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Research Article

The Existence of Solution to Fractional Boundary Value Problem with Riemann-Liouville Type History-State-Based Variable-Order Derivative

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Abstract: The paper is devoted to studying the solutions of boundary value problem for nonlinear fractional differential equation with Riemann-Liouville type history-state-based variable-order derivative. Using Schauder fixed point theorem and Banach fixed point thoerem, we prove the existence and uniqueness of solutions in the Hölder space. Lastly, two examples are given to show the applicability of the existence theorems.

***Keywords*:** fractional differential equations, Riemann-Liouville type variable-order derivative, boundary value problem, existence of solution, Hölder space

MSC: 34B15, 34A08

1. Introduction

The history of fractional calculus has more than three hundred years. There are many fractional derivatives, such as Riemann-Liouville, Caputo, Erdélyi-Kober, Hadamard fractional derivative and so on. The Riemann-Liouville derivative [1] D_{0+}^{η} $(0 < \eta < 1)$, as one of the most classical fractional derivative, is defined by

$$D_{0+}^{\eta} f(t) = \frac{d}{dt} I_{0+}^{1-\eta} f(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} f(s)ds,\ t > 0. \tag{1}$$

It is easy to obtain that $D_{0+}^{\eta} f(t)$ exists if the derivative of $I_{0+}^{1-\eta} f(t)$ exists on interval $[0, T]$ by the definition. That is to say, the existence conditions for Riemann-Liouville fractional derivative are weaker than other fractional derivatives such as Caputo derivative. For example, Riemann-Liouville fractional derivative of function $t^{-\frac{1}{2}}$ is 0 on interval [0, 1]. But Caputo fractional derivative of function $t^{-\frac{1}{2}}$ dosen't exist at point $t = 0$. Variable-Order (VO) fractional operators are the derivatives and integrals whose order are functions of some variables. VO fractional operators are conceived and mathematically formalized only in recent years and can be seen as a natural analytical extension of

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constant-order fractional operators. The first definition of VO fractional operator was given by Samko and Ross [2] in 1993. Subsequently, Lorenzo et al. [3] and Coimbra [4] studied VO calculus by discussing its possible applications in mechanics, which marked the starting point for applications of VO operators to the analysis of different complex physical problems. After that VO fractional differential equations are widely employed in mechanics and dynamics, viscoelasticity, the modelling of transport processes, control theory because VO fractional operators can describe accurately the memory and hereditary properties of many physical phenomena and processes depending on their non-stationary power-law kernel [5-10].

For Riemann-Liouville VO fractional integral, there are two forms [3, 11] as follows

$$I_{0+}^{\eta(t)} f(t) = \int_0^t \frac{(t-s)^{\eta(s)-1}}{\Gamma(\eta(s))} f(s)ds,\ t > 0 \tag{2}$$

and

$$I_{0+}^{\eta(t)} f(t) = \frac{1}{\Gamma(\eta(t))} \int_0^t (t-s)^{\eta(t)-1} f(s)ds,\ t > 0, \tag{3}$$

where $0 < \eta(t) < 1$ is a given nonconstant function. Corresponding to (2) and (3), there are two types of Riemann-Liouville VO fractional derivative [3, 11]

$$D_{0+}^{\eta(t)} f(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\eta(s)}}{\Gamma(1-\eta(s))} f(s)ds \tag{4}$$

and

$$D_{0+}^{\eta(t)} f(t) = \frac{1}{\Gamma(1-\eta(t))} \frac{d}{dt} \int_0^t (t-s)^{-\eta(t)} f(s)ds. \tag{5}$$

In (3) and (5), the current state $\eta(t)$ is used on entire interval $[0, t]$. For any fixed t, the convolution kernel can be integrated in a closed form as for the constant-order fractional derivative operator in (1), which greatly facilitates the analysis. In (2) and (4), the power $\eta(s)$ assumes its historical state at the historical time instant s. Hence, the history-state-based fractional differential equations, which accounts for the influence of the quantity of interest at the historical time instant s with the historical state $\eta(s)$, is probably more physically relevant. However, the convolution kernel in (4) can't be integrated in a closed form in mathematics, which highly complicates the mathematical analysis. That is to say, it is difficult to obtain closed form solutions to history-state-based VO fractional differential equations because the law of exponents doesn't hold for VO fractional integral (refer to [2, 12]). Therefore, many authors [13-17] have made use of the numerical methods to solve VO fractional differential equations. As an important research topic of VO fractional differential equations, boundary value problems are concerned by many professors. Since the kernel of VO fractional operators has a variable exponent, there are many difficulties for us to obtain the existence of solutions to VO fractional boundary value problem. Few authors [18, 19] have attempted to consider the existence, uniqueness and stability of solutions to VO fractional boundary value problem using standard techniques in analysis.

