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Research Article

# On the Classical Solutions for the Kuramoto-Sivashinsky Equation with Ehrilch-Schwoebel Effects

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**Abstract:** The Kuramoto-Sivashinsky equation with Ehrilch-Schwoebel effects models the evolution of surface morphology during Molecular Beam Epitaxy growth, provoked by an interplay between deposition of atoms onto the surface and the relaxation of the surface profile through surface diffusion. It consists of a nonlinear fourth order partial differential equation. Using energy methods we prove the well-posedness (i.e., existence, uniqueness and stability with respect to the initial data) of the classical solutions for the Cauchy problem, associated with this equation.

***Keywords***: existence, uniqueness, stability, the Kuramoto-Sivashinsky equation with Ehrilch-Schwoebel effects, Cauchy problem

**MSC:** 35G25, 35K55

## 1. Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) + \beta^2 \partial_x^4 u + \nu \partial_x^2 u \\ \qquad + \kappa \partial_x^3 u + a \partial_x^2 \left( \dfrac{u}{(1+u^2)^\alpha} \right) \\ \qquad + b \partial_x \left( \dfrac{\partial_x u}{(1+(\partial_x u)^2)^\gamma} \right) \\ \qquad + d \partial_x \left( \dfrac{\partial_x u}{1+|\partial_x u|^{1+\tau}} \right) = 0, \qquad 0 \le t \le T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad\qquad x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\beta$, $\nu$, $\kappa$, $a$, $b$, $d$ are real constants such that


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$$\beta \neq 0, \quad \alpha, \gamma > 0, \quad \tau \geq 0. \tag{2}$$

On the flux $f$, we assume

$$f \in C^1(\mathbb{R}), \ |f'(u)| \leq C_0\left(1+|u|^4\right), \tag{3}$$

for some positive constant $C_0$.

On the initial datum, we assume

$$u_0 \in H^{\ell}(\mathbb{R}), \ \ell \in \{1, 2\}. \tag{4}$$

From a physical point of view, (1) models the evolution of surface morphology during Molecular Beam Epitaxy (MBE) growth, provoked by an interplay between deposition of atoms onto the surface and the relaxation of the surface profile through surface diffusion (see [1, 2]).

One of the most influential factors for instabilities in the evolution of the surface morphology of a growing film is the existence of energy barriers near steps. An adsorbed atom (adatom) approaching a step from above or below may have different probabilities. This phenomena was first observed experimentally by Ehrlich and Hudda [1] and analyzed by Schwoebel and Shipsey [2]. For that reason, it is known as the Ehrilch-Schwoebel (ES) barrier (see also [3, 4]). The ES barrier induces pyramidal or mound-type structures on the growing surface (see also [5-12]). In particular, in [5], the authors study the following equation:

$$\partial_t u + b\partial_x\left(\frac{\partial_x u}{(1+(\partial_x u)^2)^{\gamma}}\right) + \beta^2\partial_x^4 u = 0, \quad b > 0, \ \gamma > 1. \tag{5}$$

They obtained analytical results on the coarsening process by inspecting the behavior of branch of the steady state periodic solutions.

Moreover, in [6], the author studies the case $\gamma = 1$. In the spirit of the Bales and Zangwill theory [13], he studied the dynamics of a MBE growing interfaces in the absence of slope selection mechanism, when the mounds slope indefinitely increases with time (see also [14]).

Assuming that the effects of adatom desorption [15], and diffusion anisotropy are neglected [16], the continuous interfacial height in one-dimensional case, is found to obey the following general phenomenological evolution equation [10]:

$$\partial_t u + \beta^2\partial_x^4 u + d\partial_x\left(\frac{\partial_x u}{1+|\partial_x u|^{1+\tau}}\right) = 0, \tag{6}$$

while its standard form is [17]:

$$\partial_t u + \beta^2\partial_x^4 u + b\partial_x\left(\frac{\partial_x u}{(1+(\partial_x u)^2)^{\gamma}}\right) = 0. \tag{7}$$

In both of equations, $d$, $b > 0$ and $\tau$, $\gamma \geq 1$. The unknown smooth function $u$ measures the film thickness above a substrate point $x$ and at time $t$. Equations (6) and (7) have a conservative form:

$$\partial_t u + \partial_x J = 0, \tag{8}$$

where,

$$J(\partial_x u) = \beta^2 \partial_x^3 u + \frac{d\partial_x u}{1+|\partial_x u|^{1+\tau}}, \quad \text{or,} \quad J(\partial_x u) = \beta^2 \partial_x^3 u + \frac{b\partial_x u}{(1+(\partial_x u)^2)^\gamma}, \tag{9}$$

respectively. The second term of the current $J$ is the destabilizing surface current, the Ehrilch-Schwoebel (ES) effect, which generalized the form discussed in [3, 4], and is characterized by different asymptotic behaviors as $|\partial_x u| \to \infty$ [6, 18-21]. The destabilizing term, which provides the nonlinear regime, is balanced by the classical stabilizing linear term [22]. The growth laws are investigated with phase diffusion approach that allows to determine the coarsening exponent for 2D growth. In [23], a classification of important unstable crystal growth dynamics in terms of universality classes are proposed and distinct properties and coarsening exponents are shown. For Equation (7) with $\gamma = 1$, the solutions are investigated both analytically and numerically. Global solutions were constructed to the parabolic evolution equation in [24]. In particular, the authors studied the initial-boundary value problem. Numerical techniques like finite difference, finite element and kinetic Monte Carlo method were used to provide approximate solutions to the equation describing crystal surface growth [25-28]. The pyramidal structure characterized by the absence of preferred slope in one-dimension was examined in [29], applying a similarity approach. Observe that similarity technique is not applicable to the nonlinear term $\frac{\partial_x u}{1+(\partial_x u)^2}$, which is present in Equation (7). In [5], the authors study the coarsening process that may result from Equation (7), with $n > 0$, $n \neq 1$, called $n-$ model. In particular, the authors give an analytical justification of solutions to $n-$ model, which correspond to, or predict, the coarsening process. They are mainly concerned with stationary solutions, which are successfully used to describe the major features of the process of coarsening in a wide class of surface growth phenomena [30, 31]. The case $n = 1$ is analyzed in [3].

Taking $f(u) = u^2$, and $a = b = d = 0$, Equation (1) reads:

$$\partial_t u + \partial_x u^2 + \beta^2 \partial_x^4 u + \nu \partial_x^2 u + \kappa \partial_x^3 u = 0. \tag{10}$$

(10) arises in interesting physical situations, for example as a model for long waves on a viscous fluid owing down an inclined plane [32] and to derive drift waves in a plasma [33]. Equation (10) was derived also independently by Kuramoto independently [34-36] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [37] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (10) also describes incipient instabilities in a variety of physical and chemical systems [38-40]. Moreover, (10), which is also known as the Benney-Lin equation [41, 42], was derived by Kuramoto in the study of phase turbulence in the Belousov-Zhabotinsky reaction [43].

The dynamical properties and the existence of exact solutions for (10) have been investigated in [44-49]. In [50-52], the control problem for (10) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [53], the problem of global exponential stabilization of (10) with periodic boundary conditions is analyzed. In [54], it is proposed a generalization of optimal control theory for (10), while in [55] the problem of global boundary control of (10) is considered. In [56], the existence of solitonic solutions for (10) is proven. In [57-60], the well-posedness of the Cauchy problem for (10) is proven, using the energy space technique, the fixed point method, a priori estimates together with an application of the Cauchy-Kovalevskaya and a priori estimates together with an application of the Aubin-Lions Lemma, respectively (see also [61]). Instead, in [63-64], the initial-boundary value problem for (10) is studied, using a priori estimates together with an application of the Cauchy-Kovalevskaya, and the energy space technique, respectively.

Here we study (1) in its quite general form, assuming only (2). We prove the well-posedness of the Cauchy problem in the $H^1$ and $H^2$ spaces. Our argument are based on energy techniques.

The main results of this paper are the following theorems.

**Theorem 1** Fix $T > 0$. Assuming (2), (3) and (4) with $\ell = 2$, there exists a unique solution $u$ of (1), such that

$$u \in H^1\left((0,T)\times\mathbb{R}\right) \cap L^\infty\left(0,T; H^2(\mathbb{R})\right) \cap L^4\left(0,T; W^{2,4}(\mathbb{R})\right) \cap L^6\left(0,T; W^{2,6}(\mathbb{R})\right), \ \partial_x^4 u \in L^2\left((0,T)\times\mathbb{R}\right). \tag{11}$$

Moreover, if $u_1$ and $u_2$ are two solutions of (1) in correspondence of the initial data $u_{1,0}$ and $u_{2,0}$, we have that

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\|u_{1,0}-u_{2,0}\|_{L^2(\mathbb{R})}, \tag{12}$$

for some suitable $C(T) > 0$, and every, $0 \le t \le T$.

**Theorem 2** Assuming (2), (3) and (4) with $\ell = 1$, there exists a solution $u$ of (1), such that

$$u \in L^\infty\left(0,T;H^1(\mathbb{R})\right)\cap L^4\left(0,T;W^{1,4}(\mathbb{R})\right)\cap L^6\left(0,T;W^{1,6}(\mathbb{R})\right),\ \partial_x^3 u \in L^2\left((0,T)\times\mathbb{R}\right). \tag{13}$$

In particular, under assumption

$$b = d = 0, \tag{14}$$

$u$ is unique and (12) holds.

Theorem 1 gives the well-posedness of the classical solutions for (1) under Assumption (4) in $H^2$, without additional assumption on the constants. Theorem 2 gives the well-posedness in $H^1$.

The paper is organized as follows. In Section 2 we prove Theorem 1, and in Section 3 we prove Theorem 2.

## 2. Proof of Theorem 1

In this section, we prove Theorem 1, under assumption

$$u_0 \in H^2(\mathbb{R}). \tag{15}$$

We prove some a priori estimates on $u$, denoting with $C_0$ the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on $T$.

We begin by proving the following lemma.

