Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Generalization of Tail Inequalities for Random Variables in the Martingale Theory

Maria Rosaria Formica[1*], **Eugeny Ostrovsky**[2], **Leonid Sirota**[2]

[1]Parthenope University of Naples, via Generale Parisi 13, Naples, Italy
[2]Bar-Ilan University, Department of Mathematics and Statistics, Ramat Gan, Israel
E-mail: mara.formica@uniparthenope.it

Received: 30 May 2022; **Revised:** 27 September 2022; **Accepted:** 30 September 2022

Abstract: We generalize the tail Doob's inequality, concerning two non-negative random variables, arising in the martingale theory, in three directions: on the more general source data, on the random variables belonging to the so-called Grand Lebesgue Spaces, as well as on the multidimensional variables. We also provide several examples. Moreover we show the exactness of the estimates obtained in the particular case of positive random variables having exponential distribution.

***Keywords*:** martingale, random variable, expectation, tail of distribution, probability space, Doob's inequalities, Grand Lebesgue spaces, Young-Fenchel transform

MSC: 60B05, 46E30, 26D15, 60G46

1. Introduction

Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$ be a non-trivial probability space with expectation $\mathbf{E}$. The classical Doob's inequality ([1, 2], see also [3]) states that if the non-negative numerical valued random variables (r.v-s) ξ, X are such that, for some positive constants β, C,

$$t\mathbf{P}(\xi > \beta t) \le C\mathbf{E}[XI(\xi > t)],\ t \ge 0, \tag{1}$$

then

$$\|\xi\|_p = [\mathbf{E}\xi^p]^{1/p} \le C\,\frac{p}{p-1}\,\beta^p\,\|X\|_p,\ p > 1, \tag{2}$$

where henceforth $I(A)$ denotes the indicator function of the random event A, and $\|\eta\|_p$ denotes, as usually, the classical Lebesgue-Riesz norm of the r.v. η:

DOI: https://doi.org/10.37256/cm.3420221587

$$\| \eta \|_p := \left[\mathbf{E} | \eta |^p \right]^{1/p} = \left(\int_\Omega |\eta(\omega)|^p \ \mathbf{P}(d\omega) \right)^{1/p}, \ p \ge 1.$$

The inequality (2) plays a very important role, in particular, in the martingale theory (see [4, 5]).

Our aim is a generalization of the Doob's inequality in three directions: on the more general source data, on the random variables belonging to the Grand Lebesgue Spaces and on the multidimensional random vectors.

1.1 *Statement of the problem*

Let us assume the following generalization of inequality (1)

$$h(t)\mathbf{P}(\xi > \beta t) \le C\mathbf{E}[XI(\xi > t)], \ t \ge 0,$$

where $h = h(t)$ is a non-negative continuous strictly increasing deterministic function and, as before, X and ξ are non-negative random variables.

We establish a generalization of the estimate (2). Moreover we provide some examples in order to show the exactness of the obtained estimates, in the particular case of positive random variables having exponential distribution.

2. Main result

Theorem 2.1 Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$ be a non-trivial probability space with expectation $\mathbf{E}$ and let ξ, X be non-negative numerical valued random variables. Let $h = h(t)$ be a non-negative continuous strictly increasing deterministic function. Let us assume that, for some positive constants β, C, the following inequality is satisfied:

$$h(t)\mathbf{P}(\xi > \beta t) \le C\mathbf{E}[XI(\xi > t)], \ t \ge 0. \tag{3}$$

Let $p > 1$ and define the random variable (measurable function):

$$\kappa_p(\xi) \stackrel{def}{=} \int_0^\xi \frac{t^{p-1}}{h(t)} dt,$$

if there exists. Assume that there exists $\theta > 1$ such that $K_p(\theta) := \|\kappa_p(\xi)\|_\theta < \infty$.

