Generalization of Tail Inequalities for Random Variables in the Martingale Theory

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Abstract: We generalize the tail Doob’s inequality, concerning two non-negative random variables, arising in the martingale theory, in three directions: on the more general source data, on the random variables belonging to the so-called Grand Lebesgue Spaces, as well as on the multidimensional variables. We also provide several examples. Moreover, we show the exactness of the estimates obtained in the particular case of positive random variables having exponential distribution.

Keywords: martingale, random variable, expectation, tail of distribution, probability space, Doob’s inequalities, Grand Lebesgue spaces, Young-Fenchel transform

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1. Introduction

Let \((\Omega = \{\omega\}, \mathcal{B}, P)\) be a non-trivial probability space with expectation \(E\). The classical Doob’s inequality ([1, 2], see also [3]) states that if the non-negative numerical valued random variables (r.v-s) \(\xi, X\) are such that, for some positive constants \(\beta, C\),

\[ P(\xi > \beta t) \leq C [E(I(\xi > t))]^p, \ t \geq 0, \]  

then

\[ \|\xi\|_p = [E(\xi^p)]^{1/p} \leq C \frac{p}{p-1} \beta^p \|X\|_p, \ p > 1, \]  

where henceforth \(I(A)\) denotes the indicator function of the random event \(A\), and \(\|\eta\|_p\) denotes, as usually, the classical Lebesgue-Riesz norm of the r.v. \(\eta\):
\[
\| \eta \|_p := \left[ \mathbb{E} | \eta |^p \right]^{1/p} = \left( \int_{\Omega} |\eta(\omega)|^p \, P(d\omega) \right)^{1/p}, \ p \geq 1.
\]

The inequality (2) plays a very important role, in particular, in the martingale theory (see [4, 5]).

Our aim is a generalization of the Doob’s inequality in three directions: on the more general source data, on the random variables belonging to the Grand Lebesgue Spaces and on the multidimensional random vectors.

1.1 Statement of the problem

Let us assume the following generalization of inequality (1)

\[
h(t)P(\xi > \beta t) \leq C \mathbb{E}[XI(\xi > t)], \ t \geq 0,
\]

where \( h = h(t) \) is a non-negative continuous strictly increasing deterministic function and, as before, \( X \) and \( \xi \) are non-negative random variables.

We establish a generalization of the estimate (2). Moreover we provide some examples in order to show the exactness of the obtained estimates, in the particular case of positive random variables having exponential distribution.

2. Main result

**Theorem 2.1** Let \((\Omega = \{\omega\}, \mathcal{B}, P)\) be a non-trivial probability space with expectation \(\mathbb{E}\) and let \(\xi, X\) be non-negative numerical valued random variables. Let \(h = h(t)\) be a non-negative continuous strictly increasing deterministic function. Let us assume that, for some positive constants \(\beta, C\), the following inequality is satisfied:

\[
h(t)P(\xi > \beta t) \leq C \mathbb{E}[XI(\xi > t)], \ t \geq 0.
\]

Let \(p > 1\) and define the random variable (measurable function):

\[
\kappa_p(\xi) \overset{def}{=} \int_0^\xi t^{p-1} \frac{dt}{h(t)}
\]

if there exists. Assume that there exists \(\theta > 1\) such that \(K_p(\theta) := \|\kappa_p(\xi)\|_\theta < \infty\).

Suppose that \(\exists r = r(\theta, p) \in [1, p), \ \exists \nu = \nu(\theta, p, r) < \infty\) such that

\[
K_p(\theta) = \|\kappa_p(\xi)\|_\theta \leq \nu(\theta, p, r) \left\| \xi \right\|_p.
\]

Then

\[
\left\| \xi \right\|_p \leq \left[C \nu(\theta, p, r) \beta'' \right]^{\frac{1}{\theta'-(p-r)}} \left\| X \right\|_\theta^{\frac{1}{\theta'-(p-r)}}, \ \theta' = \frac{\theta}{\theta-1} \in (1, \infty).
\]