In [18], authors proved the existence of solutions and presented a generalized Lyapunov-type inequality to the boundary value problem for VO fractional differential equation

$$\begin{cases} D_{0+}^{q(t)} x(t) = f(t,x),\ 0 < t < T,\ 0 < T < +\infty, \\ x(0) = x(T) = 0, \end{cases} \tag{6}$$

where $1 < q(t) < 2$ is a piecewise constant function, $f: (0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $D_{0+}^{q(t)}$ is the Riemann-Liouville type VO fractional derivative defined by

$$D_{0+}^{q(t)} x(t) = \frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))} x(s) ds, \ t > 0.$$

In [19], authors proved the wellposedness and smoothing properties of the Dirichlet boundary value problem of one dimensional VO linear space-fractional diffusion equation

$$\begin{cases} -D^2 u - D\left[c(x)\left(\zeta {}_0^l I_x^{\alpha(x)} + (1-\zeta) {}_x^r I_1^{\alpha(x)} \right) Du(x) \right] = f(x), \ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{7}$$

where $0 < \alpha(x) \le 1$, ${}_0^l I_x^{\alpha(x)}$ and ${}_x^r I_1^{\alpha(x)}$ are defined by

$${}_0^l I_x^{\alpha(x)} u = \frac{1}{\Gamma(\alpha(x))} \int_0^x \frac{u(s)}{(x-s)^{1-\alpha(x)}} ds, \ \ {}_x^r I_1^{\alpha(x)} u = \frac{1}{\Gamma(\alpha(x))} \int_x^1 \frac{u(s)}{(s-x)^{1-\alpha(x)}} ds.$$

The authors in [18] proved the existence of solutions to the problem (6) in the space of continuous functions and the variable order was a piecewise constant function. The authors in [19] proved the wellposedness of the problem (7) in the Hölder space and the order of derivative was two. It is well known that the Hölder space is the function space in which functions are Hölder continuous and it is included in the space of continuous functions. Many papers [20-25] considered the existence of solutions of Fredholm integral equations in the Hölder spaces using fixed point theorems. However, few papers [26, 27] considered the existence of solutions of boundary value problem for fractional differential equations, where the fractional order is constant order.

In [26], authors proved the existence and uniqueness of solutions for the following fractional boundary value problem in the Hölder space

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = \lambda f(t, u(t)), \ \ t \in [0,1], \\ u(0) = \gamma I_{0+}^{\rho} u(\eta) = \gamma \int_0^{\eta} \frac{(\eta - s)^{\rho - 1}}{\Gamma(\rho)} u(s) ds \end{cases} \tag{8}$$

where ${}^C D_{0+}^{\alpha}$ denotes the Caputo fractional derivative [1], $0 < \alpha \le 1$, $0 < \eta < 1$ and $\lambda, \gamma, \rho \in \mathbb{R}$.

In [27], authors studied the existence of positive solutions to the following fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), (Hu)(t)) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{9}$$

where $\alpha > 2$, $n - 1 < \alpha \le n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2, \cdots)$, $\triangle - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha - 1} > 0$, where $\triangle = (\alpha - 1)(\alpha - 2) \cdots (\alpha - i)$, H is an operator applying $C[0, 1]$ into itself satisfying certain assumptions and D_{0+}^{α} denotes the Riemann-liouville fractional derivative.

To the best of our knowledge, few papers studied the existence of solutions to boundary value problems of history-state-based VO fractional differential equations in the Hölder space. Inspired by the above excellent work, in this paper, we consider the existence of solution to the following boundary value problem for fractional differential equation involving Riemann-Liouville type history-state-based VO fractional derivative

$$\begin{cases} D_{0+}^{\alpha}x(t)+h(t)f(x, D_{0+}^{\alpha_1}x, D_{0+}^{p(t)}x), \ 0<t<T, \\ x(0)=0, \ x(T)=0, \end{cases} \tag{10}$$

where $0 < \alpha_1 < 1 < \alpha < 2$, $0 < p(t) < 1$, p, h, f are all given real-valued functions, $D_{0+}^{\alpha_1}$ denotes the constant-order Riemann-Liouville fractional derivative given by (1), D_{0+}^{α} denotes the constant-order Riemann-Liouville fractional derivative [1]

$$D_{0+}^{\alpha}x(t)=\frac{d^2}{dt^2}I_{0+}^{2-\alpha}x(t)=\frac{d^2}{dt^2}\int_0^t\frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}x(s)ds, \tag{11}$$

and $I_{0+}^{2-\alpha}$, $I_{0+}^{1-\alpha_1}$ are both the Riemann-Liouville fractional integrals (refer to [1]). $D_{0+}^{p(t)}$ denotes the Riemann-Liouville type history-state-based VO fractional derivative given by (4).