Suppose that $\exists \, r = r(\theta, p) \in [1, p)$, $\exists \, v = v(\theta, p, r) < \infty$ such that

$$K_p(\theta) = \| \kappa_p(\xi) \|_\theta \le v(\theta, p, r) \| \xi \|_p^r . \tag{4}$$

Then

$$\| \xi \|_p \le \left[Cpv(\theta, p, r)\beta^p \right]^{1/(p-r)} \cdot \| X \|_{\theta'}^{1/(p-r)}, \ \theta' = \frac{\theta}{\theta - 1} \in (1, \infty). \tag{5}$$

Of course,

$$\| \xi \|_p \le \inf_{r \in R} \inf_{\theta \in \Theta} \left\{ \left[Cpv(\theta, p, r)\beta^p \right]^{1/(p-r)} \cdot \| X \|_{\theta'}^{1/(p-r)} \right\}, \tag{6}$$

where

$$R = R(\theta, p) := \{r \in [1, p) : K_p(\theta) < \infty\},$$

$$\Theta = \Theta(p)[h, \xi] := \{\theta > 1 : K_p(\theta) < \infty\}.$$

Proof. By (3) we have

$$\mathbf{P}(\xi / \beta > t) \leq \frac{C}{h(t)} \mathbf{E}[XI(\xi > t)],\ t \geq 0,$$

hence

$$\int_0^\infty pt^{p-1}\, \mathbf{P}(\xi / \beta > t)\, dt \leq \int_0^\infty \frac{C\,p\,t^{p-1}}{h(t)} \mathbf{E}[XI(\xi > t)]\, dt,\ p > 1. \tag{7}$$

The left-hand side of (7) is equal to $\mathbf{E}\,|\,\xi / \beta\,|^p = \|\,\xi\,\|_p^p\, \beta^{-p}$. Let us investigate the right-hand side of (7). We deduce, by virtue of Fubini's theorem,

$$\int_0^\infty \frac{Cpt^{p-1}}{h(t)} \mathbf{E}[XI(\xi > t)]\, dt = Cp\mathbf{E}\left[X \int_0^\xi \frac{t^{p-1}}{h(t)}\, dt\right].$$

Then inequality (7) yields

$$\|\,\xi\,\|_p^p\, \beta^{-p} \leq Cp\mathbf{E}\,[X\kappa_p(\xi)].$$

By Hölder's inequality with conjugate exponents θ and θ' we get

$$\|\,\xi\,\|_p^p \leq Cp\beta^p\, \|\,\kappa_p(\xi)\,\|_\theta \|\,X\,\|_{\theta'},$$

and by assumption (4) we deduce

$$\|\,\xi\,\|_p^p \leq Cp\beta^p v(\theta, p, r)\, \|\,\xi\,\|_p^r \|\,X\,\|_{\theta'},$$

which implies (5). □

Example 2.1 If $h(t) = t$, inequality (3) reduces to (1):

$$t\mathbf{P}(\xi > \beta t) \leq C\mathbf{E}(XI(\xi > t)),\ t, \xi, X \geq 0,$$

and (5) reduces to (2):

$$\|\,\xi\,\|_p \leq C \frac{p}{p-1} \beta^p\, \|\,X\,\|_p,\ p > 1, \tag{8}$$

choosing $v(\theta, p, r) = \dfrac{1}{p-1}$, $r = p - 1$ and $\theta = \dfrac{p}{p-1}$, that is $\theta' = p$.

Example 2.2 A more general case. Suppose

$$t^{\Delta}\mathbf{P}(\xi > \beta t) \le C\mathbf{E}(XI(\xi > t)),\ t, \xi, X \ge 0,\ \Delta = \text{const} > 1. \tag{9}$$

We get, after simple calculations, the following estimate

$$\|\xi\|_p \le C\frac{p}{p-\Delta}\beta^p \|X\|_p,\ p > \Delta. \tag{10}$$

Remark 2.1 Let $s \in [1, p]$. By virtue of Lyapunov's inequality $\|\xi\|_s \le \|\xi\|_p$. Therefore, if (9) holds, then by (10) we have also

$$\|\xi\|_s \le C\frac{p}{p-\Delta}\beta^p \|X\|_p,\ p > \Delta,\ \forall s \in [1, p].$$

3. Unimprovability of the estimations. Lower bounds

Let us show the exactness of the obtained results (8) and (10) in a particular case. Introduce the following functionals, which are involved in the lower estimate.

$$Z[C, \beta, \Delta](\xi, X, p) \overset{def}{=} \left[\frac{\|\xi\|_p}{Cp(p-\Delta)^{-1}\beta^p \|X\|_p}\right],$$

$$K[C, \beta, \Delta] = K[C, \beta, \Delta](p) \overset{def}{=} \sup_{p>\Delta}\sup_{\xi \in L_p}\sup_{X \in L_p} Z[C, \beta, \Delta](\xi, X, p), \tag{11}$$

where all the supremums are calculated over the r.v. - s ξ, X satisfying the condition (9) and when $p > \Delta$, $\Delta = \text{const} \ge 1$.