**Lemma 1** We have that

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\|\partial_x^2 u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C(T), \tag{16}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C(T), \tag{17}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. Multiplying (1) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} u\partial_t u dx\\ &= \underbrace{-2\int_{\mathbb{R}} uf'(u)\partial_x u dx}_{=0} - 2\beta^2\int_{\mathbb{R}} u\partial_x^4 u dx - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx\\ &\quad -2\kappa\int_{\mathbb{R}} u\partial_x^3 dx - 2a\int_{\mathbb{R}} u\partial_x^2\left(\frac{u}{(1+u^2)^\alpha}\right)dx\end{aligned}$$

$$-2b\int_{\mathbb{R}} u\partial_x\left(\frac{\partial_x u}{(1+(\partial_x u)^2)^\gamma}\right)dx - 2d\int_{\mathbb{R}} u\partial_x\left(\frac{\partial_x u}{1+|\partial_x u|^{1+\tau}}\right)$$

$$= 2\beta^2\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx + 2\kappa\int_{\mathbb{R}}\partial_x u\partial_x^2 u dx$$

$$+2a\int_{\mathbb{R}}\partial_x u\partial_x\left(\frac{u}{(1+u^2)^\alpha}\right)dx + 2b\int_{\mathbb{R}}\frac{(\partial_x u)^2}{(1+(\partial_x u)^2)^\gamma}dx$$

$$+2d\int_{\mathbb{R}}\frac{(\partial_x u)^2}{1+|\partial_x u|^{1+\tau}}dx$$

$$= -2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx - 2a\int_{\mathbb{R}}\frac{u}{(1+u^2)^\alpha}\partial_x^2 u dx$$

$$+2b\int_{\mathbb{R}}\frac{(\partial_x u)^2}{(1+(\partial_x u)^2)^\gamma}dx + 2d\int_{\mathbb{R}}\frac{(\partial_x u)^2}{1+|\partial_x u|^{1+\tau}}dx.$$

Therefore, we have that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$= -2\nu\int_{\mathbb{R}} u\partial_x^2 u dx - 2a\int_{\mathbb{R}}\frac{u}{(1+u^2)^\alpha}\partial_x^2 u dx$$

$$+2b\int_{\mathbb{R}}\frac{(\partial_x u)^2}{(1+(\partial_x u)^2)^\gamma}dx + 2d\int_{\mathbb{R}}\frac{(\partial_x u)^2}{1+|\partial_x u|^{1+\tau}}dx. \tag{18}$$

Thanks to (2) and the Young inequality,

$$2|\nu|\int_{\mathbb{R}}|u||\partial_x^2 u|dx = \int_{\mathbb{R}}\left|\frac{2\nu u}{\beta}\right||\beta\partial_x^2 u|dx \le \frac{2\nu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|a|\int_{\mathbb{R}}\left|\frac{u}{(1+u^2)^\alpha}\right||\partial_x^2 u|dx = \int_{\mathbb{R}}\left|\frac{2au}{\beta(1+u^2)^\alpha}\right||\beta\partial_x^2 u|dx$$

$$\le \frac{2a^2}{\beta^2}\int_{\mathbb{R}}\frac{u^2}{(1+u^2)^{2\alpha}}dx + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le \frac{2a^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (2) and (18) that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2|b|\int_{\mathbb{R}}\frac{(\partial_x u)^2}{(1+(\partial_x u)^2)^{\gamma}}dx+2|d|\int_{\mathbb{R}}\frac{(\partial_x u)^2}{1+|\partial_x u|^{1+\tau}}dx$$

$$\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2(|b|+|d|)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{19}$$

Observe that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}\partial_x u\partial_x u dx=-\int_{\mathbb{R}}u\partial_x^2 u dx.$$

Therefore, by the Hölder inequality,

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\int_{\mathbb{R}}|u||\partial_x^2 u|dx \le \|u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{20}$$

Consequently, by (19) and (20),

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2(|b|+|d|)\|u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{21}$$

Thanks to the Young (21),

$$2(|b|+|d|)\|u(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}=\frac{2|b|+|d|}{|\beta|}\|u(t,\cdot)\|_{L^2(\mathbb{R})}|\beta|\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}$$

$$\le\frac{2(|b|+|d|)^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (21) that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

By the Gronwall Lemma and (15), we have that

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0 e^{C_0 t}\le C(T),$$

which gives (16).

Finally, we prove (17). We begin by observing that, by (16) and (20),

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Due to the Young inequality,

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T) + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on (0, $t$) and (16) give (17). □

Following [65, Lemma 2.3], or [66, Lemma 2.2], we can prove the following result.

**Lemma 2** We have that

$$\int_0^t \|u(s,\cdot)\|^2_{L^\infty(\mathbb{R})} ds \le C(T), \tag{22}$$

$$\int_0^t \|u(s,\cdot)\|^8_{L^\infty(\mathbb{R})} ds \le C(T), \tag{23}$$

$$\int_0^t \|u(s,\cdot)\|^6_{L^\infty(\mathbb{R})} ds \le C(T), \tag{24}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^\infty(\mathbb{R})} ds \le C(T), \tag{25}$$

$$\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{26}$$

$$\int_0^t \left\|u^2(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{27}$$

for every $0 \le t \le T$.

**Lemma 3** We have that

$$\|\partial_x u\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T), \tag{28}$$

$$\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{29}$$

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{30}$$

$$\int_0^t \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} ds \le C(T), \tag{31}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^6_{L^6(\mathbb{R})} ds \le C(T), \tag{32}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. We begin by observing that

$$\partial_x\left(\frac{u}{(1+u^2)^\alpha}\right) = \frac{\partial_x u}{(1+u^2)^\alpha} - \frac{2\alpha(1+u^2)^{\alpha-1}u^2}{(1+u^2)^{2\alpha}}\partial_x u$$

$$= \frac{\partial_x u}{(1+u^2)^\alpha} - \frac{2\alpha u^2}{(1+u^2)^{\alpha+1}}\partial_x u. \tag{33}$$

Multiplying (1) by $-2\partial_x^2 u$, thanks to (33) an integration on $(0, t)$ gives

$$\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} = -2\int_{\mathbb{R}}\partial_x^2 u\partial_t u dx$$

$$= 2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}}\partial_x^2 u\partial_x^4 u dx + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+2\kappa\int_{\mathbb{R}}\partial_x^2 u\partial_x^3 u dx + 2a\int_{\mathbb{R}}\partial_x^2 u\partial_x^2\left(\frac{u}{(1+u^2)^\alpha}\right)dx$$

$$+2b\int_{\mathbb{R}}\partial_x^2 u\partial_x\left(\frac{\partial_x u}{(1+(\partial_x u)^2)^\gamma}\right) + 2d\int_{\mathbb{R}}\partial_x^2 u\partial_x\left(\frac{\partial_x u}{1+|\partial_x u|^{1+\tau}}\right)$$

$$= 2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$-2a\int_{\mathbb{R}}\partial_x^3 u\partial_x\left(\frac{u}{(1+u^2)^\alpha}\right)dx - 2b\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{(1+(\partial_x u)^2)^\gamma}dx$$

$$-2d\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{1+|\partial_x u|^{1+\tau}}dx$$

$$= 2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$-2a\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{(1+u^2)^\alpha}dx + 4a\alpha\int_{\mathbb{R}}\frac{u^2\partial_x u\partial_x^3 u}{(1+u^2)^{\alpha+1}}dx$$

$$-2b\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{(1+(\partial_x u)^2)^\gamma}dx - 2d\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{1+|\partial_x u|^{1+\tau}}dx.$$

Therefore, we have that

$$\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$=2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^2 u dx+2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2a\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{(1+u^2)^{\alpha}}dx$$

$$+4a\alpha\int_{\mathbb{R}}\frac{u^2\partial_x u\partial_x^3 u}{(1+u^2)^{\alpha+1}}dx-2b\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{(1+(\partial_x u)^2)^{\gamma}}dx$$

$$-2d\int_{\mathbb{R}}\frac{\partial_x u\partial_x^3 u}{1+|\partial_x u|^{1+\tau}}dx. \tag{34}$$

Thanks to (2), (3) and the Young inequality,

$$2\int_{\mathbb{R}}|f'(u)||\partial_x u||\partial_x^2 u|dx$$

$$\leq 2C_0\int_{\mathbb{R}}|\partial_x u||\partial_x^2 u|dx+2C_0\int_{\mathbb{R}}|u||\partial_x u||\partial_x^2 u|dx+2C_0\int_{\mathbb{R}}u^2|\partial_x u||\partial_x^2 u|dx$$

$$+2C_0\int_{\mathbb{R}}|u|^3|\partial_x u||\partial_x^2 u|dx+2C_0\int_{\mathbb{R}}u^4|\partial_x u||\partial_x^2 u|dx$$

$$\leq\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|u^2(t,\cdot)u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\int_{\mathbb{R}}\left|\frac{u^3\partial_x u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}C_0\partial_x^2 u\right|dx+2\int_{\mathbb{R}}\left|\frac{u^4\partial_x u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_x^2 u\right|dx$$

$$\leq\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{1}{D_1}\int_{\mathbb{R}}u^6(\partial_x u)^2dx+\frac{1}{D_1}\int_{\mathbb{R}}u^8(\partial_x u)^2dx+C_0(1+2D_1)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{1}{D_1}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\left(\|u(t,\cdot)\|^6_{L^\infty(\mathbb{R})}+\|u(t,\cdot)\|^8_{L^\infty(\mathbb{R})}\right)$$

$$+C_0(1+2D_1)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\left\|u(t,\cdot)\right\|^6_{L^\infty(\mathbb{R})} + \left\|u(t,\cdot)\right\|^8_{L^\infty(\mathbb{R})}\right)$$

$$+C_0(1+2D_1)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|a|\int_{\mathbb{R}}\frac{|\partial_x u||\partial_x^3 u|}{(1+u^2)^\alpha}dx \leq 2|a|\int_{\mathbb{R}}|\partial_x u|\partial_x^3 u|dx = 2\int_{\mathbb{R}}\left|\frac{a\partial_x u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^3 u\right|dx$$

$$\leq \frac{a^2}{\beta^2 D_2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$4|a\alpha|\int_{\mathbb{R}}\frac{|u^2\partial_x u||\partial_x^3 u|}{(1+u^2)^{\alpha+1}}dx \leq 4|a\alpha|\int_{\mathbb{R}}|u^2\partial_x u||\partial_x^3 u|dx = 2\int_{\mathbb{R}}\left|\frac{a\alpha u^2\partial_x u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^3 u\right|dx$$

$$\leq \frac{4a^2\alpha^2}{\beta^2 D_2}\left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|b|\int_{\mathbb{R}}\frac{|\partial_x u||\partial_x^3 u|}{(1+(\partial_x u)^2)^\gamma}dx \leq 2|b|\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u|dx = 2\int_{\mathbb{R}}\left|\frac{b\partial_x u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^3 u\right|dx$$

$$\leq \frac{b^2}{\beta^2 D_2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|d|\int_{\mathbb{R}}\frac{|\partial_x u||\partial_x^3 u|}{1+|\partial_x u|^{1+\tau}}dx \leq 2|d|\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u|dx = 2\int_{\mathbb{R}}\left|\frac{d\partial_x u}{\beta\sqrt{D_2}}\right|\left|\beta\sqrt{D_2}\partial_x^3 u\right|dx$$

$$\leq \frac{2d^2}{D_2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

where $D_1$, $D_2$ are two positive constants, which will be specified later. It follows from (34) that

$$\frac{d}{dt}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2(1-2D_2)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq\left(1+\frac{a^2}{\beta^2 D_2}+\frac{4a^2\alpha^2}{\beta^2 D_2}+\frac{b^2}{\beta^2 D_2}+\frac{2d^2}{D_2}\right)\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\left[2|\nu|+C_0\left(1+2D_1\right)\right]\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left(1+\frac{4a^2\alpha^2}{\beta^2 D_2}\right)\left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\left\|u(t,\cdot)\right\|^6_{L^\infty(\mathbb{R})}+\left\|u(t,\cdot)\right\|^8_{L^\infty(\mathbb{R})}\right).$$

Choosing $D_2=\frac{1}{4}$, we have that

$$\frac{d}{dt}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq C_0\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C_0\left(1+2D_1\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+C_0\left\|u^2(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\left\|u(t,\cdot)\right\|^6_{L^\infty(\mathbb{R})}+\left\|u(t,\cdot)\right\|^8_{L^\infty(\mathbb{R})}\right).$$

Integrating on (0, $t$), thanks to (15), (16), (17), (23), (24), (26) and (27), we have that

$$\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$\leq C_0+C_0\int_0^t\left\|\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}+C_0\left(1+2D_1\right)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+C_0\int_0^t\left\|u^2(s,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+\int_0^t\left\|u(s,\cdot)\partial_x u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\int_0^t\left\|u(s,\cdot)\right\|^6_{L^\infty(\mathbb{R})}+\int_0^t\left\|u(s,\cdot)\right\|^8_{L^\infty(\mathbb{R})}ds\right)$$

$$\leq C(T)\left(1+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+D_1\right). \tag{35}$$

Consequently, by (35),

$$\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C(T)\left(1+\frac{1}{D_1}\left\|\partial_x u\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+D_1\right),$$

which gives,

$$\left(1-\frac{C(T)}{D_1}\right)\|\partial_x u\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T)\left(1+D_1\right).$$

Choosing

$$D_1 = \frac{1}{2C(T)}, \tag{36}$$

we obtain that

$$\frac{1}{2}\|\partial_x u\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T),$$

which gives (28).

