,n}(y); i \geq 0$ such that m and n any real constants$\}$ which is built using the recurrence relation below:

$$\varphi_i^{m,n}(y) = my\varphi_{i-1}^{m,n}(y) + n\varphi_{i-2}^{m,n}(y);\ i \geq 2, \tag{3}$$

with the initial values: $\varphi_0^{m,n}(y) = 0$ and $\varphi_1^{m,n}(y) = 1$.

We can express $\varphi_i^{m,n}(y)$ in its analytic form as:

$$\varphi_i^{m,n}(y) = \sum_{r=0}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \binom{i-r-1}{r} n^r (my)^{i-2r-1}, \tag{4}$$

where $\lfloor i \rfloor$ is the largest integer that is less than or equal to i.

Recognize that $\varphi_i^{m,n}(y)$ is a polynomial of degree $(i - 1)$. Then, to prevent any possible ambiguity in notation we can write the generalized Fibonacci polynomials with a degree i by the following formula.

$$Q_i^{m,n}(y) = \varphi_{i+1}^{m,n}(y);\ i \geq 0, \tag{5}$$

which implies that the sequence of polynomials $Q_i^{m,n}(y)$ is created by the recurrence relation below:

$$Q_i^{m,n}(y) = myQ_{i-1}^{m,n}(y) + nQ_{i-2}^{m,n}(y);\ i \geq 2, \tag{6}$$

with the initial values: $Q_0^{m,n}(y) = 1$, and $Q_1^{m,n}(y) = my$.

There are many popular polynomials that we can consider as special cases of the generalized Fibonacci polynomials $Q_i^{m,n}(y)$. These polynomials are Fibonacci, Pell, Fermat, second kind Chebyshev, and second kind Dickson polynomials. And we can obtain them respectively as,

$$F_{i+1}(y) = Q_i^{1,1}(y),\ P_{i+1}(y) = Q_i^{2,1}(y),$$

$$F_{i+1}(y) = Q_i^{3,-2}(y),\ U_i(y) = Q_i^{2,-1}(y),$$

$$E_i(y) = Q_i^{1,-\alpha}(y).$$

$Q_i^{m,n}(y)$ has the following analytic form:

$$Q_i^{m,n}(y) = \sum_{s=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \binom{i-s}{s} n^s (my)^{i-2s}, \tag{7}$$

and in turn, can be written as:

$$Q_i^{m,n}(y) = \sum_{l=0}^{i} m^l n^{\frac{i-l}{2}} \lambda_{i+l} \binom{\frac{i+l}{2}}{\frac{i-l}{2}} y^l, \tag{8}$$

where:

$$\lambda_k = \begin{cases} 0, & r \text{ odd}, \\ 1, & r \text{ even}. \end{cases}$$

The following form is another essential formula of the generalized Fibonacci Polynomials called Binet's form.

$$Q_i^{m,n}(y) = \frac{(my + \sqrt{m^2 y^2 + 4n})^i - (my - \sqrt{m^2 y^2 + 4n})^i}{2i\sqrt{m^2 y^2 + 4n}}; \ 0 \le y \le 1. \tag{9}$$

The following are the first generalised Fibonacci Polynomials:

$$Q_0^{m,n}(y) = 1,\ Q_1^{m,n}(y) = my,\ Q_2^{m,n}(y) = m^2 y^2,$$

$$Q_3^{m,n}(y) = m^3 y^3 + 2mny,\ Q_4^{m,n}(y) = m^4 y^4 + 3m^2 ny^2 + n^2,$$

$$Q_5^{m,n}(y) = m^5 y^5 + 3m^3 ny^3 + 3mn^2 y.$$

Now, we clarify an important theorem for the fractional derivative of the Fibonacci polynomial vectors by driving its operational matrix. We follow the proceed in [13].

Theorem 1. Let $\Phi(x) = [Q_0^{m,n}(x), Q_1^{m,n}(x), Q_2^{m,n}(x), \ldots, Q_M^{m,n}(x)]^T$ be the generalized Fibonacci polynomial vector, then for any $\alpha > 0$ the following statement holds:

$$D^{\alpha}\mathbf{\Phi}(x) = \frac{d^{\alpha}\mathbf{\Phi}(x)}{dx^{\alpha}} = x^{-\alpha} H^{(\alpha)}\mathbf{\Phi}(x), \tag{10}$$

where $H^{(\alpha)} = (h_{i,j}^{\alpha})$ is the Fibonacci operational matrix of fractional derivatives with order $(M + 1) \times (M + 1)$, and it can be expressed as:

$$H^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_\alpha(\lceil \alpha \rceil, 0) & \omega_\alpha(\lceil \alpha \rceil, \lceil \alpha \rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_\alpha(i,0) & \cdots & \omega_\alpha(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_\alpha(M,0) & \omega_\alpha(M,1) & \omega_\alpha(M,2) & \cdots & \omega_\alpha(M,M) \end{pmatrix}. \tag{11}$$