Of course,

\[
\left\| \xi \right\|_p \leq \inf_{r \in R} \inf_{\beta \in \mathbb{R}} \left\{C \nu(\theta, p, r) \beta'' \right\}^{\frac{1}{\theta'-(p-r)}} \left\| X \right\|_\theta^{\frac{1}{\theta'-(p-r)}},
\]

where

*Contemporary Mathematics* 458 | Maria Rosaria Formica, et al.
\[ R = R(\theta, p) := \{ r \in [1, p) : K_p(\theta) < \infty \}, \]
\[ \Theta = \Theta(p)[h, \xi] := \{ \theta > 1 : K_p(\theta) < \infty \}. \]

**Proof.** By (3) we have
\[ P(\xi / \beta > t) \leq \frac{C}{h(t)} E[XI(\xi > t)], \quad t \geq 0, \]
hence
\[ \int_0^\infty pt^{p-1} P(\xi / \beta > t) \, dt \leq \int_0^\infty \frac{Cp t^{p-1}}{h(t)} E[XI(\xi > t)] \, dt, \quad p > 1. \quad (7) \]

The left-hand side of (7) is equal to \[ E | \xi / \beta |^p = \| \xi \|_p^p \beta^{-p}. \]
Let us investigate the right-hand side of (7). We deduce, by virtue of Fubini’s theorem,
\[ \int_0^\infty \frac{Cpt^{p-1}}{h(t)} E[XI(\xi > t)] \, dt = C\rho E \left[ X \int_0^\xi t^{p-1} \frac{1}{h(t)} \, dt \right]. \]

Then inequality (7) yields
\[ \| \xi \|_p^p \beta^{-p} \leq C \rho E [X\kappa_p(\xi)]. \]

By Hölder’s inequality with conjugate exponents \( \theta \) and \( \theta' \) we get
\[ \| \xi \|_p^p \leq C \rho \kappa_p(\xi) \| X \|_{\theta'}, \]
and by assumption (4) we deduce
\[ \| \xi \|_p^p \leq C \rho \kappa_p(\xi) \nu(\theta, p, r) \| \xi \|_{\theta'} \| X \|_{\theta'}, \]
which implies (5). \( \square \)

**Example 2.1** If \( h(t) = t \), inequality (3) reduces to (1):
\[ P(\xi > \beta t) \leq C E(\xi | I(\xi > t)), \quad t, \xi, X \geq 0, \]
and (5) reduces to (2):
\[ \| \xi \|_p^p \leq C \frac{p}{p-1} \beta^{-p} \| X \|_p, \quad p > 1, \quad (8) \]
choosing \( \nu(\theta, p, r) = \frac{1}{p-1}, \quad r = p-1 \) and \( \theta = \frac{p}{p-1} \), that is \( \theta' = p \).

**Example 2.2** A more general case. Suppose
\[ t^A \mathbf{P} (\xi > \beta t) \leq C \mathbf{E} (X^I (\xi > t)), \ t, \xi, X \geq 0, \ A = \text{const} > 1. \] (9)

We get, after simple calculations, the following estimate

\[ \| \xi \|_p \leq \frac{P}{p-\Delta} \| \beta^p X \|_p, \ p > \Delta. \] (10)

**Remark 2.1** Let \( s \in [1, p] \). By virtue of Lyapunov’s inequality \( \| \xi \|_s \leq \| \xi \|_p \). Therefore, if (9) holds, then by (10) we have also

\[ \| \xi \|_s \leq \frac{P}{p-\Delta} \beta^p \| X \|_p, \ p > \Delta, \ \forall s \in [1, p]. \]

### 3. Unimprovability of the estimations. Lower bounds

Let us show the exactness of the obtained results (8) and (10) in a particular case. Introduce the following functionals, which are involved in the lower estimate.

\[
Z[C, \beta, \Delta](\xi, X, \rho) \overset{\text{def}}{=} \left[ \frac{\| \xi \|_p}{Cp(p-\Delta)^{-1} \beta^p \| X \|_p} \right].
\]

\[
K[C, \beta, \Delta] = K[C, \beta, \Delta](p) = \sup_{p > \Delta} \sup_{\xi \in X, X \in X} Z[C, \beta, \Delta](\xi, X, \rho).
\] (11)

where all the supremums are calculated over the r.v. - s \( \xi, X \) satisfying the condition (9) and when \( p > \Delta, \Delta = \text{const} \geq 1 \).