(29) follows from (28), (35) and (36), while (16) gives (30).

We prove (31). [61, Lemma 2.3] says that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 9\int_{\mathbb{R}} u^2(\partial_x^2 u)^2\,dx.$$

Therefore, by (30), we have that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 9\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on (0, $t$) and (16) give (31).

Finally, we prove (32). We begin by observing that

$$\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} = \int_{\mathbb{R}} \partial_x u(\partial_x u)^5\,dx = -5\int_{\mathbb{R}} u(\partial_x u)^4\partial_x^2 u dx. \tag{37}$$

Thanks to (30) and the Young inequality,

$$\begin{aligned}
5\int_{\mathbb{R}} |u|(\partial_x u)^4|\partial_x^2 u|\,dx &= \int_{\mathbb{R}} |\partial_x u|^3 |5u\partial_x u\partial_x^2 u|\,dx \\
&\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + \frac{25}{2}\int_{\mathbb{R}} u^2(\partial_x u)^2\partial_x^2 u^2\,dx \\
&\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + \frac{25}{2}\|u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^2 u)^2\,dx \\
&\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + C(T)\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^2 u)^2\,dx.
\end{aligned}$$

Consequently, by (37),

$$\frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} \leq C(T)\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 \, dx. \tag{38}$$

Observe that

$$\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 \, dx = \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x^2 u dx = -\int_{\mathbb{R}} \partial_x u \partial_x \left( (\partial_x u)^2 \partial_x^2 u \right) dx$$

$$= -2\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 \, dx - \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^3 u dx.$$

Therefore,

$$\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^2 u)^2 \, dx = -\frac{1}{3}\int_{\mathbb{R}} (\partial_x u)^3 \partial_x^3 u dx. \tag{39}$$

It follows from (38) and (39) that

$$\frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} \leq -C(T)\int_{\mathbb{R}} (\partial_x u)^3 \partial_x^3 u dx \leq C(T)\int_{\mathbb{R}} |\, \partial_x u \,|^3 \, |\, \partial_x^3 u \,| \, dx. \tag{40}$$

Due to the Young inequality,

$$C(T)\int_{\mathbb{R}} |\, \partial_x u \,|^3 \, |\, \partial_x^3 u \,| \, dx = 2\int_{\mathbb{R}} \left| \frac{(\partial_x u)^3}{\sqrt{3}} \right| \left| \frac{C(T)\sqrt{3}\partial_x^3 u}{2} \right| dx$$

$$\leq \frac{1}{3}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently by (40),

$$\frac{1}{6}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} \leq C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on (0, $t$) and (29) give (32). □

**Lemma 4** We have that

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{41}$$

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$

$$+ 2\,|\, d(\tau+1) \,|\, \int_0^t \int_{\mathbb{R}} \frac{|\, \partial_x^2 u \,||\, \partial_x^4 u \,|}{(1+|\, \partial_x u \,|^{1+\tau})^2} ds dx \leq C(T), \tag{42}$$

$$\int_0^t \left\| \partial_x^2 u(t,\cdot) \right\|_{L^4(\mathbb{R})}^4 ds \le C(T), \tag{43}$$

$$\int_0^t \left\| \partial_x^3 u(t,\cdot) \right\|_{L^\infty(\mathbb{R})}^2 ds \le C(T), \tag{44}$$

$$\int_0^t \left\| \partial_x^2 u(t,\cdot) \right\|_{L^6(\mathbb{R})}^6 ds \le C(T), \tag{45}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. We begin by observe that, by (33), we have that

$$\begin{aligned}
\partial_x^2 \left( \frac{u}{(1+u^2)^\alpha} \right) &= \frac{\partial_x^2 u}{(1+u^2)^\alpha} - \frac{\alpha(1+u^2)^{\alpha-1} 2u}{(1+u^2)^{2\alpha}} (\partial_x u)^2 \\
&\quad - \frac{4\alpha u(1+u^2)^\alpha (1-\alpha u^2)}{(1+u^2)^{2\alpha+2}} (\partial_x u)^2 - \frac{2\alpha u^2}{(1+u^2)^{\alpha+1}} \partial_x^2 u \\
&= \frac{\partial_x^2 u}{(1+u^2)^\alpha} - \frac{2\alpha u}{(1+u^2)^{\alpha+1}} (\partial_x u)^2 - \frac{4\alpha u(1-\alpha u^2)}{(1+u^2)^{\alpha+2}} (\partial_x u)^2 - \frac{2\alpha u^2}{(1+u^2)^{\alpha+1}} \partial_x^2 u \\
&= \frac{\partial_x^2 u}{(1+u^2)^\alpha} - \frac{2\alpha u}{(1+u^2)^{\alpha+1}} (\partial_x u)^2 - \frac{4\alpha u}{(1+u^2)^{\alpha+2}} (\partial_x u)^2 \\
&\quad + \frac{4\alpha^2 u^3}{(1+u^2)^{\alpha+2}} (\partial_x u)^2 - \frac{2\alpha u^2}{(1+u^2)^{\alpha+1}} \partial_x^2 u.
\end{aligned} \tag{46}$$

Moreover, we have that

$$\begin{aligned}
\partial_x \left( \frac{\partial_x u}{(1+(\partial_x u)^2)^\gamma} \right) &= \frac{\partial_x^2 u}{(1+(\partial_x u)^2)^\gamma} - \frac{2\gamma((1+(\partial_x u)^2)^{\gamma-1}(\partial_x u)^2}{(1+(\partial_x u)^2)^{2\gamma}} \partial_x^2 u \\
&= \frac{\partial_x^2 u}{(1+(\partial_x u)^2)^\gamma} - \frac{2\gamma(\partial_x u)^2}{(1+(\partial_x u)^2)^{\gamma+1}} \partial_x^2 u,
\end{aligned}$$

$$\partial_x \left( \frac{\partial_x u}{1+|\partial_x u|^{1+\tau}} \right) = \frac{\partial_x^2 u}{1+|\partial_x u|^{1+\tau}} - \frac{(\tau+1)\partial_x u\, |\partial_x u|^\tau \operatorname{sign} \partial_x u}{(1+|\partial_x u|^{1+\tau})^2} \partial_x^2 u. \tag{47}$$

Multiplying (1) by $2\partial_x^4 u$, thanks to (46) and (47), an integration on $\mathbb{R}$ gives

$$\frac{d}{dt} \left\| \partial_x^2 u(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 = 2\int_{\mathbb{R}} \partial_x^4 u \partial_t u dx$$

$$= -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^4 u dx - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 - 2\nu\int_{\mathbb{R}} \partial_x^2 u\partial_x^4 u dx$$

$$-2\kappa\int_{\mathbb{R}} \partial_x^3 u\partial_x^4 u dx - 2a\int_{\mathbb{R}} \partial_x^4 u\partial_x^2 \left(\frac{u}{(1+u^2)^{\alpha}}\right)dx$$

$$-2b\int_{\mathbb{R}} \partial_x^4 u\partial_x \left(\frac{\partial_x u}{(1+(\partial_x u)^2)^{\gamma}}\right)dx$$

$$-2d\int_{\mathbb{R}} \partial_x^4 u\partial_x \left(\frac{\partial_x u}{1+|\partial_x u|^{1+\tau}}\right)dx$$

$$= -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^4 u dx - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + 2\nu \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$-2a\int_{\mathbb{R}} \frac{\partial_x^2 u\partial_x^4 u}{(1+u^2)^{\alpha}}dx + 4a\alpha\int_{\mathbb{R}} \frac{u(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+1}}dx$$

$$+8a\alpha\int_{\mathbb{R}} \frac{u(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+2}}dx - 8a\alpha^2\int_{\mathbb{R}} \frac{u^3(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+2}}$$

$$+2a\alpha\int_{\mathbb{R}} \frac{u^2\partial_x^2 u\partial_x^4 u}{(1+u^2)^{\alpha+1}}dx - 2b\int_{\mathbb{R}} \frac{\partial_x^2 u\partial_x^4 u}{(1+(\partial_x u)^2)^{\gamma}}dx$$

$$+4b\gamma\int_{\mathbb{R}} \frac{(\partial_x u)^2\partial_x^2 u\partial_x^4 u}{(1+(\partial_x u)^2)^{\gamma+1}}dx - 2d\int_{\mathbb{R}} \frac{\partial_x^2 u\partial_x^4 u}{1+|\partial_x u|^{1+\tau}}dx$$

$$+2d(\tau+1)\int_{\mathbb{R}} \frac{\partial_x u\partial_x^2 u\partial_x^4 u\,|\partial_x u|^{\tau}\,\mathrm{sign}(\partial_x u)}{(1+|\partial_x u|^{1+\tau})^2}dx.$$

Consequently, we have that

$$\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$= -2\int_{\mathbb{R}} f'(u)\partial_x u\partial_x^4 u dx + 2\nu \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 - 2a\int_{\mathbb{R}} \frac{\partial_x^2 u\partial_x^4 u}{(1+u^2)^{\alpha}}dx$$

$$+4a\alpha\int_{\mathbb{R}} \frac{u(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+1}}dx + 8a\alpha\int_{\mathbb{R}} \frac{u(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+2}}dx - 8a\alpha^2\int_{\mathbb{R}} \frac{u^3(\partial_x u)^2\partial_x^4 u}{(1+u^2)^{\alpha+2}}$$

$$+2a\alpha\int_{\mathbb{R}}\frac{u^2\partial_x^2u\partial_x^4u}{(1+u^2)^{\alpha+1}}dx-2b\int_{\mathbb{R}}\frac{\partial_x^2u\partial_x^4u}{(1+(\partial_xu)^2)^{\gamma}}dx$$

$$+4b\gamma\int_{\mathbb{R}}\frac{(\partial_xu)^2\partial_x^2u\partial_x^4u}{(1+(\partial_xu)^2)^{\gamma+1}}dx-2d\int_{\mathbb{R}}\frac{\partial_x^2u\partial_x^4u}{1+|\partial_xu|^{1+\tau}}dx$$

$$+2d(\tau+1)\int_{\mathbb{R}}\frac{\partial_xu\partial_x^2u\partial_x^4u\,|\partial_xu|^{\tau}\ \mathrm{sign}(\partial_xu)}{(1+|\partial_xu|^{1+\tau})^2}dx. \tag{48}$$