In addition, the elements $(h_{i,j}^{\alpha})$ are given explicitly by the form

$$h_{i,j}^{\alpha} = \begin{cases} \omega_\alpha(i,j), & i \ge \lceil \alpha \rceil,\ i \ge j; \\ 0, & \text{otherwise}. \end{cases}$$

And,

$$\omega_\alpha(i,j)=\sum_{\lambda=\lceil\alpha\rceil}^{i}\frac{i\lambda!\upsilon_{i+\lambda}\upsilon_{j+\lambda}\xi_j(-1)^{\frac{\lambda-j}{2}}b^{\frac{i-j}{2}}\left(\frac{i+\lambda}{2}-1\right)!}{\left(\frac{i-\lambda}{2}\right)!\left(\frac{\lambda-j}{2}\right)!\left(\frac{j+\lambda}{2}\right)!\Gamma(1+\lambda-\alpha)}. \tag{12}$$

Proof. See [11]. □

3. Numerical spectral solution of Volterra-Fredholm integral equations

In this section, we will investigate the method of solving the Volterra-Fredholm integro-differential equation numerically using the Generalized Fibonacci tau method (GFTM).

According to Eq.(1), we consider the approximate solution of $u(x)$ as:

$$u(x)\simeq\sum_{i=0}^{M+1}c_iQ_i^{m,n}(x), \tag{13}$$

where,

$$C_i=[c_0,c_1,c_2,\ldots,c_{M+1}]^T,$$

and

$$\Phi(x)=[Q_0^{m,n}(x),Q_1^{m,n}(x),Q_2^{m,n}(x),\ldots,Q_{M+1}^{m,n}(x)].$$

The residual of the Volterra-Fredholm integro-differential equation can be written as:

$$R(x)=A(D^\alpha u)(x)+Bu(x)-Cf(x)-\lambda\int_0^x\kappa_1(x,t,u(t))dt-\mu\int_0^1\kappa_2(x,t,u(t))dt. \tag{14}$$

For the first integral we set: $t=xz$.

Consequently,

$$R(x)=A(D^\alpha u)(x)+Bu(x)-Cf(x)-\lambda x\int_0^1\kappa_1(x,xz,u(xz))dz-\mu\int_0^1\kappa_2(x,z,u(z))dz. \tag{15}$$

Or, in other words

$$R(x)=A(D^\alpha u)(x)+Bu(x)-Cf(x)-\int_0^1\kappa(x,z,u(z),\ u(xz))dz, \tag{16}$$

where:

$$\kappa=\lambda x\kappa_1+\mu\kappa_2.$$

Now apply Romberg's integration rule:

$$\int_0^1 g(z)dz = \Gamma_{m,n}(g) + E_{m,n}, \tag{17}$$

where:

$$r_{0,0} = \frac{g(0)+g(1)}{2},\ r_{n,0} = \frac{1}{2}r_{n-1,0} + 2^{-n}\sum_{k=1}^{2^{n-1}} g\left(\frac{2k-1}{2^n}\right),$$

$$r_{n,m}(g) = r_{n,m-1} + \frac{1}{4^m-1}(r_{n,m-1} - r_{n-1,m}),\ E_{m,n} = \Phi(4^{-n(m+1)}).$$

Consequently,

$$R(x) = A(D^\alpha u_M)(x) + Bu_M(x) - Cf(x) - r(k)(x). \tag{18}$$

The final discretization of the Volterra-Fredholm integro-differential equation is given by

$$A(D^\alpha u_M)(x_j) + Bu_M(x_j) = Cf(x_j) + r(k)(x_j);\ 0 \le j \le M+1. \tag{19}$$

With the aid of the initial condition, we get $u_M(0) = u_0$.
Where:

$$u_M(x) = \sum_{i=0}^{M+1} c_i Q_i^{m,n}(x). \tag{20}$$

By applying the spectral tau method to find the numerical solution for $u(x)$ we get,

$$\int_0^l x^\alpha R(x) Q_i^{m,n}(x)dx = 0;\ i = 0,\ 1,\ 2,\ ...,\ M-1. \tag{21}$$

Now we have a system of algebraic equations of dimensions $M + 2$ that by solving it using Newton iterative method we will get the numerical solution of $u(x)$.

4. Convergence and error analysis

In this section, we study the investigation of the convergence and error analysis for the expansion of generalized Fibonacci polynomials. The following two theorems are satisfied.