**Proposition 3.1** Let \( \xi_0, X_0 \) be positive random variables such that \( X_0 = \xi_0 \) and having standard exponential distribution

\[ \mathbf{P} (\xi_0 > t) = \mathbf{P} (X_0 > t) = e^{-t}, \ t > 0. \]

Then

\[ K[C, \beta, \Delta] = 1. \] (12)

**Proof.** The upper estimate \( K[C, \beta, \Delta] \leq 1 \) is contained in (10). We deduce the lower one in the particular case \( C = \beta = 1 \).

So we have to prove

\[ K[1, 1, \Delta] = 1. \] (13)

The natural generating function for these r.v. - s has the form

\[ \nu(p) = \| \xi_0 \|_p = \| X_0 \|_p = \Gamma(p+1), \ p > 1, \]
where, as usually, $\Gamma(\cdot)$ is the Euler’s Gamma function, since
\[
\| \xi \|_p = \left(p \int_0^\infty t^{p-1} \mathbb{P}(\xi \geq t)dt\right)^{1/p} = \left(p \int_0^\infty t^{p-1}e^{-t}dt\right)^{1/p} = (p\Gamma(p))^{1/p} = \Gamma^{1/p}(p+1).
\]

Note on the way that as $p \to \infty \Rightarrow \nu(p) \sim p^\beta$. Note also that the condition (9) is satisfied. We have
\[
Z[C, \beta, \Delta](\xi_0, X_0, p) = \frac{p^{-\Delta}}{p}, \quad p > \Delta.
\]

Following,
\[
K[1, 1, \Delta] \geq \sup_{p > \Delta} \left\{ \frac{p^{-\Delta}}{p} \right\}.
\]

So (13) follows immediately from the fact that
\[
\lim_{p \to \infty} \left\{ \frac{p^{-\Delta}}{p} \right\} = 1.
\]

Remark 3.1 The cases $C, \beta \neq 1$ may be considered quite analogously.

4. Generalization on the Grand Lebesgue Spaces approach. Examples

We intend in this section to extend the previous results upon the so-called Grand Lebesgue Spaces (GLS) of random variables.

Let $1 \leq a < b \leq \infty$. Let $\psi = \psi(p), \ p \in (a, b)$ be a numerical valued measurable strictly positive function, such that $\inf_{p \in (a, b)} \psi(p) > 0$, not necessarily bounded.

We use the following notations:
\[
\text{Dom}(\psi) := \{ p : \psi(p) < \infty \},
\]
\[
\Psi(a, b) := \{ \psi : \text{supp}(\psi) = (a, b) \},
\]
\[
\Psi := \bigcup_{(a, b)} \Psi(a, b).
\]

Definition 4.1 (see e.g. [6, 7]) Let $\varphi(\cdot) \in \Psi(a, b)$. The Grand Lebesgue Space $G_{\psi}$ is defined as the set of all random variables (measurable functions) $\tau$ having finite norm:
\[
\| \tau \|_{G_{\psi}} \overset{def}{=} \sup_{p \in (a, b)} \left\{ \frac{\| \tau \|_{L_p(\Omega)}}{\psi(p)} \right\} = \sup_{p \in (a, b)} \left\{ \frac{\| \tau \|_p}{\psi(p)} \right\} .
\]

The function $\varphi(\cdot)$ is named generating function for the space $G_{\psi}$.

The particular case of these spaces, under some additional restrictions on the generating function $\psi = \psi(p)$,
corresponds to the so-called Yudovich spaces ([8, 9], see also [10, 11]). These spaces was applied at first in the theory of Partial Differential Equations (PDEs), see [10, 11].

The Grand Lebesgue spaces $G_\psi$ are rearrangement invariant Banach function spaces; they and their particular cases were investigated in many works, with applications in probability, interpolation theory, PDEs (see e.g. [6, 7, 12-24]).