Due to (2), (29), (30) and the Young inequality,

$$2\int_{\mathbb{R}}|f'(u)||\partial_xu||\partial_x^4u|\,dx\le 2\|f'\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_xu||\partial_x^4u|\,dx$$

$$\le 2C(T)\int_{\mathbb{R}}|\partial_xu||\partial_x^4u|\,dx=2\int_{\mathbb{R}}\left|\frac{C(T)\partial_xu}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4u\right|dx$$

$$\le\frac{C(T)}{D_3}\|\partial_xu(t,\cdot)\|^2_{L^2(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le\frac{C(T)}{D_3}+D_3\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \tag{49}$$

$$2|a|\int_{\mathbb{R}}\frac{|\partial_x^2u||\partial_x^4u|}{(1+u^2)^{\alpha}}dx\le 2|a|\int_{\mathbb{R}}|\partial_x^2u||\partial_x^4u|\,dx=2\int_{\mathbb{R}}\left|\frac{a\partial_x^2u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4u\right|dx$$

$$\le\frac{a^2}{D_3\beta^2}\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$4|a\alpha|\int_{\mathbb{R}}\frac{|u|(\partial_xu)^2|\partial_x^4u|}{(1+u^2)^{\alpha+1}}dx\le 4|a\alpha|\int_{\mathbb{R}}|u|(\partial_xu)^2|\partial_x^4u|\,dx$$

$$\le 4|a\alpha|\|u\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}(\partial_xu)^2|\partial_x^4u|\,dx\le 2C(T)\int_{\mathbb{R}}(\partial_xu)^2|\partial_x^4u|\,dx$$

$$\le 2\int_{\mathbb{R}}\left|\frac{C(T)(\partial_xu)^2}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4u\right|dx$$

$$\le\frac{C(T)}{D_3}\|\partial_xu(t,\cdot)\|^4_{L^4(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$
\begin{aligned}
8|a\alpha|\int_{\mathbb{R}}\frac{|u|(\partial_x u)^2|\partial_x^4 u|}{(1+u^2)^{\alpha+2}}dx &\leq 8|a\alpha|\int_{\mathbb{R}}|u|(\partial_x u)^2|\partial_x^4 u|dx\\
&\leq 8|a\alpha|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|dx \leq 2C(T)\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|dx\\
&= 2\int_{\mathbb{R}}\left|\frac{C(T)(\partial_x u)^2}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|dx\\
&\leq \frac{C(T)}{D_3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
8|a\alpha^2|\int_{\mathbb{R}}\frac{|u|^3(\partial_x u)^2|\partial_x^4 u|}{(1+u^2)^{\alpha+2}} &\leq 8|a\alpha^2|\int_{\mathbb{R}}|u|^3(\partial_x u)^2|\partial_x^4 u|dx\\
&\leq 8|a\alpha^2|\|u\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|dx \leq 2C(T)\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^4 u|dx\\
&= 2\int_{\mathbb{R}}\left|\frac{C(T)(\partial_x u)^2}{\beta\sqrt{D_3}}\right|\left|\sqrt{D_3}\beta\partial_x^4 u\right|dx\\
&\leq \frac{C(T)}{D_3}\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|a\alpha|\int_{\mathbb{R}}\frac{u^2|\partial_x^2 u||\partial_x^4 u|}{(1+u^2)^{\alpha+1}}dx &\leq 2|a\alpha|\int_{\mathbb{R}}u^2|\partial_x^2 u||\partial_x^4 u|dx\\
&\leq 2|a\alpha|\|u\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\partial_x^2 u||\partial_x^4 u|dx \leq 2C(T)\int_{\mathbb{R}}\partial_x^2 u||\partial_x^4 u|dx\\
&= 2\int_{\mathbb{R}}\left|\frac{C(T)\partial_x^2 u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|dx\\
&\leq \frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\beta\left\|{}^2\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|b|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{(1+(\partial_x u)^2)^{\gamma}}dx &\leq 2|b|\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u|dx = 2\int_{\mathbb{R}}\left|\frac{b\partial_x^2 u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|dx\\
&\leq \frac{b^2}{D_3\beta^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$4|b\gamma|\int_{\mathbb{R}}\frac{(\partial_x u)^2|\partial_x^2 u||\partial_x^4 u|}{(1+(\partial_x u)^2)^{\gamma+1}}dx \le 4|b\gamma|\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^2 u||\partial_x^4 u|dx$$

$$=2\int_{\mathbb{R}}\left|\frac{2b\gamma(\partial_x u)^2|\partial_x^2 u|}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|dx$$

$$\le \frac{4b^2\gamma^2}{\beta^2 D_3}\int_{\mathbb{R}}(\partial_x u)^4(\partial_x^2 u)^2dx + D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|d|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{1+|\partial_x u|^{1+\tau}}dx \le 2|d|\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u|dx = 2\int_{\mathbb{R}}\left|\frac{d\partial_x^2 u}{\beta\sqrt{D_3}}\right|\left|\beta\sqrt{D_3}\partial_x^4 u\right|dx$$

$$\le \frac{d^2}{D_3\beta^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_3\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

where $D_3$ is a positive constant, which will specified later. Observe that, thanks to (2),

$$2d(\tau+1)\int_{\mathbb{R}}\frac{\partial_x u\partial_x^2 u\partial_x^4 u|\partial_x u|^{\tau}\ \mathrm{sign}(\partial_x u)}{(1+|\partial_x u|^{1+\tau})^2}dx \le 2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u||\partial_x u|^{\tau+1}}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$=2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u||\partial_x u|^{\tau+1}+|\partial_x^2 u||\partial_x^4 u|-|\partial_x^2 u||\partial_x^4 u|}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$=2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|\left(1+|\partial_x u|^{\tau+1}\right)}{(1+|\partial_x u|^{1+\tau})^2}dx - 2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$=2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{1+|\partial_x u|^{1+\tau}}dx - 2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$\le 2|d(\tau+1)|\int_{\mathbb{R}}|\partial_x^2 u||\partial_x^4 u|dx - 2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{(1+|\partial_x u|^{1+\tau})^2}dx. \tag{50}$$

It follows from (48), (49) and (50) that

$$\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left(2-9D_3\right)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_x^4 u|}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$\le \frac{C(T)}{D_3} + 2|\nu|\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C(T)}{D_3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{C(T)}{D_3}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})}$$

$$+\frac{4b^2\gamma^2}{\beta^2 D_3}\int_{\mathbb{R}}(\partial_x u)^4(\partial_x^2 u)^2\,dx+2\,|\,d(\tau+1)\,|\int_{\mathbb{R}}|\,\partial_x^2 u\,||\,\partial_x^4 u\,|\,dx.$$

Choosing $D_3=\frac{1}{9}$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+2\,|\,d(\tau+1)\,|\int_{\mathbb{R}}\frac{|\,\partial_x^2 u\,||\,\partial_x^4 u\,|}{(1+|\,\partial_x u\,|^{1+\tau})^2}\,dx\\
&\leq C(T)+2\,|\,\nu\,|\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+C(T)\left\|\partial_x u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+C_0\int_{\mathbb{R}}(\partial_x u)^4(\partial_x^2 u)^2\,dx\\
&\quad+2\,|\,d(\tau+1)\,|\int_{\mathbb{R}}|\,\partial_x^2 u\,||\,\partial_x^4 u\,|\,dx.
\end{aligned}\tag{51}$$

Due to the Young inequality,

$$\begin{aligned}
2\,|\,d(\tau+1)\,|\int_{\mathbb{R}}|\,\partial_x^2 u\,||\,\partial_x^4 u\,|\,dx&=\int_{\mathbb{R}}\left|\frac{2d(\tau+1)\partial_x^2 u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx\\
&\leq 2d^2(\tau+1)^2\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Consequently, by (51),

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+2\,|\,d(\tau+1)\,|\int_{\mathbb{R}}\frac{|\,\partial_x^2 u\,||\,\partial_x^4 u\,|}{(1+|\,\partial_x u\,|^{1+\tau})^2}\,dx\\
&\leq C(T)+2\,|\,\nu\,|\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+C(T)\left\|\partial_x u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+C_0\int_{\mathbb{R}}(\partial_x u)^4(\partial_x^2 u)^2\,dx.
\end{aligned}\tag{52}$$

Observe that

$$\left\|\partial_x^2 u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4=\int_{\mathbb{R}}(\partial_x^2 u)^3\partial_x^2 u dx=-3\int_{\mathbb{R}}\partial_x u(\partial_x^2 u)^2\partial_x^3 u dx.\tag{53}$$

Thanks to the Young inequality,

$$3\int_{\mathbb{R}}|\,\partial_x u(\partial_x^2 u)^2\,|\,\partial_x^3 u dx=\frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+\frac{9}{2}\int_{\mathbb{R}}(\partial_x u)^2(\partial_x^3 u)^2\,dx.$$

Therefore, by (53),

$$\left\|\partial_x^2 u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 \le 9\int_{\mathbb{R}} (\partial_x u)^2 (\partial_x^3 u)^2\, dx \le 9\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2. \tag{54}$$

Due to (54) and the Young inequality,

$$\begin{aligned} C_0\int_{\mathbb{R}} (\partial_x u)^4 (\partial_x^2 u)^2\, dx &\le C_0\int_{\mathbb{R}} (\partial_x u)^8 dx + C_0\int_{\mathbb{R}} (\partial_x^2 u)^4 dx \\ &\le C_0\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\|\partial_x u(t,\cdot)\right\|_{L^6(\mathbb{R})}^6 + C_0\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows from (52) that

$$\begin{aligned} &\frac{d}{dt}\partial_x^2 u(t,\cdot)_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + 2\,|\,d(\tau+1)\,|\int_{\mathbb{R}} \frac{|\,\partial_x^2 u\,||\,\partial_x^4 u\,|}{(1+|\,\partial_x u\,|^{1+\tau})^2} dx \\ &\le C(T) + C_0\left(1+\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + C(T)\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ &+C(T)\left\|\partial_x u(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 + C_0\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\|\partial_x u(t,\cdot)\right\|_{L^6(\mathbb{R})}^6. \end{aligned}$$

Integrating on (0, *t*), thanks to (15), (16), (29), (31) and (32), we have that

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2}\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds \\ &+2\,|\,d(\tau+1)\,|\int_0^t\int_{\mathbb{R}} \frac{|\,\partial_x^2 u\,||\,\partial_x^4 u\,|}{(1+|\,\partial_x u\,|^{1+\tau})^2}\, dsdx \\ &\le C_0 + C(T)t + C_0\left(1+\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right)\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds \\ &+C(T)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|_{L^2(\mathbb{R})}^2 ds + C(T)\int_0^t\left\|\partial_x u(s,\cdot)\right\|_{L^4(\mathbb{R})}^4 ds \\ &+ C_0\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \int_0^t\left\|\partial_x u(s,\cdot)\right\|_{L^6(\mathbb{R})}^6 ds \\ &\le C(T)\left(1+\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\right). \end{aligned} \tag{55}$$

We prove (41). Thanks to (29) and (53).

$$\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \le C(T)\sqrt{1+\left\|\partial_x u\right\|_{L^\infty((0,T)\times\mathbb{R})}^2}.$$

Hence,

$$\|\partial_x u\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which give (41).

(42) follows from (41) and (55).

We prove (43). Thanks to (41) and (54),

$$\left\|\partial_x^2 u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on $(0, t)$ and (29) give (43).