Theorem 2. Let $u(x)$ be defined on the interval [0, 1] and $|u^{(i)}(0)| \le N^i$, $i \ge 0$, where N is a positive integer constant, and let $u(x)$ expands as:

$$u(x) = \sum_{r=0}^{\infty} c_r Q_r^{m,n}(x).$$

Then:

1. $| c_r | \leq \sigma \dfrac{(\frac{N}{|a|})^{r+1}}{r!}$,

2. The series is absolutely convergent.

Proof. See [13]. □

Theorem 3. If $u(x)$ satisfies the same hypothesis of Theorem (1), and if $e_l(x)$ be the truncation error such that:

$$e_l(x) = \sum_{i=l+1}^{\infty} c_i Q_i^{m,n}(x).$$

Then the truncation error estimates as follows:

$$| e_l(x) | < \tilde{c} \frac{\tau^l}{(l-1)!}.$$

Proof. See [13]. □

In the following theorem we investigate the global error of the numerical solution of equation (1) and give an estimated value for it.

Theorem 4. If $e_l(x)$ be the truncation error, and $E_l = \max_{x \in [0,1]} \eta_l(x)$ be the global error, assuming that: $|A| \leq A_1$, $|B| \leq B_1$ and $L^2 \leq L_1$. Where A_1, B_1 and L_1 are positive constants. Then we obtain the global error estimate as follows:

$$E_l \leq A_1 L_1 \sigma g e^g \frac{g^{i+1}}{(i+1)!} + \gamma \tilde{c} \frac{\tau^l}{(l-1)!} \tag{22}$$

such that:

$$\gamma = \max(A_1, B_1, \lambda, \kappa_1, \kappa_2, \mu)$$

Proof. Let

$$\eta_l(x) = | A(D^\alpha u_l)(x) + Bu_l(x) - Cf(x) - \lambda \int_0^x \kappa_1(x,t)u(t))dt - \mu \int_0^1 \kappa_2(x,t)u(t))dt |, \tag{23}$$

where

$$\kappa_1(x,t,u(t)) = \kappa_1(x,t)u(t).$$

From eq. (1) we have,

$$Cf(x) = AD^\alpha u(x) + Bu(x) - \lambda \int_0^x \kappa_1(x,t)u(t))dt - \mu \int_0^1 \kappa_2(x,t)u(t))dt. \tag{24}$$

Then

$$\eta_l(x) \leq |A|L^2 \sum_{r=i+1}^{\infty} \frac{\sigma(\frac{N}{|a|})^{r+1}}{r!} + |B||e_l(x)| - |\lambda||\kappa_1(x,t)||e_l(t)||x| - |\mu||\kappa_2(x,t)||e_l(t)|. \tag{25}$$

Let

$$\frac{N}{|a|} = g,$$

then

$$\sum_{r=i+1}^{\infty} \frac{\sigma g^{r+1}}{r!} = \sigma g \sum_{r=i+1}^{\infty} \frac{g^r}{r!} = \sigma g e^g (1 - \frac{\Gamma(1+i,g)}{\Gamma(1+i)}) = \sigma g e^g (\frac{\Gamma(1+i) - \Gamma(1+i,g)}{\Gamma(1+i)}) = \sigma g e^g \frac{g^{i+1}}{(i+1)!}. \tag{26}$$

Where

$$\frac{\Gamma(1+i) - \Gamma(1+i,g)}{\Gamma(1+i)} = \frac{1}{\Gamma(1+i)} \int_0^l z^i e^{-z} dz \leq \frac{1}{\Gamma(1+i)} \int_0^l z^i dz \leq \frac{1}{\Gamma(1+i)} \frac{g^{i+1}}{i+1} \leq \frac{g^i + 1}{(i+1)!}. \tag{27}$$

According to the theorem's assumption, we have:

$$\eta_l(x) \leq A_1 L_1 \sigma \frac{N}{|a|} e(\frac{N}{|a|}) \frac{(\frac{N}{|a|})^{i+1}}{(i+1)!} + (|B|) + |\lambda||\kappa_1| + |\mu||\kappa_2|)e_l, \tag{28}$$

then the global error estimated as:

$$E_l \leq A_1 L_1 \sigma g e^g \frac{g^{i+1}}{(i+1)!} + \gamma \tilde{c} \frac{\tau^l}{(l-1)!} \tag{29}$$

□

5. Numerical examples

In this section we are going to present some numerical experiments to check the applicability and accuracy of the method, also, we will test our results with some schemes appeared in previous papers by comparing them with the results in [19, 20, 22-24]. All of the numerical calculations were carried out using Mathematica software.

Example 1. [22] Consider the following linear Fredholm integral equation problem:

$$\int_0^1 \sin(xt) u(t) dt = \frac{\sin(x) - x cos(x)}{x^2}. \tag{30}$$

This equation has the exact solution $u(x) = x$.

Table 1 compares our numerical results with those in [22]. We observe that the absolute errors obtained by our method at different values of M are better than obtained by the other method. Table 2 gives the maximum absolute error of Eq. (30) for different values of m and n. Table 3 clarifies the time used for the running program (CPU time). Figure 1 shows the absolute error in case of $m = n = 1$.