Proposition 3.1 Let ξ_0, X_0 be positive random variables such that $X_0 = \xi_0$ and having standard exponential distribution

$$\mathbf{P}(\xi_0 > t) = \mathbf{P}(X_0 > t) = e^{-t},\ t > 0.$$

Then

$$K[C, \beta, \Delta] = 1. \tag{12}$$

Proof. The upper estimate $K[C, \beta, \Delta] \le 1$ is contained in (10). We deduce the lower one in the particular case $C = \beta = 1$.

So we have to prove

$$K[1, 1, \Delta] = 1. \tag{13}$$

The natural generating function for these r.v.- s has the form

$$\nu(p) \overset{def}{=} \|\xi_0\|_p = \|X_0\|_p = \Gamma^{1/p}(p+1),\ p > 1,$$

where, as usually, $\Gamma(\cdot)$ is the Euler's Gamma function, since

$$\| \xi_0 \|_p = \left(p \int_0^\infty t^{p-1} \mathbf{P}(\xi_0 \ge t) dt \right)^{1/p} = \left(p \int_0^\infty t^{p-1} e^{-t}\, dt \right)^{1/p} = (p\Gamma(p))^{1/p} = \Gamma^{1/p}(p+1).$$

Note on the way that as $p \to \infty \Rightarrow v(p) \sim p / e$. Note also that the condition (9) is satisfied. We have

$$Z[C, \beta, \Delta](\xi_0, X_0, p) = \frac{p - \Delta}{p},\ p > \Delta.$$

Following,

$$K[1,1,\Delta] \ge \sup_{p > \Delta} \left\{ \frac{p - \Delta}{p} \right\}. \tag{14}$$

So (13) follows immediately from the fact that

$$\lim_{p \to \infty} \left\{ \frac{p - \Delta}{p} \right\} = 1. \qquad \square$$

Remark 3.1 The cases $C, \beta \neq 1$ may be considered quite analogously.

4. Generalization on the Grand Lebesgue Spaces approach. Examples

We intend in this section to extend the previous results upon the so-called Grand Lebesgue Spaces (GLS) of random variables.

Let $1 \le a < b \le \infty$. Let $\psi = \psi(p)$, $p \in (a, b)$ be a numerical valued measurable strictly positive function, such that $\inf_{p \in (a,b)} \psi(p) > 0$, not necessarily bounded.

We use the following notations:

$$\mathrm{Dom}(\psi) := \{\, p : \psi(p) < \infty \,\},$$

$$\Psi(a,b) := \{\, \psi : \mathrm{supp}(\psi) = (a,b) \,\},$$

$$\Psi := \bigcup_{(a,b)} \Psi(a,b).$$

Definition 4.1 (see e.g. [6, 7]) Let $\psi(\cdot) \in \Psi(a, b)$. The Grand Lebesgue Space $G\psi$ is defined as the set of all random variables (measurable functions) τ having finite norm:

$$\| \tau \|_{G\psi} \overset{def}{=} \sup_{p \in (a,b)} \left\{ \frac{\| \tau \|_{L_p(\Omega)}}{\psi(p)} \right\} = \sup_{p \in (a,b)} \left\{ \frac{\| \tau \|_p}{\psi(p)} \right\}. \tag{15}$$

The function $\psi(\cdot)$ is named generating function for the space $G\psi$.

The particular case of these spaces, under some additional restrictions on the generating function $\psi = \psi(p)$, corresponds to the so-called *Yudovich spaces* ([8, 9], see also [10, 11]). These spaces was applied at first in the theory of Partial Differential Equations (PDEs), see [10, 11].

The Grand Lebesgue spaces $G\psi$ are rearrangement invariant Banach function spaces; they and their particular cases were investigated in many works, with applications in probability, interpolation theory, PDEs (see e.g. [6, 7, 12-24]).