The paper is organized as follows. In section 2, we present some basic notations, propositions and lemmas, which will play an important role in obtaining the existence of solution to the boundary value problem (10). In section 3, main results are stated based on our previous analysis, then two examples are given to substantiate the theoretical results.

2. Preliminary

In this section, we introduce some basic notations, propositions and lemmas which are used throughout this paper.

Let $[0, b]$ be a finite interval. I_{0+}^{δ}, I_{0+}^{γ} and D_{0+}^{δ} denote the Riemann-Liouville fractional integral and derivative [1] respectively. $I_{0+}^{1-p(t)}$ is the Riemann-Liouville type history-state-based VO fractional integral given by (2).

Proposition 2.1 ([1] Lemma 2.9). If $\gamma > 0$, $\delta > 0$, then the equality $I_{0+}^{\gamma}I_{0+}^{\delta}f(t) = I_{0+}^{\delta}I_{0+}^{\gamma}f(t) = I_{0+}^{\gamma+\delta}f(t)$ holds at any point of $t \in [0, b]$ for $f \in C[0, b]$.

Proposition 2.2 ([1] Lemma 2.9). If $\gamma > 0$, $f \in C[0, b]$, then the equality $D_{0+}^{\delta}I_{0+}^{\delta}f(t) = f(t)$ holds at any point of $t \in [0, b]$.

Proposition 2.3 ([1] Lemma 2.9). If $f \in C[0, b]$ and $D_{0+}^{\delta}f \in C[0, b]$, then

$$I_{0+}^{\delta}D_{0+}^{\delta}f(t)=f(t)+c_1t^{\delta-1}+c_2t^{\delta-2}+\cdots+c_nt^{\delta-n},$$

where $n - 1 < \delta \leq n$, $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

Let $H_\eta[0, T](0 < \eta < 1)$ be the space of functions $x(t)$ such that $x(t)$ is Hölder continuous with index η on the interval $[0, T]$ with respect to the norm

$$\|x\|_\eta=|x(0)|+\sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}}\frac{|x(t_1)-x(t_2)|}{|t_1-t_2|^\eta}, \ x\in H_\eta[0,T].$$

In [28], $(H_\eta[0, T], \|\cdot\|_\eta)$ is a Banach space.

Lemma 2.1 ([28]). For $x \in H_\eta[0, T]$, the following inequality is satisfied

$$\max_{0\le t\le T}|x(t)|\le \max\{1,T^\eta\}\|x\|_\eta.$$

Lemma 2.2 ([28]). For $0 < \gamma < \eta < 1$, we have

$$H_\eta[0,T] \subset H_\gamma[0,T] \subset C[0,T].$$

Moreover, for $x \in H_\gamma[0, T]$, the following inequality is satisfied

$$\| x \|_\gamma \le \max\{1, T^{\eta-\gamma}\} \| x \|_\eta .$$

Lemma 2.3 ([27]). Suppose that $0 < \gamma < \eta < 1$ and that Ω_η is a bounded subset in $H_\eta[0, T]$ (this means that $\|x\|_\eta \le M$ for certain constant $M > 0$, for any $x \in \Omega_\eta$), then Ω_η is a relatively compact subset of $H_\gamma[0, T]$.

We assume that

(A_1) $0 < \alpha_1 < \alpha - 1 < 1 < \alpha < 2$, $p \in C^1[0,T]$, $p_* = \min_{0 \le t \le T} p(t), p^* = \max_{0 \le t \le T} p(t)$ and $0 < p_* \le p^* < \alpha_1$.

Lemma 2.4 ([29]). If $0 \le \gamma \le 1$ and $0 < a < b$, then

$$b^\gamma - a^\gamma \le (b-a)^\gamma .$$

Lemma 2.5 ([30, 31]). If $0 < \beta^* < \eta < 1$, then

$$\int_0^\infty t^\eta \left[t^{-1-\beta^*} - (t+1)^{-1-\beta^*} \right] dt < \infty.$$

Lemma 2.6 Assume that (A_1) holds and $x \in H_\beta[0, T]$ $(p^* < \beta < 1)$ with $x(0) = 0$, then $I_{0+}^{1-p(t)}x(t) \in C^1(0, T] \cap C[0, T]$ and

$$\frac{d}{dt} I_{0+}^{1-p(t)} x(t) = \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} \left[-p'(s)\ln(t-s) + \frac{p(s)-p(t)}{t-s} \right] ds$$

$$+ \int_0^t \left[\frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right] (t-s)^{-p(s)-1} ds + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}. \tag{12}$$

Proof. We take $\alpha(t) = 1 - p(t)$. Since $\min_{0 \le t \le T}(1 - p(t)) = 1 - p^*$, then $\beta + 1 - p^* > 1$ from $\beta > p^*$. According to Lemma 2.2 in [32], $I_{0+}^{1-p(t)}x(t) \in C[0, T]$. According to Theorem 3.3 in [29], we obtain that (12) holds. But in [29], the detailed proof of $\frac{d}{dt} I_{0+}^{1-p(t)} x(t) \in C(0, T]$ is not presented.