Table 1. Comparison between GFTM and [22] for Example 1

M	2	3	4	5	6	7	8
E	$5 .10^{-15}$	$3.5 .10^{-11}$	$1.2 .10^{-9}$	2.10^{-9}	$1 .10^{-8}$	$1 .10^{-9}$	$1.5 .10^{-9}$
Results in [22]	$2 .10^{-9}$	$5.3 .10^{-5}$	$1.3 .10^{-4}$	$2.2 .10^{-4}$	$3 .10^{-4}$	$5.9 .10^{-4}$	$3.8 .10^{-4}$

Table 2. Maximum absolute error E for different values of m and n for Example 1

m	n	M	E	M	E	M	E	M	E	M	E	M	E
1	1	3	$5 .10^{-15}$	4	$3.5 .10^{-11}$	5	$2 .10^{-9}$	6	$1 .10^{-8}$	7	$1 .10^{-9}$	8	$1.5 .10^{-9}$
2	1		$5 .10^{-15}$		$2 .10^{-9}$		$1.5 .10^{-8}$		$2 .10^{-9}$		$2 .10^{-9}$		$2.5 .10^{-8}$
3	-2		$2 .10^{-9}$		$5 .10^{-3}$		$5 .10^{-3}$		$1.2 .10^{-3}$		$1.4 .10^{-3}$		$1.5 .10^{-3}$
3	-1		$2 .10^{-6}$		$1 .10^{-3}$		$1 .10^{-3}$		$1 .10^{-3}$		$1 .10^{-3}$		$1 .10^{-3}$

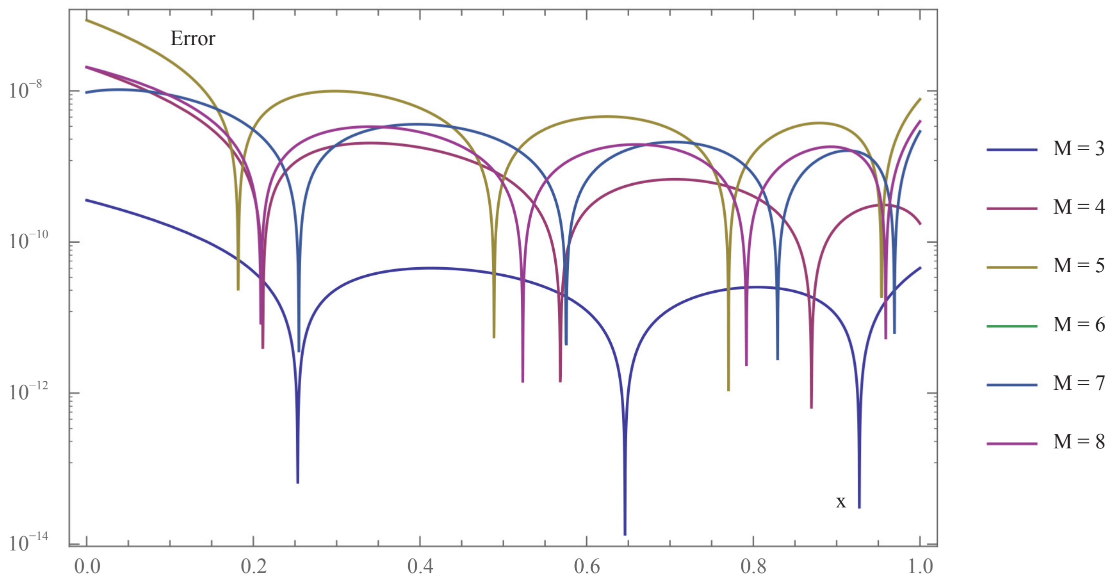

Figure 1. Graph of the error at M = 3, 4, 5, 6, 7 and 8

Table 3. CPU time for Example 1

M	CPU time
2	8.953
3	19.375
4	31.971
5	45.766
6	67.671
7	93.142
8	182.36

Example 2. [19] Consider the following linear Volterra integral equation problem:

$$\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt = f(x),\ x \in [0,1], \tag{31}$$

where

$$f(x) = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3).$$

This equation has the exact solution $u(x) = x^3 - x^2 + 1$.

Table 4. Comparison between GFTM and [19] for Example 2

M	2	3	4	5	6	7	8	9	10
E	$4 .10^{-2}$	$4 .10^{-13}$	$4 .10^{-12}$	$2 .10^{-12}$	$1 .10^{-10}$	$1.5 .10^{-9}$	$2 .10^{-8}$	$1 .10^{-8}$	$1 .10^{-9}$
Results in [19]	$6.4 .10^{-2}$	$2.4 .10^{-2}$	$3.3 .10^{-3}$	$1.2 .10^{-3}$	$1.9 .10^{-4}$	$2.7 .10^{-4}$	$1.7 .10^{-5}$	$5.6 .10^{-5}$	$4.5 .10^{-7}$