We prove (44). Thanks to the Hölder inequality,

$$(\partial_x^3 u(t,x))^2 = 2\int_{-\infty}^{x} \partial_x^3 u \partial_x^4 u dy \le 2\int_{\mathbb{R}} |\partial_x^3 u||\partial_x^4 u| dx \le 2\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \le 2\left\|\partial_x^3 u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Thanks to the Young inequality,

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \le \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0, t)$, by (29) and (42), we have that

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^\infty(\mathbb{R})} ds \le \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),$$

which gives (44).

Finally, we prove (45). We begin by observing that

$$\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} = \int_{\mathbb{R}} \partial_x^2 u(\partial_x^2 u)^5 dx = -5\int_{\mathbb{R}} \partial_x u(\partial_x^2 u)^4 \ \partial_x^3 u dx. \tag{56}$$

Due to (41), (42), (44) and the Young inequality,

$$\begin{aligned} 5\int_{\mathbb{R}} |\partial_x u|(\partial_x^2 u)^4 |\partial_x^3 u| dx &\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + \frac{25}{2}\int_{\mathbb{R}} (\partial_x u)^2(\partial_x^2 u)^2(\partial_x^3 u)^2 dx \\ &\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + \frac{25}{2}\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} (\partial_x^2 u)^2(\partial_x^3 u)^2 dx \\ &\le \frac{1}{2}\|\partial_x u(t,\cdot)\|^6_{L^6(\mathbb{R})} + C(T)\int_{\mathbb{R}} (\partial_x^2 u)^2(\partial_x^3 u)^2 dx \end{aligned}$$

$$\leq \frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^6_{L^6(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq \frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^6_{L^6(\mathbb{R})} + C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}.$$

Therefore, by (56),

$$\frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^6_{L^6(\mathbb{R})} \leq C(T)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}.$$

An integration on (0, $t$) and (44) give (45). □

**Lemma 5** We have that

$$\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2}{7}\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{57}$$

for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Consider $A$ a positive constant, which will be specified later. Multiplying (1) by $2A\partial_t u$, thanks to (46) and (47), an integration on $\mathbb{R}$ gives

$$A\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2A\beta^2\int_{\mathbb{R}} \partial_x^4 u \partial_t u dx$$

$$= -2A\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2A\int_{\mathbb{R}} f'(u)\partial_x u \partial_t u dx$$

$$-2A\nu\int_{\mathbb{R}} \partial_x^2 u \partial_t u dx - 2A\kappa\int_{\mathbb{R}} \partial_x^3 u \partial_t u dx$$

$$-2Aa\int_{\mathbb{R}} \partial_t u \partial_x^2\left(\frac{u}{(1+u^2)^\alpha}\right)dx$$

$$-2Ab\int_{\mathbb{R}} \partial_t u \partial_x\left(\frac{\partial_x u}{(1+(\partial_x u)^2)^\gamma}\right)dx$$

$$-2Ad\int_{\mathbb{R}} \partial_t u \partial_x\left(\frac{\partial_x u}{1+|\partial_x u|^{1+\tau}}\right)dx$$

$$= -2A\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2A\int_{\mathbb{R}} f'(u)\partial_x u \partial_t u dx$$

$$-2A\nu\int_{\mathbb{R}} \partial_x^2 u \partial_t u dx - 2A\kappa\int_{\mathbb{R}} \partial_x^3 u \partial_t u dx$$

$$-2Aa\int_{\mathbb{R}} \frac{\partial_x^2 u \partial_t u}{(1+u^2)^\alpha} dx + 4Aa\alpha\int_{\mathbb{R}} \frac{u(\partial_x u)^2 \partial_t u}{(1+u^2)^{\alpha+1}} dx$$

$$+8Aa\alpha\int_{\mathbb{R}}\frac{u\partial_t u(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx-8Aa\alpha^2\int_{\mathbb{R}}\frac{u^3\partial_t u(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx$$

$$+4Aa\alpha\int_{\mathbb{R}}\frac{u^2\partial_x^2 u\partial_t u}{(1+u^2)^{\alpha+1}}dx-2Ab\int_{\mathbb{R}}\frac{\partial_x^2 u\partial_t u}{(1+(\partial_x u)^2)^{\gamma}}dx$$

$$+4Ab\gamma\int_{\mathbb{R}}\frac{(\partial_x u)^2\partial_x^2 u\partial_t u}{(1+(\partial_x u)^2)^{\gamma+1}}dx-2Ad\int_{\mathbb{R}}\frac{\partial_x^2 u\partial_t u}{1+|\partial_x u|^{1+\tau}}dx$$

$$+2Ad(\tau+1)\int_{\mathbb{R}}\frac{\partial_x u|\partial_x u|^{\tau}\ \mathrm{sign}(\partial_x u)\partial_x^2 u\partial_t u}{(1+|\partial_x u|^{1+\tau})^2}dx.$$

Therefore, we have that

$$A\beta^2\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+2A\left\|\partial_t u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$=-2A\int_{\mathbb{R}}f'(u)\partial_x u\partial_t u dx-2A\nu\int_{\mathbb{R}}\partial_x^2 u\partial_t u dx-2A\kappa\int_{\mathbb{R}}\partial_x^3 u\partial_t u dx$$

$$-2Aa\int_{\mathbb{R}}\frac{\partial_x^2 u\partial_t u}{(1+u^2)^{\alpha}}dx+4Aa\alpha\int_{\mathbb{R}}\frac{u(\partial_x u)^2\partial_t u}{(1+u^2)^{\alpha+1}}dx$$

$$+8Aa\alpha\int_{\mathbb{R}}\frac{u\partial_t u(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx-8Aa\alpha^2\int_{\mathbb{R}}\frac{u^3\partial_t u(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx$$

$$+4Aa\alpha\int_{\mathbb{R}}\frac{u^2\partial_x^2 u\partial_t u}{(1+u^2)^{\alpha+1}}dx-2Ab\int_{\mathbb{R}}\frac{\partial_x^2 u\partial_t u}{(1+(\partial_x u)^2)^{\gamma}}dx$$

$$+4Ab\gamma\int_{\mathbb{R}}\frac{(\partial_x u)^2\partial_x^2 u\partial_t u}{(1+(\partial_x u)^2)^{\gamma+1}}dx-2Ad\int_{\mathbb{R}}\frac{\partial_x^2 u\partial_t u}{1+|\partial_x u|^{1+\tau}}dx$$

$$+2Ad(\tau+1)\int_{\mathbb{R}}\frac{\partial_x u|\partial_x u|^{\tau}\ \mathrm{sgin}(\partial_x u)\partial_x^2 u\partial_t u}{(1+|\partial_x u|^{1+\tau})^2}dx. \tag{58}$$

Due to (2), (29), (30), (41), (42) and the Young inequality,

$$2A\int_{\mathbb{R}}|f'(u)||\partial_x u||\partial_t u|dx\le 2A\left\|f'\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx$$

$$\le 2C(T)A\int_{\mathbb{R}}|\partial_x u||\partial_t u|dx\le A^2C(T)\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\left\|\partial_t u(t,\cdot)\right\|_{L^2(\mathbb{R})}^2$$

$$\leq A^2C(T)+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2A|\nu|\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx \leq A^2\nu^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq A^2C(T)+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2A|\kappa|\int_{\mathbb{R}}|\partial_x^3 u||\partial_t u|\,dx \leq A^2\kappa^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2A|a|\int_{\mathbb{R}}\frac{|\partial_x^2 u\|\partial_t u|}{(1+u^2)^\alpha}dx \leq 2A|a|\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx \leq A^2a^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$4A|a\alpha|\int_{\mathbb{R}}\frac{|u|(\partial_x u)^2|\partial_t u|}{(1+u^2)^{\alpha+1}}dx \leq 4A|a\alpha|\int_{\mathbb{R}}|u|(\partial_x u)^2|\partial_t u|\,dx$$

$$\leq 4A|a\alpha|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u|\partial_t u|\,dx$$

$$\leq 2AC(T)\int_{\mathbb{R}}|\partial_x u|\partial_t u|\,dx \leq A^2C(T)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq A^2C(T)+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$8|Aa\alpha|\int_{\mathbb{R}}\frac{|u\|\partial_t u|(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx \leq 8|Aa\alpha|\int_{\mathbb{R}}|u||\partial_t u|(\partial_x u)^2\,dx$$

$$\leq 8|Aa\alpha|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|\,dx$$

$$\leq 2AC(T)\int_{\mathbb{R}}|\partial_x u||\partial_t u|\,dx \leq A^2C(T)\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\leq A^2C(T)+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$8A|a|\alpha^2\int_{\mathbb{R}}\frac{|u|^3|\partial_t u|(\partial_x u)^2}{(1+u^2)^{\alpha+2}}dx \leq 8A|a|\alpha^2\int_{\mathbb{R}}|u|^3|\partial_t u|(\partial_x u)^2\,dx$$

$$\leq 8A|a|\alpha^2\|u\|^3_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u||\partial_t u|\,dx$$

$$\leq 2AC(T)\int_{\mathbb{R}}|\partial_x u||\partial_t u|\,dx \leq A^2C(T)\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$4A|a\alpha|\int_{\mathbb{R}}\frac{u^2|\partial_x^2 u||\partial_t u|}{(1+u^2)^{\alpha+1}}dx \leq 4A|a\alpha|\int_{\mathbb{R}}u^2|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq 4A|a\alpha|\left\|u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx \leq 2AC(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq A^2C(T)\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2A|b|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_t u|}{(1+(\partial_t u)^2)^{\gamma}}dx \leq 2A|b|\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq A^2b^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$4A|b\gamma|\int_{\mathbb{R}}\frac{(\partial_x u)^2|\partial_x^2 u||\partial_t u|}{(1+(\partial_x u)^2)^{\gamma+1}}dx \leq 4A|b\gamma|\int_{\mathbb{R}}(\partial_x u)^2|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq 4A|b\gamma|\left\|\partial_x u\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx \leq 2AC(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq A^2C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2A|d|\int_{\mathbb{R}}\frac{|\partial_x^2 u||\partial_t u|}{1+|\partial_t u|^{1+\tau}}dx \leq 2A|d|\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq A^2d^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2Ad(\tau+1)\int_{\mathbb{R}}\frac{\partial_x u|\partial_x u|^{\tau}\,\mathrm{sign}(\partial_x u)\partial_x^2 u\partial_t u}{(1+|\partial_x u|^{1+\tau})^2}dx \leq A|d(\tau+1)|\int_{\mathbb{R}}\frac{|\partial_x u|^{\tau+1}|\partial_x^2 u||\partial_t u|}{(1+|\partial_x u|^{1+\tau})^2}dx$$

$$\leq A|d(\tau+1)|\left\|\partial_x u\right\|^{\tau+1}_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq 2AC(T)\int_{\mathbb{R}}|\partial_x^2 u||\partial_t u|\,dx$$

$$\leq A^2C(T)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq A^2C(T)+\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (58) that

$$A\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2(A-6)\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le A^2 C(T) + A^2\kappa^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Choosing $A = 7$, we have that

$$7\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C(T) + 49\kappa^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

which gives

$$\beta^2 \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2}{7}\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C(T) + 7\kappa^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on (0, $T$), by (15) and (29), we get

$$\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2}{7}\int_0^t \left\|\partial_t u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$

$$\le C_0 + C(T)t + 7\kappa^2 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),$$

which gives (57). □

We are ready for the proof of Theorem 1.