It is important in particular to note that there is an exact interrelation, under certain natural conditions on the generating function, between the r.v. $\tau$ belonging to this space and its tail behavior, of course up to a finite multiplicative constant (see e.g. [20, p.5], [25, p.336]). Indeed, assume that $\tau \in G_\psi$ and moreover $\|\tau\|_{G_\psi} = 1$; then

$$T_\tau(t) = \mathbb{P}[\tau > t] \leq \exp\{-h^*(\ln t)\}, \quad t \geq e,$$

where

$$h(p) = h[\psi](p) := p \ln \psi(p)$$

and $h^*(\cdot)$ is the Young-Fenchel (Legendre) transform of the function $h(\cdot)$, defined by

$$h^*(u) = \sup_{p \in \text{Dom}(\psi)} (pu - h(p)).$$

Inversely, let the tail function $T_\tau(t)$, $t \geq 0$, be given. Introduce the following so-called natural function generated by $\tau$

$$\psi_\tau(p) \overset{\text{def}}{=} \left[p \int_0^{\infty} t^{p-1} T_\tau(t) \, dt\right]^{1/p} = \|\tau\|_{\text{Dom}(\psi)}^{1/p},$$

if it is finite for some value $b \in (a, \infty]$ and consequently, it is finite at least for all the values $p \in (a, b)$.

As long as

$$\mathbb{E} \left[\tau\right]^p = p \int_0^{\infty} t^{p-1} T_\tau(t) \, dt = \psi_\tau^p(p), \quad p \in [1, b),$$

we conclude that if the last natural function $\psi_\tau(p)$ for the r.v. $\tau$ is finite inside some non-trivial segment $p \in (1, b), 1 < b \leq \infty$, then

$$\tau \in G_\psi; \quad \|\tau\|_{G_\psi} = 1.$$

Furthermore, let the estimate (16) be given in the following version:

$$T_\tau(t) \leq \exp\{-h^*(\ln t / K)\}, \quad t \geq e, \quad K = K[\psi] = \text{const} > 0,$$

for some generating function $\psi(\cdot) \in \Psi(a, b)$. Assume in addition that $\psi = \psi(p), p \in \text{Dom}(\psi)$, is continuous and suppose, in the case $b = \infty$,

$$\lim_{p \to \infty} \frac{\psi(p)}{p} = 0.$$

Then the r.v. $\tau$ belongs to the Grand Lebesgue Space $G_\psi$ and
\[ \| \tau \|_{G_{\psi}} \leq K[\psi] < \infty, \]  

(see e.g. [21]).

These conditions on the generating function \( \psi(\cdot) \) are satisfied for example for the functions \( \psi_{m,L}(p) \) of the form

\[ \psi_{m,L}(p) = p^\alpha L(p), \quad m = \text{const} > 1, \quad b = \infty, \]  

where \( L = L(p) \) is some continuous strictly positive slowly varying at infinity function such that

\[ \sup_{p>1} \left[ \frac{L(p^\theta)}{L(p)} \right] = C(\theta) < \infty, \quad \forall \theta > 0. \]  

(21)

For instance, \( L(p) = [\ln(p + 1)]^r, \quad r \in \mathbb{R}. \)

We conclude that under the formulated restrictions the r.v. \( \tau \) belongs to the space \( G_{\psi_{m,L}}: \)

\[ \sup_{p>1} \left( \frac{\| \tau \|_p}{\psi_{m,L}(p)} \right) = C(m,L) < \infty \]  

(22)

if and only if

\[ \exists C_2(m,L) > 0 : T_{\gamma}(u) \leq \exp\left( -C_2(m,L) u^m / L(u) \right), \quad u \geq e. \]  

(23)

A very popular example of these spaces forms the so-called subgaussian space \( \text{Sub} = \text{Sub}(\Omega) \); it consists on the subgaussian random variables, for which \( \psi(p) = \psi_{\gamma}(p) = \sqrt{p}: \)

\[ \| \tau \|_{\text{Sub}} = \| \tau \|_{\text{Sub}_2} = \sup_{p>1} \left( \frac{\| \tau \|_p}{\sqrt{p}} \right). \]  

(24)

The r.v. \( \tau \) belongs to the subgaussian space \( \text{Sub}(\Omega) \) if

\[ \exists C > 0 : T_{\gamma}(u) \leq \exp(-Cu^2), \quad u \geq 0. \]  