Table 5. Maximum absolute error E for Example 2

m	n	M	E	M	E	M	E	M	E	M	E
1	1		$4 .10^{-13}$		$2 .10^{-12}$		$1.5 .10^{-9}$		$1 .10^{-8}$		$1 .10^{-9}$
2	1	3	$5 .10^{-13}$	5	$8 .10^{-12}$	7	$3.5 .10^{-11}$	9	$6 .10^{-9}$	10	$2 .10^{-8}$
3	-2		$8 .10^{-14}$		$2.5 .10^{-12}$		$3 .10^{-11}$		$4 .10^{-10}$		$2 .10^{-9}$
3	-1		$5 .10^{-14}$		$4 .10^{-13}$		$7 .10^{-11}$		$1.2 .10^{-9}$		$4 .10^{-9}$

Table 6. CPU time for Example 2

M	CPU time
2	3
3	4.921
4	7.687
5	11.093
6	16.689
7	20.438
8	30.75
9	38.89
10	49.97

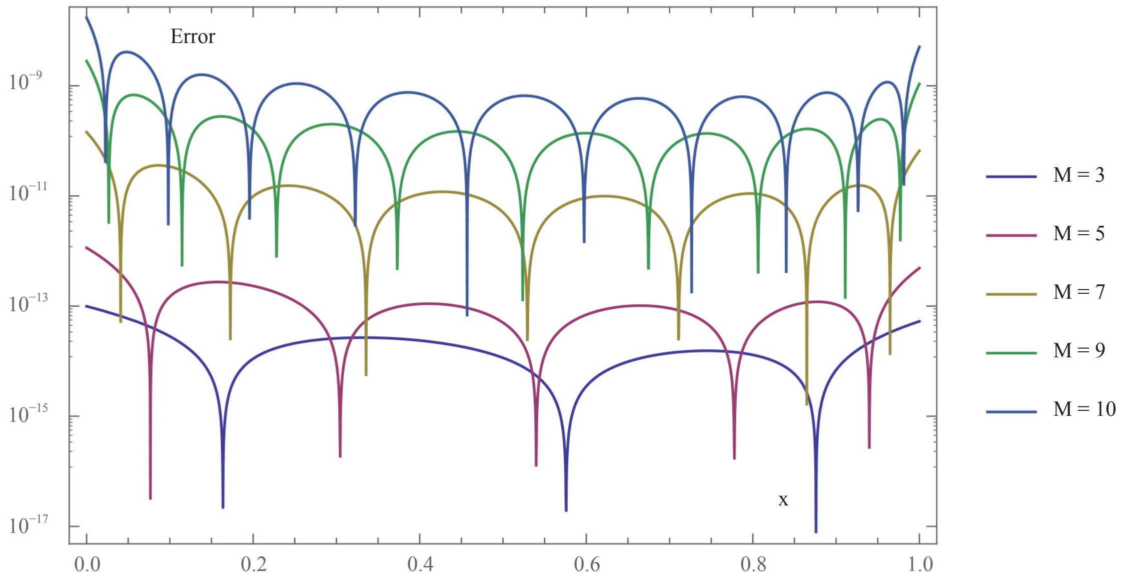

Figure 2. Graph of the error at M = 3, 5, 7, 9 and 10

Table 4 compares our results with in [19]. It is clear that the absolute errors decrease drastically with increasing the number of steps. We notice that the obtained errors are the best and the least. Table 5 lists the maximum approximate error of Eq. (31) for different values of m and n. Table 6 illustrates the time used for the running program (CPU time). Figure 2 shows the absolute error in case of $m = n = 1$.

Example 3. [20] Consider the following non-linear Volterra integral equation problem:

$$\int_0^x (1+x-t)^2 (u(t)-u^3(t))dt = x^2,\ x \in [0,10]. \tag{32}$$

The exact solution for this equation is not known. We will try to find the exact solution by using the series solution method for solving Volterra integral equation of the first kind.

Let the solution $u(x)$ be an analytic function and apply Taylor series at $x = 0$ as

$$u(x)=\sum_{i=0}^{\infty}a_i x^i. \tag{33}$$

Then substituting in the Eq. (32) as

$$\int_0^x (1+x-t)^2 (\sum_{i=0}^{\infty}a_i t^i - (\sum_{i=0}^{\infty}a_i t^i)^3)dt = T[x^2], \tag{34}$$

where $T[x^2]$ is the Taylor series for x^2.

We solve the Eq. (32) by proceeding with a few terms of Taylor series, integrating the right side and equating the coefficients of x in both sides, then finally obtain the solution as

$$u(x)=\frac{2}{5}-\frac{2}{5}x$$

Now to find the numerical solution we apply the GFCM (Generalized Fibonacci collocation method).

Table 7 illustrates the maximum approximate error of Eq. (32) for different values of m and n at three different values for M. Table 8 shows the time used for the running program (CPU time). Figure 3 shows the absolute error for different values of a and b. From this Figure, we observe that the convergence is exponential.