**Proof of Theorem 1** Thanks to Lemmas 1, 2, 3, 4 and the Cauchy-Kovalevskaya Theorem 67], we have that $u$ is solution of (1) and (11) holds.

We prove (12). Let $u_1$ and $u_2$ be two solutions of (1), which verify (11), that is,

$$\begin{cases} \partial_t u_i + \partial_x f(u_i) + \beta^2 \partial_x^4 u_i + \nu\partial_x^2 u_i & \\ \qquad + \kappa\partial_x^3 u_i + a\partial_x^2\left(\dfrac{u_i}{(1+u_i^2)^\alpha}\right) & \\ \qquad + b\partial_x\left(\dfrac{\partial_x u_i}{(1+(\partial_x u_i)^2))^\gamma}\right) & i = 1, 2. \\ \qquad + d\partial_x\left(\dfrac{\partial_x u_i}{1+|\partial_x u_i|^{1+\tau}}\right) = 0, \qquad 0 \le t \le T, \quad x \in \mathbb{R}, & \\ u_i(0,x) = u_{i,0}(x), \qquad x \in \mathbb{R}, & \end{cases}$$

Then, the function

$$\omega(t,x) = u_1(t,x) - u_2(t,x), \tag{59}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + \partial_x \left( f(u_1) - f(u_2) \right) + \beta^2 \partial_x^4 \omega + \nu \partial_x^2 \omega \\ \qquad + \kappa \partial_x^3 \omega + a \partial_x^2 \left( \dfrac{u_1}{(1+u_1^2)^\alpha} - \dfrac{u_2}{(1+u_2^2)^\alpha} \right) \\ \qquad + b \partial_x \left( \dfrac{\partial_x u_1}{(1+(\partial_x u_1)^2))^\gamma} - \dfrac{\partial_x u_2}{(1+(\partial_x u_2)^2))^\gamma} \right) \\ \qquad + d \partial_x \left( \dfrac{\partial_x u_1}{1+|\partial_x u_1|^{1+\tau}} - \dfrac{\partial_x u_2}{1+|\partial_x u_2|^{1+\tau}} \right) = 0, & 0 \le t \le T, \quad x \in \mathbb{R}, \\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{60}$$

Fixed $T > 0$, since $u_1(t, \cdot), u_2(t, \cdot) \in H^2(\mathbb{R})$, for every $0 \le t \le T$, we have that

$$\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \ \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T),$$

$$\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{61}$$

We define the following function:

$$F_1(v_1) = \frac{v_1}{(1+v_1^2)^\alpha}. \tag{62}$$

Since $F_1 \in C^1(\mathbb{R})$, thanks to (59), there exists $\xi_1$ between $u_1$ and $u_2$, such that

$$F_1(u_1) - F_1(u_2) = F_1'(\xi_1)(u_1 - u_2) = F_1'(\xi_1)\omega, \ \ u_1 < \xi_1 < u_2, \text{ or, } u_2 < \xi_1 < u_1. \tag{63}$$

Moreover, by (61),

$$|F_1'(\xi_1)| \le \|F_2'\|_{L^\infty(-C(T),C(T))} \le C(T). \tag{64}$$

We define the following function:

$$F_2(v_2) = \frac{v_2}{(1+(v_2)^2))^\gamma}, \ \ v_2 := \partial_x u. \tag{65}$$

Since $F_2 \in C^1(\mathbb{R})$, thanks to (59), there exists $\xi_2$ between $\partial_x u_1$ and $\partial_x u_2$, such that

$$F_2(\partial_x u_1) - F_2(\partial_x u_2) = F_2'(\xi_2)(\partial_x u_1 - \partial_x u_2) = F_2'(\xi_2)\partial_x \omega,$$

$$\partial_x u_1 < \xi_2 < \partial_x u_2, \text{ or, } \partial_x u_2 < \xi_2 < \partial_x u_1. \tag{66}$$

Moreover, by (61),

$$| F_2'(\xi_2) | \leq \|F_2'\|_{L^\infty(-C(T), C(T))} \leq C(T). \tag{67}$$

We defined the following function:

$$F_3(v_3) = \frac{v_3}{1 + v_3^{1+\tau}}, \; v_3 := | \partial_x u |. \tag{68}$$

Since $F_3 \in C^1(\mathbb{R})$, there exists $\xi_3$ between $|\partial_x u_1|$ and $|\partial_x u_2|$, such that

$$F_3(| \partial_x u_1 |) - F_3(| \partial_x u_2 |) = F_3'(\xi_3)\big(| \partial_x u_1 | - | \partial_x u_2 |\big),$$

$$| \partial_x u_1 | < \xi_3 < | \partial_x u_2 |, \text{ or, } | \partial_x u_2 | < \xi_3 < | \partial_x u_1 |. \tag{69}$$

Moreover, by (61), we have that

$$| F_3(\xi) | \leq \|F_3\|_{L^\infty(-C(T), C(T))} \leq C(T). \tag{70}$$

Observe that, since thanks to (59),

$$\big|| \partial_x u_1 | - | \partial_x u_2 |\big| \leq | \partial_x u_1 - \partial_x u_2 | = | \partial_x \omega |, \tag{71}$$

it follows from (68), (69), (70) and (71) that

$$\begin{aligned} \big| F_3(| \partial_x u_1 |) - F_3(| \partial_x u_2 |) \big| &= | F_3(\xi) | \big|| \partial_x u_1 | - | \partial_x u_2 |\big| \\ &\leq C(T) \big|| \partial_x u_1 | - | \partial_x u_2 |\big| \\ &\leq C(T) | \partial_x u_1 - \partial_x u_2 | = C(T) | \partial_x \omega |. \end{aligned} \tag{72}$$

Since $f \in C^1(\mathbb{R})$, thanks to (59), there exists $\xi_4$ between $u_1$ and $u_2$, such that

$$f(u_1) - f(u_2) = f'(\xi_4)(u_1 - u_2) = f'(\xi_4)\omega, \quad u_1 < \xi_4 < u_2, \text{ or, } u_2 < \xi_4 < u_1. \tag{73}$$

Moreover, by (61), we have that

$$| f'(\xi_4) | \leq \|f'\|_{L^\infty(-C(T), C(T))} \leq C(T). \tag{74}$$

Observe that, thanks to (62), (63), (65), (66), (68), (69) and (73), Equation (60) is equivalent to the following one:

$$\partial_t\omega+\partial_x\left(f'(\xi_4)\omega\right)+\beta^2\partial_x^4\omega+\nu\partial_x^2\omega+\kappa\partial_x^3\omega$$

$$+a\partial_x^2\left(F_1'(\xi_1)\omega\right)+b\partial_x\left(F_2'(\xi_2)\partial_x\omega\right)+d\partial_x\left(F_3'(\xi_3)\left(|\partial_x u_1|-|\partial_x u_2|\right)\right)=0. \tag{75}$$

Multiplying (75) by $2\omega$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}}\partial_t\omega\omega dx\\
&= -2\int_{\mathbb{R}}\omega\partial_x\left(f'(\xi_4)\omega\right)dx-2\beta^2\int_{\mathbb{R}}\omega\partial_x^4\omega dx-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx\\
&\quad -2\kappa\int_{\mathbb{R}}\omega\partial_x^3\omega dx-2a\int_{\mathbb{R}}\omega\partial_x^2\left(F_1'(\xi_1)\omega\right)dx\\
&\quad -2b\int_{\mathbb{R}}\omega\partial_x\left(F_2'(\xi_2)\partial_x\omega\right)dx-2d\int_{\mathbb{R}}\omega\partial_x\left(F_3'(\xi_3)\left(|\partial_x u_1|-|\partial_x u_2|\right)\right)\\
&= -2\int_{\mathbb{R}}f'(\xi_4)\omega\partial_x\omega dx+2\beta^2\int_{\mathbb{R}}\partial_x\omega\partial_x^3\omega dx-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx\\
&\quad +2\kappa\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega dx+2a\int_{\mathbb{R}}\partial_x\omega\partial_x\left(F_1'(\xi_1)\omega\right)dx\\
&\quad +2b\int_{\mathbb{R}}F_2'(\xi_2)(\partial_x\omega)^2dx+2d\int_{\mathbb{R}}F_3'(\xi_3)\left(|\partial_x u_1|-|\partial_x u_2|\right)\partial_x\omega dx\\
&= -2\int_{\mathbb{R}}f'(\xi_4)\omega\partial_x\omega dx-2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx\\
&\quad -2a\int_{\mathbb{R}}F_1'(\xi_1)\omega\partial_x^2\omega dx+2b\int_{\mathbb{R}}F_2'(\xi_2)(\partial_x\omega)^2dx\\
&\quad +2d\int_{\mathbb{R}}F_3'(\xi_3)\left(|\partial_x u_1|-|\partial_x u_2|\right)\partial_x\omega dx.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = -2\int_{\mathbb{R}}f'(\xi_4)\omega\partial_x\omega dx-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx-2a\int_{\mathbb{R}}F_1'(\xi_1)\omega\partial_x^2\omega dx\\
&\quad +2b\int_{\mathbb{R}}F_2'(\xi_2)(\partial_x\omega)^2dx+2d\int_{\mathbb{R}}F_3'(\xi_3)\left(|\partial_x u_1|-|\partial_x u_2|\right)\partial_x\omega dx.
\end{aligned} \tag{76}$$

Due to (64), (67), (72) and the Young inequality,

$$2\int_{\mathbb{R}} |f'(\xi_4)||\omega||\partial_x \omega|dx \le 2C(T)\int_{\mathbb{R}} |\omega||\partial_x \omega|dx$$

$$\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|\nu|\int_{\mathbb{R}} |\omega||\partial_x^2 \omega|dx = \int_{\mathbb{R}} \left|\frac{2\nu\omega}{\beta}\right| \left|\beta\partial_x^2 \omega\right| dx$$

$$\le \frac{2\nu^2}{\beta^2}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|a|\int_{\mathbb{R}} |F_1'(\xi_1)||\omega||\partial_x^2 \omega|dx \le 2C(T)\int_{\mathbb{R}} |u||\partial_x^2 u|dx = \int_{\mathbb{R}} \left|\frac{C(T)u}{\beta}\right| \left|\beta\partial_x^2 u\right| dx$$

$$\le C(T)\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|b|\int_{\mathbb{R}} |F_2'|(\xi_2)(\partial_x \omega)^2 dx \le C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|d|\int_{\mathbb{R}} |F_3'(\xi_3)|\left||\partial_x u_1| - |\partial_x u_2|\right||\partial_x \omega|dx \le C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

It follows from (76) that

$$\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{77}$$

Observe that

$$C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} = C(T)\int_{\mathbb{R}} \partial_x \omega \partial_x \omega dx = -C(T)\int_{\mathbb{R}} \omega \partial_x^2 \omega dx.$$

Therefore, by the Young inequality,

$$C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\int_{\mathbb{R}} |\omega||\partial_x^2 \omega|dx = \int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| \left|\beta\partial_x^2 \omega\right| dx$$

$$\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (77),

$$\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (60) give

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2e^{C(T)t}}{2}\int_0^t e^{-C(T)s}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le e^{C(T)t}\|\omega_0\|^2_{L^2(\mathbb{R})}. \tag{78}$$

(12) follows from (59) and (78). □

## 3. Proof of Theorem 2

In this section, we prove Theorem 2, under assumption

$$u_0\in H^1(\mathbb{R}). \tag{79}$$

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1). The presence of the high order perturbation allows us to make rigorous all the computations in the following lemmas.