(25)

Example 4.1 Introduce the following function \( \nu \in \Psi: \)

\[ \nu(\gamma)(p) = \nu(p) = \exp(0.5 p \gamma), \quad p \geq 1, \quad \gamma = \text{const} > 0. \]  

(26)

If the r.v. \( \zeta \) belongs to the space \( G\nu[\gamma] \) and has unit norm: \( \| \zeta \|_{G\nu[\gamma]} = 1, \) then

\[ T_{\zeta}(t) \leq \exp\left( -0.5 \gamma^{-1}(\ln^2 t) \right), \quad t \geq e. \]  

(27)

Conversely, let the estimation (27) holds true for some r.v. \( \zeta; \) then this r.v. \( \zeta \) belongs to the Grand Lebesgue Space \( G\nu: \| \zeta \|_{G\nu[\gamma]} \leq C(\gamma) < \infty. \)

Remark 4.1 As a rule, on the the r.v. \( \tau \) from the spaces \( G_{\psi_{m,L}} \) is imposed the condition of centering: \( \mu = 0. \)

Example 4.2 Suppose that the r.v. \( \tau \) be such that

\[ \| \tau \|_{G_{\psi_{m,L}}} \leq K[\psi] < \infty, \]  

(19)
$T_r(t) \leq T_{\beta,\gamma,L}(t)$, $\beta > 1$, $\gamma > -1$, $L = L(t)$,

where

$$T_{\beta,\gamma,L}(t) \overset{\text{def}}{=} r^\beta (\ln t)^\gamma L(\ln t), \ t \geq e$$

and $L = L(t)$, $t \geq e$ be, as before, slowly varying at infinity positive continuous function. It is known (see [21]) that as $p \in [1, \beta)$

$$\psi_r(p) = \|r\|_p \leq C_r(\beta, \gamma, L) (\beta - p)^{-(\gamma + 1)/\beta} L^{1/\beta} (1/ (\beta - p)),$$

and conversely, if the relation (28) holds, then

$$T_r(t) \leq C_r(\beta, \gamma, L) T_{\beta,\gamma,r,L}(t).$$

Herewith both this estimations are unimprovable.

Let us return to the problem exposed at the beginning of this Section. Indeed, we assume that the r.v. $X$ belongs to a certain Grand Lebesgue Space (GLS) $G_{\psi} = G_{\psi}(a, b)$:

$$\|X\|_{G_{\psi}} = \|X\|_{G_{\psi}(a, b)} < \infty; \ 1 \leq a < b \leq \infty.$$  

Of course, the generating function $\psi(\cdot)$ can be chosen as natural for the r.v. $X : \psi(p) := \|X\|_p$, if it is finite.

Let $\Delta = \text{const} \in [a, b]$; we introduce a new generating function

$$\psi_{\Delta, \beta}(p) = \psi_{\Delta, \beta} (\psi)(p) \overset{\text{def}}{=} \frac{p}{p - \Delta} \beta^p \psi(p), \ \Delta < p \leq b, \ \beta > 1.$$  

so that $\psi_{\Delta, \beta}(\cdot) \in \Psi(\Delta, b)$

The next Proposition holds also in the case of general generating function $\psi$, but we will state and prove it in the particular case of the generating functions considered above in (29).

**Proposition 4.1** Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbb{P})$ be a non-trivial probability space with expectation $E$ and let $\zeta, X$ be non-negative numerical valued random variables. Let $\psi$ and $\psi_{\Delta, \beta}$ the functions introduced above and let $X \in G_{\psi}(a, b)$. Under condition (9), we have

$$\|\zeta\|_{G_{\psi_{\Delta, \beta}}(a, b)} \leq C \|X\|_{G_{\psi}(a, b)},$$

with the corresponding tail estimation (16).

Herewith the estimate (30) is, in the general case, essentially non-improvable.