Table 7. Maximum absolute error E for different values of m and n for Example 3

m	n	M	E	M	E	M	E
1	1		2.10^{-1}		4.10^{-1}		4.10^{-1}
2	1		2.10^{-1}		4.10^{-1}		4.10^{-1}
3	-2	1	2.10^{-1}	3	4.10^{-1}	5	4.10^{-1}
3	-1		2.10^{-1}		4.10^{-1}		4.10^{-1}
Results in [20]			1.10^{-1}		1.10^{-3}		1.10^{-4}

Table 8. CPU time for Example 3

M	CPU time
1	80.813
3	5.875
5	10.471

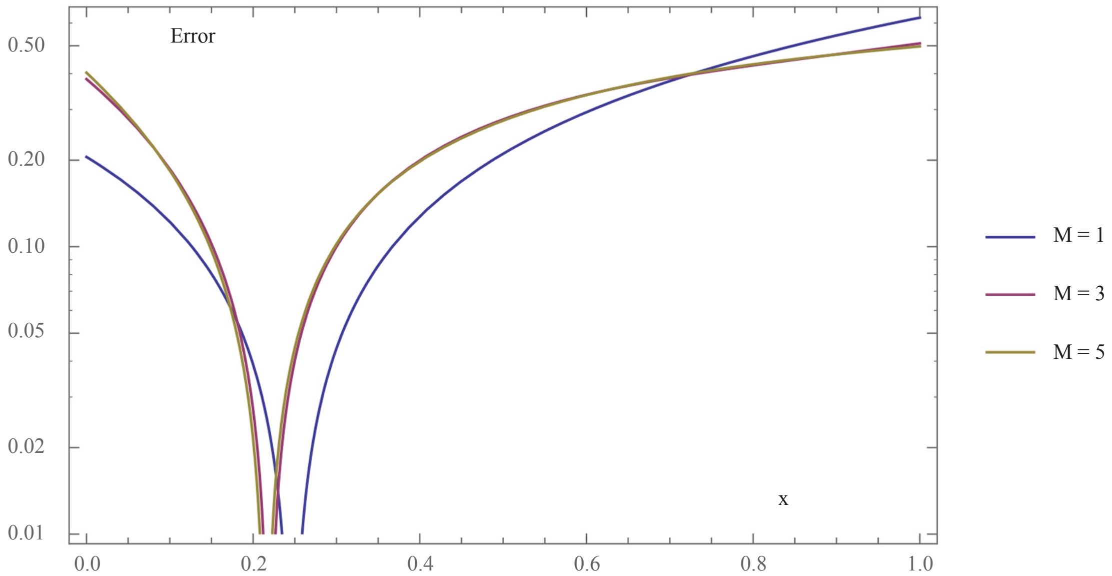

Figure 3. Graph of the error at M = 1, 3 and 5

Example 4. [23] Consider the following non-linear Volterra-Fredholm integral equation problem:

$$u(x)-3\int_0^x \sin(x-t)u^2(t)dt-\frac{6}{7-6\cos(1)}\int_0^1(1-t)cos^2(x)(t+u(t))dt=f(x), \tag{35}$$

where

$$f(x)=cos(x)+\frac{6(-\frac{6}{7}+cos(1))cos(x)^2}{7-6cos(1)}-2(2+cos(x))sin(\frac{x}{2})^2.$$

This equation has the exact solution $u(x) = \cos(x)$.

Table 9 compares our results to those observed in [23]. Table 10 lists the maximum absolute error of Eq. (35) for different values of m and n. Table 11 shows the time used for the running program (CPU time). In Figure 4 the results are displayed at $M = 2, 4, 6$, and 8 in case of $m = n = 1$.

Table 9. Comparison between GFTM and [23] for Example 4

M	2	4	6	8
E	$8 .10^{-1}$	$9 .10^{-1}$	$9 .10^{-1}$	$2 .10^{-1}$
Results in [23]	$2 .10^{-5}$	-	-	$1.4 .10^{-8}$

Table 10. Maximum absolute error E for Example 4

m	n	M	E	M	E	M	E	M	E
1	1		8.10^{-1}		9.10^{-1}		9.10^{-1}		2.10^{-1}
2	1	2	8.10^{-1}	4	9.10^{-1}	6	$2.5.10^{-6}$	8	2.10^{-4}
3	-2		8.10^{-1}		9.10^{-1}		9.10^{-1}		2.10^{-4}
3	-1		8.10^{-1}		9.10^{-1}		$2.5.10^{-6}$		2.10^{-4}