It is important in particular to note that there is an exact interrelation, under certain natural conditions on the generating function, between the r.v. τ belonging to this space and its tail behavior, of course up to a finite multiplicative constant (see e.g. [20, p.5], [25, p.336]). Indeed, assume that $\tau \in G\psi$ and moreover $\|\tau\|_{G\psi} = 1$; then

$$T_\tau(t) = \mathbf{P}(|\tau| > t) \le \exp\{-h^*(\ln t)\},\ t \ge e, \tag{16}$$

where

$$h(p) = h[\psi](p) := p \ln \psi(p)$$

and $h^*(\cdot)$ is the Young-Fenchel (Legendre) transform of the function $h(\cdot)$, defined by

$$h^*(u) \stackrel{def}{=} \sup_{p \in \mathrm{Dom}(\psi)} (pu - h(p)).$$

Inversely, let the tail function $T_\tau(t)$, $t \ge 0$, be given. Introduce the following so-called *natural function* generated by τ

$$\psi_\tau(p) \stackrel{def}{=} \left[p \int_0^\infty t^{p-1}\, T_\tau(t)\, dt \right]^{1/p} = \|\tau\|_{L_p(\Omega)}, \tag{17}$$

if it is finite for some value $b \in (a, \infty]$ and consequently, it is finite at least for all the values $p \in (a, b)$.

As long as

$$\mathbf{E}|\tau|^p = p\int_0^\infty t^{p-1}\, T_\tau(t)\, dt = \psi_\tau^p(p),\ p \in [1, b),$$

we conclude that if the last *natural function* $\psi_\tau(p)$ for the r.v. τ is finite inside some non-trivial segment $p \in (1, b)$, $1 < b \le \infty$, then

$$\tau \in G\psi_\tau;\ \ \|\tau\|_{G\psi_\tau} = 1.$$

Furthermore, let the estimate (16) be given in the following version:

$$T_\tau(t) \le \exp\{-h^*(\ln t / K)\},\ t \ge e,\ K = K[\psi] = \mathrm{const} > 0,$$

for some generating function $\psi(\cdot) \in \Psi(a, b)$. Assume in addition that $\psi = \psi(p)$, $p \in \mathrm{Dom}(\psi)$, is continuous and suppose, in the case $b = \infty$,

$$\lim_{p \to \infty} \frac{\psi(p)}{p} = 0. \tag{18}$$

Then the r.v. τ belongs to the Grand Lebesgue Space $G\psi$ and

$$\| \tau \|_{G\psi} \le K[\psi] < \infty, \tag{19}$$

(see e.g. [21]).

These conditions on the generating function $\psi(\cdot)$ are satisfied for example for the functions $\psi_{m,L}(p)$ of the form

$$\psi_{m,L}(p) \stackrel{def}{=} p^{1/m} L(p),\ m = \text{const} > 1,\ b = \infty, \tag{20}$$

where $L = L(p)$ is some continuous strictly positive *slowly varying* at infinity function such that

$$\sup_{p \ge 1} \left[\frac{L(p^{\theta})}{L(p)} \right] = C(\theta) < \infty,\ \forall \theta > 0. \tag{21}$$

For instance, $L(p) = [\ln(p+1)]^r$, $r \in \mathbb{R}$.

We conclude that under the formulated restrictions the r.v. τ belongs to the space $G\psi_{m,L}$:

$$\sup_{p \ge 1} \left\{ \frac{\| \tau \|_p}{\psi_{m,L}(p)} \right\} = C(m, L) < \infty \tag{22}$$

if and only if

$$\exists\, C_2(m,L) > 0\ :\ T_{\tau}(u) \le \exp\Big(-C_2(m,L)\, u^m / L(u) \Big),\ u \ge e. \tag{23}$$

A very popular example of these spaces forms the so-called subgaussian space Sub = Sub(Ω); it consists on the subgaussian random variables, for which $\psi(p) = \psi_2(p) := \sqrt{p}$:

$$\| \tau \|_{\text{Sub}} = \| \tau \|_{G\psi_2} \stackrel{def}{=} \sup_{p \ge 1} \left[\frac{\| \tau \|_p}{\sqrt{p}} \right]. \tag{24}$$

The r.v. τ belongs to the subgaussian space Sub(Ω) if

$$\exists C > 0\ :\ T_{\tau}(u) \le \exp(-Cu^2),\ u \ge 0. \tag{25}$$

Example 4.1 Introduce the following function $\nu \in \Psi$:

$$\nu[\gamma](p) = \nu(p) := \exp(0.5\gamma\ p),\ p \ge 1,\ \gamma = \text{const} > 0. \tag{26}$$