**Proof.** One can take, without loss of generality, $\|X\|_{G_{\psi}} = 1$; then

$$\forall p \in (\Delta, b) \Rightarrow \|X\|_p \leq \psi(p).$$

We apply the estimation (10) for these values $p$, hence

$$\|\zeta\|_p \leq C \frac{p}{p - \Delta} \beta^p \psi(p) = C \psi_{\Delta, \beta}(p),$$
or equally by means of the direct definition of the norm in the Grand Lebesgue Space $G_{\psi, \beta}$

$$\| \xi \|_{G_{\psi, \beta}} \leq C = C \| X \|_{G_{\psi}} .$$

The non-improvability of the last estimate may be grounded, as in Proposition 3.1, considering analogously as in the Example 4.1:

$$T_{\xi}(t) = \exp \left( -0.5 \gamma^{-1} (\ln^2 t) \right), \quad t \geq e, \quad (31)$$

where $\gamma = 21\beta$, $\beta = \text{const} > 1$ and $X = 1$.

In detail, it is easy to verify that the inequality (9) for these variables is satisfied for all the values $\beta > 1$. It remains to note as before that

$$\lim_{p \to \infty} \frac{\| \xi \|_{G_{\psi, \beta}}}{\| X \|_{G_{\psi}}} = 1.$$

See for details the relation (10).

5. Multivariate case

We extend the obtained results on the multidimensional case. We introduce the following notations and restrictions:

$$d = \dim \tilde{\tau} = \dim \tilde{\xi} = 2, 3, \ldots; \quad \beta, C, \Delta = \text{const} > 0; \quad \tilde{\tau} = \{ t_1, t_2, \ldots, t_d \}, \quad \tilde{\xi} = \{ \xi_1, \xi_2, \ldots, \xi_d \};$$

$$t_j \geq 0, \xi_j \geq 0, \quad \forall j = 1, 2, \ldots, d; \quad \tilde{\xi} > \tilde{\tau} \Leftrightarrow \xi_j > t_j, \forall j$$

$$|\tilde{\tau}| = \frac{d}{\sqrt{\sum_{j=1}^{d} t_j^2}}, \quad \| \tilde{\xi} \|_p = \left[ \sum_{j=1}^{d} \| \xi_j \|^p \right]^{1/p}, \quad p \geq 1.$$

**Proposition 5.1** Let $(\Omega = \{ \omega \}, B, P)$ be a non-trivial probability space with expectation $E$ and let $X$ be a non-negative numerical valued random variable. Let $\tilde{\tau}, \tilde{\xi}, \beta, C, \Delta$ be the quantities defined above. Assume that

$$|\tilde{\tau}|^p P(\tilde{\xi} > \beta \tilde{\tau}) \leq CE[X(\tilde{\xi} > \tilde{\tau})]. \quad (32)$$

We state that, for the values $p > \max(1, \Delta)$,

$$\| \tilde{\xi} \|_p \leq C \frac{p}{p - \Delta} \beta^p \| X \|_p. \quad (33)$$

**Proof.** We have, for $j = 1, 2, ..., d,$
\( t \mathbf{j} ^{\top} \mathbf{P}(\xi_j > \beta t_j) \leq C \mathbb{E}[\mathcal{X} (\xi_j > t_j)]. \)

It follows from the one-dimensional estimates

\[
\| \xi_j \|_p \leq C \frac{p}{p - \Delta} \beta^p \| X \|_p.
\]

Further,

\[
\| \xi \|_p = \left[ \sum_{j=1}^{d} \| \xi_j \|_p \right]^{1/p} \leq \left[ d \left( \frac{C p}{p - \Delta} \right)^p \beta^p \| X \|_p \right]^{1/p} = C d^{1/p} \frac{p}{p - \Delta} \beta^p \| X \|_p.
\]

\( \square \)

**Remark 5.1** One can replace the classical Euclidean norm \(| t |\) for the vector \( \vec{t} \) with another one, for instance the \( l_s \)-norm

\[
| \vec{t} |_s := \left( \sum_{j=1}^{d} | t_j |^s \right)^{1/s}, \quad s = \text{const} \geq 1.
\]

**6. Concluding remarks**

It is interesting, in our opinion, to apply the estimates obtained in the previous Sections, in particular, in the martingale theory. We will investigate this problem further, in future projects.

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**Conflicts of interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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