Table 11. CPU time for Example 4

M	CPU time
2	1.546
4	5.813
6	8.03
8	13

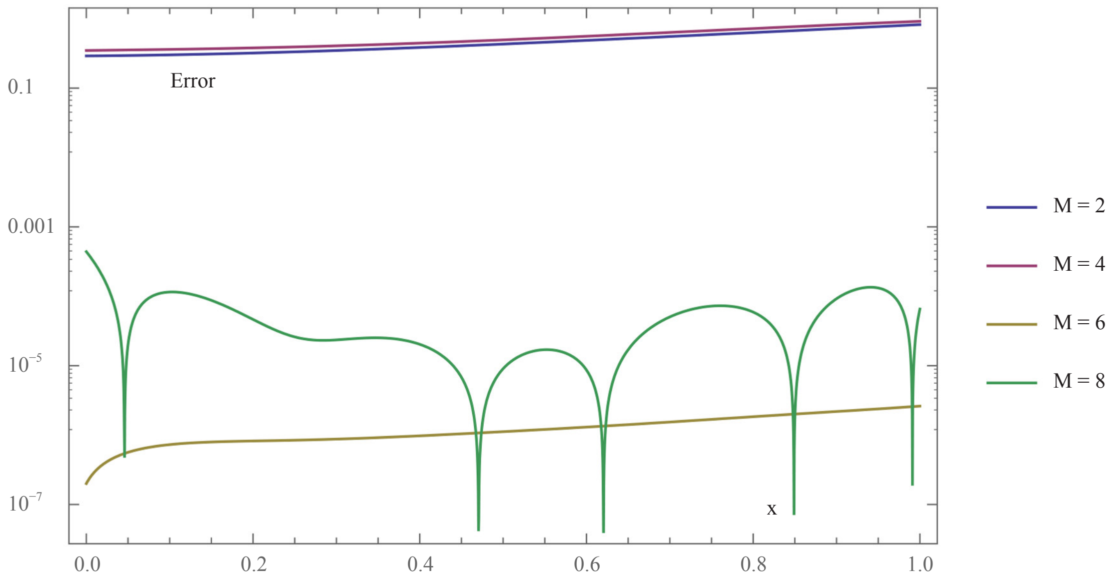

Figure 4. Graph of the error at M = 2, 4, 6 and 8

Example 5. [24] Consider the following non-linear Fredholm fractional integro-differential equation problem:

$$D^{\alpha} u(x) - \int_0^1 x e^t u^2(t) dt = f(x), \tag{36}$$

where

$$f(x) = (674 - 248\varrho)x - \frac{x^{1-\alpha}(-6 + 6x^2 + 5\alpha - 5\alpha^2)}{(-6 + 11\alpha - 6\alpha^2 + \alpha^3)\Gamma(1-\alpha)}.$$

This equation has the exact solution $u(x) = x - x^3$.

Table 12. Absolute error E for Example 5

x	α = 0.7	α = 0.8	α = 0.9	Exact
0.1	0.099	0.099	0.99	0.302649
0.2	0.192	0.191	0.109	0.286029
0.3	0.273	0.273	0.272	0.252505
0.4	0.336	0.336	0.335	0.202076
0.5	0.375	0.365	0.357	0.134742
0.6	0.384	0.384	0.384	0.0505042
0.7	0.375	0.375	0.375	-0.0506381
0.8	0.288	0.288	0.288	-0.168685
0.9	0.171	0.171	0.171	-0.303636

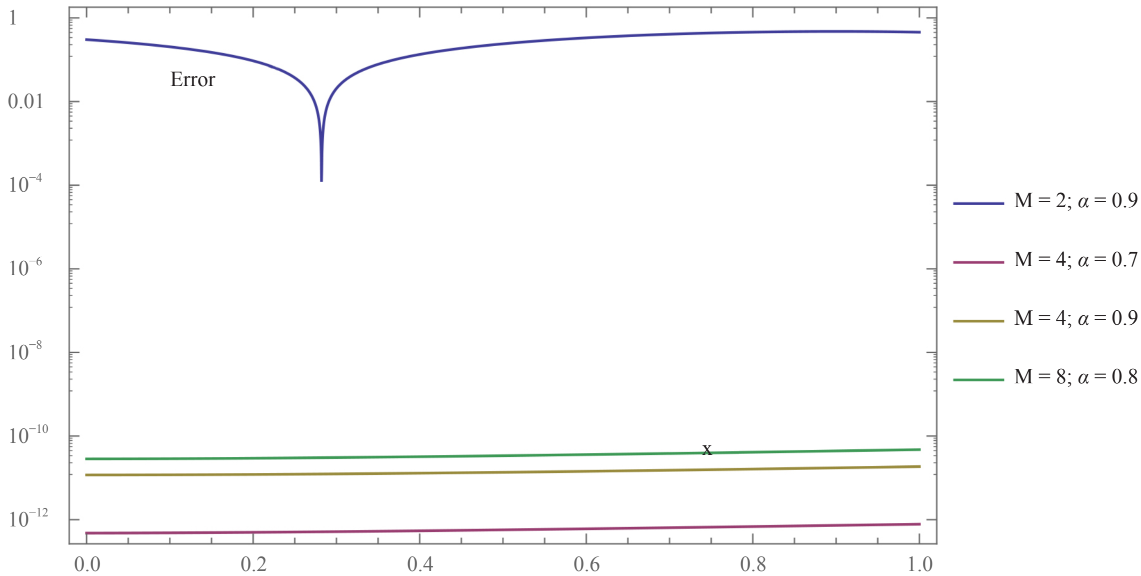

Figure 5. Graph of the error at M = 2, 4 and 8

Table 12 illustrates the approximate solutions for various values of α and the exact solution in [24]. Figure 5 shows the approximate solutions for various values of α.