If the r.v. ζ belongs to the space $G\nu[\gamma]$ and has unit norm: $\|\zeta\|_{G\nu[\gamma]} = 1$, then

$$T_{\zeta}(t) \le \exp\Big(-0.5\, \gamma^{-1} (\ln^2 t) \Big),\ t \ge e. \tag{27}$$

Conversely, let the estimation (27) holds true for some r.v. ζ; then this r.v. ζ belongs to the Grand Lebesgue Space $G\nu : \|\zeta\|_{G\nu[\gamma]} \le C_1(\gamma) < \infty$.

Remark 4.1 As a rule, on the the r.v. τ from the spaces $G\psi_{m,L}$ is imposed the condition of centering: $\mathbf{E}\tau = 0$.

Example 4.2 Suppose that the r.v. τ be such that

$$T_\tau(t) \le T^{\beta,\gamma,L}(t),\ \beta > 1,\ \gamma > -1,\ L = L(t),$$

where

$$T^{\beta,\gamma,L}(t) \stackrel{def}{=} t^{-\beta}(\ln t)^\gamma L(\ln t),\ t \ge e$$

and $L = L(t)$, $t \ge e$ be, as before, slowly varying at infinity positive continuous function. It is known (see [21]) that as $p \in [1, \beta)$

$$\psi_\tau(p) = \| \tau \|_p \le C_1(\beta,\gamma,L)\,(\beta - p)^{-(\gamma+1)/\beta} L^{1/\beta}(1/(\beta - p)), \tag{28}$$

and conversely, if the relation (28) holds, then

$$T_\tau(t) \le C_7(\beta,\gamma,L)\, T^{\beta,\gamma+1,L}(t).$$

Herewith both this estimations are unimprovable.

Let us return to the problem exposed at the beginning of this Section. Indeed, we assume that the r.v. X belongs to a certain Grand Lebesgue Space (GLS) $G\psi = G\psi(a, b)$:

$$\| X \|_{G\psi} = \| X \|_{G\psi(a,b)} < \infty;\ 1 \le a < b \le \infty.$$

Of course, the generating function $\psi(\cdot)$ can be chosen as natural for the r.v. $X : \psi(p) := \|X\|_p$, if it is finite.

Let $\Delta = \text{const} \in [a, b]$; we introduce a new generating function

$$\psi_{\Delta,\beta}(p) = \psi_{\Delta,\beta}[\psi](p) \stackrel{def}{=} \frac{p}{p - \Delta}\beta^p \psi(p),\ \Delta < p \le b,\ \beta > 1. \tag{29}$$

so that $\psi_{\Delta,\beta}(\cdot) \in \Psi(\Delta, b)$

The next Proposition holds also in the case of general generating function ψ, but we will state and prove it in the particular case of the generating functions considered above in (29).

Proposition 4.1 Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$ be a non-trivial probability space with expectation $\mathbf{E}$ and let ξ, X be non-negative numerical valued random variables. Let ψ and $\psi_{\Delta,\beta}$ the functions introduced above and let $X \in G\psi(a, b)$. Under condition (9), we have

$$\| \xi \|_{G\psi_{\Delta,\beta}(\Delta,b)} \le C \| X \|_{G\psi(a,b)}, \tag{30}$$

with the corresponding tail estimation (16).

Herewith the estimate (30) is, in the general case, essentially non-improvable.

Proof. One can take, without loss of generality, $\|X\|_{G\psi} = 1$; then

$$\forall p \in (\Delta, b) \Rightarrow \| X \|_p \le \psi(p).$$

We apply the estimation (10) for these values p, hence

$$\| \xi \|_p \le C \frac{p}{p - \Delta}\beta^p \psi(p) = C\psi_{\Delta,\beta}(p),$$

or equally by means of the direct definition of the norm in the Grand Lebesgue Space $G\psi_{\Delta,\beta}$

$$\| \xi \|_{G\psi_{\Delta,\beta}} \le C = C \| X \|_{G\psi} .$$

The non-improvability of the last estimate may be ground, as in Proposition 3.1, considering analogously as in the Example 4.1:

$$T_{\xi}(t) = \exp\left(-0.5\gamma^{-1}(\ln^2 t)\right), \ t \ge e, \tag{31}$$

where $\gamma = 2\ln\beta$, $\beta = \text{const} > 1$ and $X = 1$.