Fix a small number $0<\varepsilon<1$, and let $u_\varepsilon=u_\varepsilon(t,x)$ be the unique classical solution of the following problem (see Section 2, [61]):

$$\begin{cases}\partial_t u_\varepsilon+\partial_x f(u_\varepsilon)+\beta^2\partial_x^4u_\varepsilon+\nu\partial_x^2u_\varepsilon\\ \qquad+\kappa\partial_x^3u_\varepsilon+a\partial_x^2\left(\dfrac{u_\varepsilon}{(1+u_\varepsilon{}^2)^\alpha}\right)\\ \qquad+b\partial_x\left(\dfrac{\partial_x u_\varepsilon}{(1+(\partial_x u_\varepsilon)^2))^\gamma}\right)\\ \qquad+d\partial_x\left(\dfrac{\partial_x u_\varepsilon}{1+|\partial_x u_\varepsilon|^{1+\tau}}\right)=\varepsilon\partial_x^6u_\varepsilon, & 0\le t\le T,\quad x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R},\end{cases} \tag{80}$$

where $u_{\varepsilon,0}$ is a $C^\infty$ approximation of $u_0$ such that

$$\|u_{\varepsilon,0}\|_{H^1(\mathbb{R})}\le\ \|u_0\|_{H^1(\mathbb{R})},\ \sqrt{\varepsilon}\left\|\partial_x^2u_{\varepsilon,0}\right\|_{L^2(\mathbb{R})}\le C_0, \tag{81}$$

where $C_0$ is a positive constant, independent on $\varepsilon$.

Some a priori estimates on $u_\varepsilon$ are needed.

**Lemma 6** We have that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2e^{C_0t}}{2}\int_0^t e^{-C_0s}\left\|\partial_x^2u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+2\varepsilon e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^3u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T), \tag{82}$$

$$\int_0^t \left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{83}$$

$$\int_0^t \left\|u_\varepsilon(s,\cdot)\right\|^2_{L^\infty(\mathbb{R})} ds \le C(T),\ \int_0^t \left\|u_\varepsilon(s,\cdot)\right\|^8_{L^\infty(\mathbb{R})} ds \le C(T), \tag{84}$$

$$\int_0^t \left\|u_\varepsilon(s,\cdot)\right\|^6_{L^\infty(\mathbb{R})} ds \le C(T),\ \int_0^t \left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^\infty(\mathbb{R})} ds \le C(T), \tag{85}$$

$$\int_0^t \left\|u_\varepsilon(s,\cdot)\ \partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T),\ \int_0^t \left\|u_\varepsilon^2(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{86}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. Observe that

$$2\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon dx = -2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, arguing as in Lemma 1, we have that

$$\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C_0 \left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

By the Gronwall Lemma and (81), we have that

$$\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$

$$+2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0 e^{C_0 t} \le C(T),$$

which gives (82).

Finally, arguing as in Lemma 1, we have (83), while arguing as in Lemma 2, we have (84), (85), (86). □

**Lemma 7** We have that

$$\left\|\partial_x u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T), \tag{87}$$

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds$$

$$+2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{88}$$

$$\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{89}$$

$$\int_0^t \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} ds \le C(T), \tag{90}$$

$$\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})}\, ds \le C(T), \tag{91}$$

$$\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^\infty(\mathbb{R})}\, ds \le C(T), \tag{92}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. Observe that

$$-2\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx = -2\varepsilon \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, arguing as in Lemma 3, we have that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C_0 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\left(1+2D_1\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&+ C_0 \left\|u_\varepsilon^2(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&+ \frac{1}{D_1}\|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\|u_\varepsilon(t,\cdot)\|^6_{L^\infty(\mathbb{R})} + \|u(t,\cdot)\|^8_{L^\infty(\mathbb{R})}\right).
\end{aligned}$$

Integrating on (0, $t$), by (82), (83), (84), (85), and (86), we have that

$$\begin{aligned}
&\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&\le C_0 + C_0\int_0^t \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0\left(1+2D_1\right)\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&+ C_0\int_0^t \left\|u_\varepsilon^2(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \int_0^t \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\
&+ \frac{1}{D_1}\|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left(\int_0^t \|u_\varepsilon(s,\cdot)\|^6_{L^\infty(\mathbb{R})} ds + \int_0^t \|u(s,\cdot)\|^8_{L^\infty(\mathbb{R})} ds\right) \\
&\le C(T)\left(1+\frac{1}{D_1}\right)\|\partial_x u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned} \tag{93}$$

Consequently, by (93),

$$\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T)\left(1+\frac{1}{D_1}\right)\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}.$$

Choosing $D_1$ an in (36), we have (87).

(88) follows from (87) and (93), while arguing as in Lemma 3, we have (89), (90) and (91).

Finally, we prove (92). Observe that, thanks to (88), we have that

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \le C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})} \le C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on (0, $t$), by (82), we have (92). □

**Lemma 8** We have that

$$\sqrt{\varepsilon}\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T), \tag{94}$$

$$\begin{aligned}
&\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&+2\varepsilon^2\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\
&+2\,|\,d(\tau+1)\,|\,\varepsilon\int_0^t\int_{\mathbb{R}}\frac{|\,\partial_x^2 u_\varepsilon\,||\,\partial_x^4 u_\varepsilon\,|}{(1+|\,\partial_x u_\varepsilon\,|^{1+\tau})^2}\,dsdx \le C(T),
\end{aligned} \tag{95}$$

for every $0 \le t \le T$.

**Proof.** Let $0 \le t \le T$. We begin by observing that

$$2\varepsilon^2\int_{\mathbb{R}}\partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon dx = -2\varepsilon^2\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, arguing as in Lemma 4, we have that

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon^2\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= -2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\nu\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&-2a\varepsilon\int_{\mathbb{R}}\frac{\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon}{(1+u_\varepsilon^2)^\alpha}dx + 4a\alpha\varepsilon\int_{\mathbb{R}}\frac{u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon}{(1+u_\varepsilon^2)^{\alpha+1}}dx
\end{aligned}$$

$$+8a\alpha\varepsilon\int_{\mathbb{R}}\frac{u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon}{(1+u_\varepsilon^2)^{\alpha+2}}dx-8a\alpha^2\varepsilon\int_{\mathbb{R}}\frac{u_\varepsilon^3(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon}{(1+u_\varepsilon^2)^{\alpha+2}}$$

$$+2a\alpha\varepsilon\int_{\mathbb{R}}\frac{u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon}{(1+u^2)^{\alpha+1}}dx-2b\varepsilon\int_{\mathbb{R}}\frac{\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon}{(1+(\partial_x u_\varepsilon)^2))^{\gamma}}dx$$

$$+4b\gamma\varepsilon\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon}{(1+(\partial_x u_\varepsilon{}^2))^{\gamma+1}}dx-2d\varepsilon\int_{\mathbb{R}}\frac{\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon}{1+|\partial_x u_\varepsilon|^{1+\tau}}dx$$

$$+2d(\tau+1)\varepsilon\int_{\mathbb{R}}\frac{\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\,|\partial_x u_\varepsilon|^{\tau}\,\mathrm{sign}(\partial_x u_\varepsilon)}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx. \tag{96}$$

Since $0\le\varepsilon<1$, thanks to (2), (88), (89) and the Young inequality,

$$\begin{aligned}2\varepsilon\int_{\mathbb{R}}|f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx&\le 2\|f'\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx\\&\le 2C(T)\varepsilon\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\le C(T)+\varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned} \tag{97}$$

$$\begin{aligned}2|a|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+u_\varepsilon^2)^{\alpha}}dx&\le 2|a|\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx\\&\le a^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\le a^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$\begin{aligned}4|a\alpha|\varepsilon\int_{\mathbb{R}}\frac{|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|}{(1+u_\varepsilon^2)^{\alpha+1}}dx&\le 4|a\alpha|\varepsilon\int_{\mathbb{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx\\&\le 4|a\alpha|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx\le 2C(T)\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx\\&\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+\varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$8|a\alpha|\varepsilon\int_{\mathbb{R}}\frac{|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|}{(1+u_\varepsilon^2)^{\alpha+2}}dx \le 8|a\alpha|\varepsilon\int_{\mathbb{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx$$

$$\le 8|a\alpha|\|u_\varepsilon\|_{L^\infty(\mathbb{R})}\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx \le 2C(T)\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx$$

$$\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$8|a|\alpha^2\varepsilon\int_{\mathbb{R}}\frac{|u_\varepsilon|^3(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|}{(1+u_\varepsilon^2)^{\alpha+2}} \le 8|a|\alpha^2\varepsilon\int_{\mathbb{R}}|u_\varepsilon|^3(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx$$

$$\le 8|a|\alpha^2\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx \le 2C(T)\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx$$

$$\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|a\alpha|\varepsilon\int_{\mathbb{R}}\frac{u_\varepsilon{}^2|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+u^2)^{\alpha+1}}dx \le 2|a\alpha|\varepsilon\int_{\mathbb{R}}u_\varepsilon{}^2|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx$$

$$\le 2|a\alpha|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx \le 2C(T)\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx$$

$$\le C(T)\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\le C(T)\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|b|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+(\partial_x u_\varepsilon)^2))^\gamma}dx \le 2|b|\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx$$

$$\le b^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$\le b^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$

$$4|b\gamma|\varepsilon\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+(\partial_x u_\varepsilon)^2))^{\gamma+1}}dx \le 4|b\gamma|\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx$$

$$= 2\varepsilon\int_{\mathbb{R}}\left|\frac{b\gamma(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon}{\sqrt{D_4}}\right|\left|\sqrt{D_4}\partial_x^4 u_\varepsilon\right|dx$$

$$\le \frac{b^2\gamma^2\varepsilon}{D_4}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4(\partial_x^2 u_\varepsilon)^2 dx + D_4\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le \frac{b^2\gamma^2\varepsilon}{D_4}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_4\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\le \frac{b^2\gamma^2\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})} + D_4\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

where $D_4$ is a positive constant, which will be specified later. Observe that, thanks to (2),

$$2d(\tau+1)\varepsilon\int_{\mathbb{R}}\frac{\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon|\partial_x u_\varepsilon|^\tau\,\mathrm{sign}(\partial_x u)}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx \le 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon||\partial_x u_\varepsilon|^{\tau+1}}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx$$

$$= 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon||\partial_x u_\varepsilon|^{\tau+1}+|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|-|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx$$

$$= 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|\left(1+|\partial_x u_\varepsilon|^{\tau+1}\right)}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx - 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx$$

$$= 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{1+|\partial_x u_\varepsilon|^{1+\tau}}dx - 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx$$

$$\le 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|dx - 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx. \tag{98}$$

Since, $0 < \varepsilon < 1$, it follows from (96), (97) and (98) that

$$\varepsilon\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+2\varepsilon^2\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}dx$$

$$\leq C(T) + C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\nu|\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \left(7 + D_4\right)\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ \frac{b^2\gamma^2\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})}$$

$$+2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|\,dx. \tag{99}$$

Due to the Young inequality,

$$2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|\,dx$$

$$\leq d^2(\tau+1)^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$\leq d^2(\tau+1)^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (99),

$$\varepsilon\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ 2\varepsilon^2\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|d(\tau+1)|\varepsilon\int_{\mathbb{R}}\frac{|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}\,dx$$

$$\leq C(T) + C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\nu|\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \left(8 + D_4\right)\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$+ \frac{b^2\gamma^2\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})}.$$

Integrating on $(0, t)$, thanks to (81), (88), (90) and (92),

$$\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+2\varepsilon^2\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2\,|\,d(\tau+1)\,|\,\varepsilon\int_0^t\int_{\mathbb{R}}\frac{|\,\partial_x^2 u_\varepsilon\,\|\,\partial_x^4 u_\varepsilon\,|}{(1+|\,\partial_x u_\varepsilon\,|^{1+\tau})^2}dsdx$$

$$\leq C_0+C(T)t+C(T)\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2\,|\,\nu\,|\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+C(T)\int_0^t\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^4_{L^4(\mathbb{R})}ds+\left(8+D_4\right)\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds$$

$$+\frac{b^2\gamma^2\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\int_0^t\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^4_{L^\infty(\mathbb{R})}ds$$

$$\leq C(T)\left(1+\frac{\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+D_4\right). \tag{100}$$

Therefore, by (100), we have that

$$\varepsilon\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C(T)\left(1+\frac{\varepsilon}{D_4}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+D_4\right).$$

Hence,

$$\left(1-\frac{C(T)}{D_4}\right)\varepsilon\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C(T)\left(1+D_4\right).$$

Choosing

$$D_4=2C\left(T\right), \tag{101}$$

we obtain that

$$\frac{\varepsilon}{2}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C(T),$$

which gives (94).