Example 6. [25] Consider the following linear Volterra-Fredholm integral equation problem:

$$u(x) - \int_0^x tu(t)dt - \int_{-1}^1 t^2 u(t)dt = f(x), \tag{37}$$

where

$$f(x) = x - \frac{1}{3}x^3.$$

This equation has the exact solution $u(x) = x$.

Table 13 gives the maximum absolute error of Eq. (37) for different values of m and n. Table 14 shows the time used for the running program (CPU time). In Figure 6 the results are displayed at $M = 3, 5, 7, 9$, and 10 in case of $m = 2$ and $n = 1$.

Table 13. Maximum absolute error E for Example 6

m	n	M	E	M	E	M	E	M	E	M	E
1	1	3	$4.5 .10^{-13}$	5	$2.4 .10^{-9}$	7	$3 .10^{-5}$	9	$3 .10^{-3}$	10	$3 .10^{-2}$
2	1		$2 .10^{-13}$		$4.5 .10^{-10}$		$3.5 .10^{-8}$		$6 .10^{-4}$		$3 .10^{-2}$
3	-2		$2.6 .10^{-12}$		$4 .10^{-10}$		$5 .10^{-7}$		$1 .10^{-4}$		$2 .10^{-3}$
3	-1		$1.1 .10^{-13}$		$4 .10^{-10}$		$1.2 .10^{-6}$		$1 .10^{-5}$		$2 .10^{-3}$

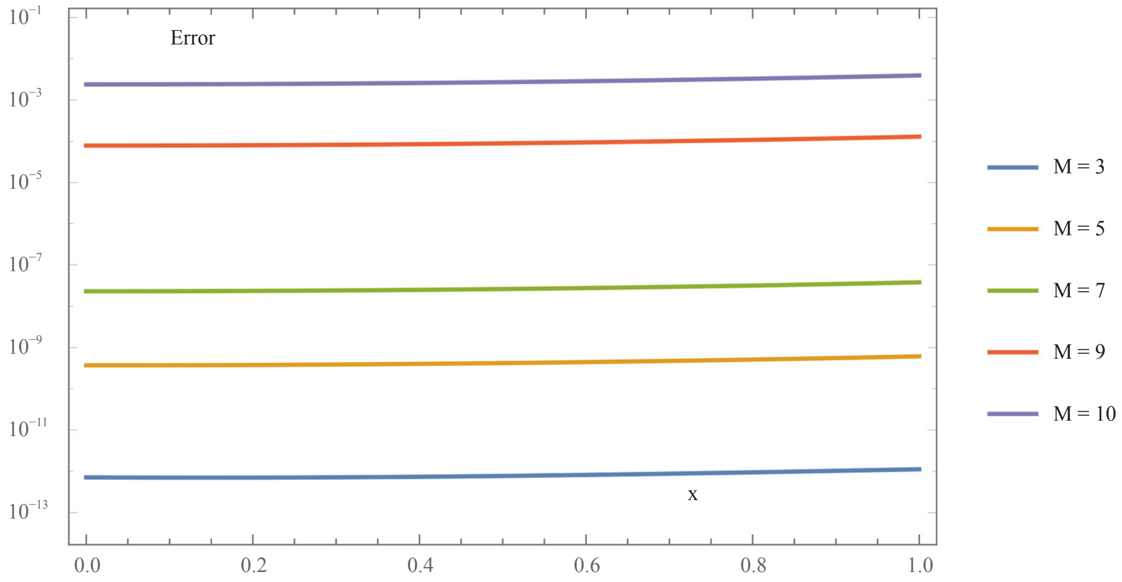

Figure 6. Graph of the error at M = 3, 5, 7, 9 and 10

Table 14. CPU time for Example 6

M	CPU time
3	0.422
5	0.781
7	0.921
9	1.093
10	16.689

6. Conclusions

In this paper, we presented an efficient numerical technique for solving Volterra-Fredholm integro-differential equation. The proposed numerical method is based on using the operational matrix of fractional derivatives of the generalized Fibonacci polynomial and a suitable spectral method to transform Volterra-Fredholm equation into a system of algebraic equations that can be solved by mathematica software. We also discussed the convergence and error analysis of the generalized Fibonacci polynomial. Finally we illustrated some numerical examples to show the accuracy of the method.

Conflict of interest

The authors declare no conflicts of interest.

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