In detail, it is easy to verify that the inequality (9) for these variables is satisfied for all the values $\beta > 1$. It remains to note as before that

$$\lim_{p\to\infty} \frac{\| \xi \|_{G\psi_{\Delta,\beta}}}{\| X \|_{G\psi}} = 1.$$

See for details the relation (10). □

5. Multivariate case

We extend the obtained results on the multidimensional case. We introduce the following notations and restrictions:

$$d = \dim \vec{t} = \dim \vec{\xi} = 2, 3, \ldots; \ \beta, C, \Delta = \text{const} > 0;$$

$$\vec{t} = \{ t_1, t_2, \ldots, t_d \}, \ \vec{\xi} = \{ \xi_1, \xi_2, \ldots, \xi_d \};$$

$$t_j \ge 0, \xi_j \ge 0, \ \forall j = 1, 2, \ldots, d; \ \vec{\xi} > \vec{t} \Leftrightarrow \xi_j > t_j, \forall j$$

$$| \vec{t} | \overset{def}{=} \sqrt{\sum_{j=1}^{d} t_j^2} , \ \| \vec{\xi} \|_p \overset{def}{=} \left[\sum_{j=1}^{d} \| \xi_j \|_p^p \right]^{1/p}, \ p \ge 1.$$

Proposition 5.1 Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$ be a non-trivial probability space with expectation $\mathbf{E}$ and let X be a non-negative numerical valued random variable. Let $\vec{t}$, $\vec{\xi}$, β, C, Δ be the quantities defined above. Assume that

$$| \vec{t} |^{\Delta} \mathbf{P}(\vec{\xi} > \beta\vec{t}) \le C\mathbf{E}[XI(\vec{\xi} > \vec{t})]. \tag{32}$$

We state that, for the values $p > \max(1, \Delta)$,

$$\| \vec{\xi} \|_p \le C \frac{p}{p-\Delta} d^{1/p} \beta^p \| X \|_p . \tag{33}$$

Proof. We have, for $j = 1, 2, \ldots, d$,

$$t_j^{\Delta}\mathbf{P}(\xi_j > \beta\, t_j) \le C\mathbf{E}[XI(\xi_j > t_j)].$$

It follows from the one-dimensional estimates

$$\|\xi_j\|_p \le C\frac{p}{p-\Delta}\beta^p \|X\|_p .$$

Further,

$$\|\vec{\xi}\|_p = \left[\sum_{j=1}^{d}\|\xi_j\|_p^p\right]^{1/p} \le \left[d\left(\frac{Cp}{p-\Delta}\right)^p \beta^{p^2}\|X\|_p^p\right]^{1/p} = Cd^{1/p}\frac{p}{p-\Delta}\beta^p\|X\|_p . \qquad \square$$

Remark 5.1 One can replace the classical Euclidean norm $|\vec{t}\,|$ for the vector $\vec{t}$ with another one, for instance the l_s-norm

$$|\vec{t}\,|_s := \left(\sum_{j=1}^{d}|t_j|^s\right)^{1/s}, \quad s = \text{const} \ge 1.$$

6. Concluding remarks

It is interesting, in our opinion, to apply the estimates obtained in the previous Sections, in particular, in the martingale theory. We will investigate this problem further, in future projects.

Acknowledgements

M.R. Formica is partially supported by University of Naples "Parthenope", Dept. of Economic and Legal Studies, project CoRNDiS, DM MUR 737/2021, CUP I55F21003620001. M.R. Formica is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilit`a e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and member of the UMI group "Teoria dell'Approssimazione e Applicazioni (T.A.A.)".

Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Doob JL. *Stochastic Processes*. New York: Wiley; 1953.
[2] Doob JL. Semimartingales and subharmonic functions. *Transactions of the American Mathematical Society*. 1954; 77: 86-121.
[3] Pollard D. *The Burkholder Inequality*. Available from: http://www.stat.yale.edu/~ypng/yale-notes/Burkholder.pdf [Accessed 15 June 2007].
[4] Burkholder DL. Distribution functions inequalities for martingales. *Annals of Probability*. 1973; 1: 19-42.
[5] Burkholder DL, Davis BJ. Integral inequalities for convex functions of operators on martingales. *Berkeley Symposium on Mathematical Statistics and Probability*. 1972; 2: 223-240.