Finally, (95) follows from (100) and (101). □

Now, we prove the following result.

**Lemma 9** Fix $T > 0$. Then,

$$\text{the sequence } \{u_\varepsilon\}_{\varepsilon>0} \text{ is compact in } L^2_{loc}((0,\infty)\times\mathbb{R}). \tag{102}$$

Consequently, there exists a subsequence $\{\,u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u\in L^2_{loc}((0,\infty)\times\mathbb{R})$ such that, for each compact subset $K$ of $(0, \infty) \times \mathbb{R})$,

$$u_{\varepsilon_k} \to u \text{ in } L^2(K) \text{ and a.e.} \tag{103}$$

Moreover, $u$ is a solution of (1), satisfying (13).

**Proof.** We begin by proving (102). To prove (102), we rely on the Aubin-Lions Lemma (see [68]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin-Lions Lemma [68], to prove (102), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{104}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})), \tag{105}$$

We prove (104). Thanks to Lemmas 82 and 7,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^1(\mathbb{R})),$$

which gives (104). We prove (105). Observe that, by (80),

$$\partial_t u_\varepsilon = -\partial_x\left(G(u_\varepsilon)\right) - f'(u_\varepsilon)\partial_x u_\varepsilon,$$

where, thanks to (33),

$$\begin{aligned} G(u_\varepsilon) = {}& \nu\partial_x u_\varepsilon + \kappa\partial_x^2 u_\varepsilon + \beta^2\partial_x^3 u_\varepsilon + \varepsilon\partial_x^5 u_\varepsilon + \frac{a\partial_x u_\varepsilon}{(1+u_\varepsilon^2)^\alpha} \\ & - \frac{2a\alpha u_\varepsilon^2\partial_x u_\varepsilon}{(1+u_\varepsilon^2)^{\alpha+1}} + \frac{b\partial_x u_\varepsilon}{(1+(\partial_x u_\varepsilon)^2))^\gamma} + \frac{d\partial_x u_\varepsilon}{1+|\partial_x u_\varepsilon|^{1+\tau}}. \end{aligned} \tag{106}$$

Thanks to (82), (88) and (95), we have that

$$\begin{aligned} & \nu^2\left\|\partial_x u_\varepsilon^2\right\|_{L^2((0,T)\times\mathbb{R})}, \ \kappa^2\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T), \\ & \beta^4\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})}, \ \varepsilon^2\left\|\partial_x^5 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \end{aligned} \tag{107}$$

We claim that

$$a^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+u_\varepsilon^2)^{2\alpha}}\,dtdx \le C(T),$$

$$4a^2\alpha^2\int_0^T\int_{\mathbb{R}}\frac{u_\varepsilon^4(\partial_x u_\varepsilon)^2}{(1+u_\varepsilon^2)^{2(\alpha+1)}}\,dtdx \le C(T),$$

$$b^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+(\partial_x u_\varepsilon)^2))^{2\gamma}}\,dtdx \le C(T),$$

$$d^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}\,dtdx \le C(T). \tag{108}$$

Thanks to (2), (83), (84), (85), and (86),

$$a^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+u_\varepsilon^2)^{2\alpha}}\,dsdx \le a^2\int_0^T\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 dt \le C(T),$$

$$4a^2\alpha^2\int_0^T\int_{\mathbb{R}}\frac{u_\varepsilon^4(\partial_x u_\varepsilon)^2}{(1+u_\varepsilon^2)^{2(\alpha+1)}}\,dsdx \le 4a^2\alpha^2\int_0^T\left\|u_\varepsilon^2(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 dt \le C(T),$$

$$b^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+(\partial_x u_\varepsilon)^2))^{2\gamma}}\,dtdx \le b^2\int_0^T\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 dt \le C(T),$$

$$d^2\int_0^T\int_{\mathbb{R}}\frac{(\partial_x u_\varepsilon)^2}{(1+|\partial_x u_\varepsilon|^{1+\tau})^2}\,dtdx \le d^2\int_0^T\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 dt \le C(T).$$

Therefore, by (106), (107) and (108), we have that

$$\{\partial_x\left(G(u_\varepsilon)\right)\}_{\varepsilon>0} \text{ is bounded in } H^1((0,T)\times\mathbb{R}). \tag{109}$$

We have that

$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2\,dtdx \le C(T). \tag{110}$$

Thanks to (88) and (89),

$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2\,dtdx \le \left\|f'\right\|_{L^\infty(-C(T),C(T))}^2\int_0^T\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 dt \le C(T).$$

Therefore, (105) follows from (109) and (110).
Thanks to the Aubin-Lions Lemma, (102) and (103) hold.
Consequently, $u$ is solution of (1) and, thanks to Lemmas 6 and 7, (13) holds. □
**Lemma 10** The stability estimate (12) holds.

**Proof.** We begin by observing that, by (14), (1) reads:

$$\begin{cases} \partial_t u + \partial_x f(u) + \beta^2 \partial_x^4 u + \nu \partial_x^2 u \\ \qquad + \kappa \partial_x^3 u + a \partial_x^2 \left( \dfrac{u}{(1+u^2)^\alpha} \right) = 0, & 0 \le t \le T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{111}$$

Lemma 9 gives the existence of a solution (111) satisfying (13).
We prove (12). Let $u_1$ and $u_2$ two solutions of (111), which verify (13), that is

$$\begin{cases} \partial_t u_i + \partial_x f(u_i) + \beta^2 \partial_x^4 u_i + \nu \partial_x^2 u_i \\ \qquad + \kappa \partial_x^3 u_i + a \partial_x^2 \left( \dfrac{u_i}{(1+u_i^2)^\alpha} \right) = 0, & 0 \le t \le T, \quad x \in \mathbb{R}, \quad i = 1, 2. \\ u_i(0,x) = u_{i,0}(x), & x \in \mathbb{R}, \end{cases}$$

Then, the function $\omega$, defined in (59), is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + \partial_x \left( f(u_1) - f(u_2) \right) + \beta^2 \partial_x^4 \omega + \nu \partial_x^2 \omega \\ \qquad + \kappa \partial_x^3 \omega + a \partial_x^2 \left( \dfrac{u_1}{(1+u_1^2)^\alpha} - \dfrac{u_2}{(1+u_2^2)^\alpha} \right) = 0, & 0 \le t \le T, \quad x \in \mathbb{R}, \\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{112}$$

Fixed $T > 0$, since $u_1$, $u_2 \in H^1(\mathbb{R})$, for every $0 \le t \le T$, we have that

$$\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \ \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{113}$$

Consequently, thanks to (113), and arguing as in Section 2, we have (63), (64), (73) and (74). Therefore, by (63) and (73), Equation (112) is equivalent to the following one:

$$\partial_t \omega + \partial_x \left( f'(\xi_4)\omega \right) + \beta^2 \partial_x^4 \omega + \nu \partial_x^2 \omega + \kappa \partial_x^3 \omega + a \partial_x^2 \left( F_1'(\xi_1)\omega \right) = 0. \tag{114}$$

Multiplying (114) by $2\omega$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt} \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} \omega \partial_t \omega dx \\ &= -2\int_{\mathbb{R}} \omega \partial_x \left( f'(\xi_4)\omega \right) dx - 2\beta^2 \int_{\mathbb{R}} \omega \partial_x^4 \omega dx - 2\nu \int_{\mathbb{R}} \omega \partial_x^2 \omega dx \\ &\quad -2\kappa \int_{\mathbb{R}} \omega \partial_x^3 \omega dx - 2a \int_{\mathbb{R}} \omega \partial_x^2 \left( F_1'(\xi_1)\omega \right) dx \\ &= 2\int_{\mathbb{R}} f'(\xi_4) \omega \partial_x \omega dx + 2\beta^2 \int_{\mathbb{R}} \partial_x \omega \partial_x^3 \omega dx - 2\nu \int_{\mathbb{R}} \omega \partial_x^2 \omega dx \end{aligned}$$

$$+2\kappa\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega dx+2a\int_{\mathbb{R}}\partial_x\omega\partial_x\left(F_1'(\xi_1)\omega\right)dx$$

$$=2\int_{\mathbb{R}}f'(\xi_4)\omega\partial_x\omega dx-2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx$$

$$-2a\int_{\mathbb{R}}F_1'(\xi_1)\omega\partial_x^2\omega dx.$$

Consequently, we have that

$$\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$

$$=2\int_{\mathbb{R}}f'(\xi_4)\omega\partial_x\omega dx-2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx-2a\int_{\mathbb{R}}F_1'(\xi_1)\omega\partial_x^2\omega dx. \tag{115}$$

Due to (64), (74) and the Young inequality,

$$2\int_{\mathbb{R}}|f'(\xi_4)||\omega||\partial_x\omega|dx\le 2C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|dx$$

$$\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$2|\nu|\int_{\mathbb{R}}|\omega|\partial_x^2\omega dx=\int_{\mathbb{R}}\left|\frac{2\nu\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx$$

$$\le\frac{2\nu^2}{\beta^2}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$2|a|\int_{\mathbb{R}}|F_1'(\xi_1)||\omega||\partial_x^2\omega|dx\le C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx$$

$$=\int_{\mathbb{R}}\left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (115) that

$$\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Arguing as in Section 2, we have (12). □

**Proof of Theorem 2** Theorem 2 follows from Lemmas 9 and 10. □

## 4. Conclusion

We considered the Kuramoto-Sivashinsky equation with Ehrilch–Schwoebel effects in its full generality. We proved that the Cauchy problem is well-posed in the Hadamard sense in both $H^1$ and $H^2$ globally in time. Our arguments are based on energy estimates.

## Conflict of interest

The authors declare that they do not have any conflict of interest.

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