[6] Kozachenko YV, Ostrovsky EI. Banach Spaces of random variables of sub-Gaussian type. *Teor Veroyatnost i Mat Statist* (Theory of Probability and Mathematical Statistics). 1985; 32(134): 42-53.
[7] Ermakov SV, Ostrovsky EI. Central limit theorem for weakly dependent Banach-space valued random variables. *Theory of Probability and Its Applications*. 1986; 30(2): 391-394.
[8] Yudovich VI. Nonstationary flow of an ideal incompressible liquid. *Computational Mathematics and Mathematical Physics*. 1963; 3: 1407-1456.
[9] Yudovich VI. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Mathematical Research Letters*. 1995; 2: 27-38.
[10] Chen Q, Miao C. The two-dimensional Euler equation in Yudovich and bmo-type spaces. *Revista Matematica Iberoamericana*. 2019; 35(1): 195-240.
[11] Crippa G, Stefani G. An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces. Available from: doi: 10.48550/arXiv.2110.15648.
[12] Ermakov SV, Ostrovsky EI. *Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields*. Moscow, VINITY (in Russian); 1986.
[13] Fiorenza A, Karadzhov GE. Grand and small Lebesgue spaces and their analogs. *Journal of Analysis and its Applications*. 2004; 23(4): 657-681.
[14] Capone C, Formica MR, Giova R. Grand Lebesgue spaces with respect to measurable functions. *Nonlinear Analysis*. 2013; 85: 125-131. Available from: doi: 10.1016/j.na.2013.02.021.
[15] Fiorenza A, Formica MR, Gogatishvili A. On grand and small Lebesgue and Sobolev spaces and some applications to PDE's. *Journal of Difference Equations and Applications*. 2018; 10(1): 21-46.
[16] Fiorenza A, Formica MR, Gogatishvili A, Kopaliani T, Rakotoson JM. Characterization of interpolation between grand, small or classical Lebesgue spaces. *Nonlinear Analysis*. 2018; 177(part B): 422-453. Available from: doi: 10.1016/j.na.2017.09.005.
[17] Fiorenza A, Formica MR, Rakotoson JM. Pointwise estimates for GΓ-functions and applications. *Differential Integral Equations*. 2017; 30: 809-824.
[18] Formica MR, Giova R. Boyd indices in generalized grand Lebesgue spaces and applications. *Mediterranean Journal of Mathematics*. 2015; 12(3): 987-995.
[19] Formica MR, Ostrovsky E, Sirota L. Grand quasi Lebesgue spaces. *Journal of Mathematical Analysis and Applications*. 2021; 504(1): 125369.
[20] Kozachenko YV, Ostrovsky EI, Sirota L. Relations between exponential tails, moments and moment generating functions for random variables and vectors. Available from: doi: https://doi.org/10.48550/arXiv.1701.01901.
[21] Kozachenko YV, Ostrovsky EI, Sirota L. Equivalence between tails, Grand Lebesgue Spaces and Orlicz norms for random variables without Kramer's condition. *Bulletin of the Taras Shevchenko National University of Kyiv. Public Administration*. 2018; 4: 20-29. Available from: doi: 10.17721/1812-5409.2018/4.3.
[22] Liflyand E, Ostrovsky EI, Sirota L. Structural properties of bilateral grand lebesgue spaces. *Turkish Journal of Mathematics*. 2010; 34: 207-219.
[23] Ostrovsky EI. *Exponential Estimations for Random Fields*. Moscow-Obninsk, OINPE; 1999.
[24] Fiorenza A, Formica MR. On the factor opposing the lebesgue norm in generalized grand lebesgue spaces. *Results in Mathematics*. 2021; 76(2). Available from: doi: 10.1007/s00025-021-01375-9.
[25] Formica MR, Kozachenko YV, Ostrovsky E, Sirota L. Exponential tail estimates in the law of ordinary logarithm (LOL) for triangular arrays of random variables. *Lithuanian Mathematical Journal*. 2020; 60(3): 330-358.