Next, we prove $\frac{d}{dt} I_{0+}^{1-p(t)} x(t) \in C(0, T]$.

For $0 < t \le T$, we write

$$\psi(t) = \frac{d}{dt} I_{0+}^{1-p(t)} x(t)$$

$$= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} (-p'(s)) \ln(t-s) ds$$

$$+ \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-1-p(s)} (p(s) - p(t)) ds$$

$$+\int_0^t \left[\frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))}\right](t-s)^{-1-p(s)} ds + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}$$

$$= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-p(t-\tau)} (-p'(t-\tau)) \ln \tau d\tau$$

$$+ \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-1-p(t-\tau)} [p(t-\tau) - p(t)] d\tau$$

$$+\int_0^t \left[\frac{x(t-\tau)}{\Gamma(-p(t-\tau))} - \frac{x(t)}{\Gamma(-p(t))}\right] \tau^{-1-p(t-\tau)} d\tau + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}$$

$$= \psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t). \tag{13}$$

Since $1 - \beta > 0$, there exists $Q_1 > 0$ such that

$$\left| t^{1-\beta} \ln t \right| \le Q_1, \ 0 < t \le T.$$

We write $Q_2 = \max\{1, T\}$, $Q_3 = \max\limits_{0 \le t \le T} \left| \frac{1}{\Gamma(1-p(t))} \right|$, $Q_4 = \max\limits_{0 \le t \le T} |p'(t)|$, $Q_5 = \max\limits_{1-p^* \le \omega \le 1-p_*} \left| \left(\frac{1}{\Gamma(\omega)}\right)' \right|$, $Q_6 = \max\limits_{0 \le t \le T} \left| \frac{1}{\Gamma(-p(t))} \right|$, $Q_7 = \max\limits_{p_* \le \vartheta \le p^*} \left| \left(\frac{1}{\Gamma(-\vartheta)}\right)' \right|$, $Q_8 = \int_0^\infty t^\beta \left[t^{-1-p^*} - (t+1)^{-1-p^*} \right] dt$, $Q = Q(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \|x\|_\beta, \beta, p^*)$. Here Q is a varying constant and is related to $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \|x\|_\beta, \beta$ and p^*.

For any $h \in \mathbb{R}$ such that $t + h \in (0, T]$, without loss of generality, we assume that $h > 0$, then

$$\left| \psi_1(t+h) - \psi_1(t) \right|$$

$$= \left| \int_0^t \left[\frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) - \frac{x(t)}{\Gamma(1-p(t))} \tau^{-p(t-\tau)} p'(t-\tau) \right] \ln \tau d\tau \right.$$

$$\left. + \int_t^{t+h} \frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau d\tau \right|$$

$$\le \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} - \frac{x(t+h)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau \right| d\tau$$

$$+ \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t))} - \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau \right| d\tau$$

$$+ \int_0^t \left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| p'(t+h-\tau) \ln \tau \right| \left| \tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)} \right| d\tau$$

$$+\int_0^t \left|\frac{x(t)}{\Gamma(1-p(t))}\right| \left|\tau^{-p(t-\tau)} \ln \tau\right| \left|p'(t+h-\tau) - p'(t-\tau)\right| d\tau$$

$$+\int_t^{t+h} \left|\frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau\right| d\tau$$

$$= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}. \tag{14}$$

In (14), we have

$$\left|\tau^{-p(t+h-\tau)} \ln \tau\right| = \left|\tau^{p^*-p(t+h-\tau)} \tau^{\beta-p^*-1} \tau^{1-\beta} \ln \tau\right| \le Q\tau^{\beta-p^*-1},\ 0 < \tau \le t, \tag{15}$$

$$\left|\frac{1}{\Gamma(1-p(t+h))} - \frac{1}{\Gamma(1-p(t))}\right| = \left|\left(\frac{1}{\Gamma(\omega)}\right)'\Bigg|_{\omega=1-p(\zeta)}\right| \left|p'(\zeta)\right| h \le Qh, \tag{16}$$

$$|x(t+h) - x(t)| \le \|x\|_\beta h^\beta \le Qh^\beta \tag{17}$$

and

$$\left|\left(\tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)}\right) \ln \tau\right| = \left|\tau^{p^*-p(t+h-\tau)} - \tau^{p^*-p(t-\tau)}\right| \left|\tau^{1-\beta} \ln \tau\right| \tau^{\beta-p^*-1}$$

$$= h\left|\tau^{p^*-p(\xi)} p'(\xi) \tau^{\frac{1-\beta}{2}} \ln \tau\right| \left|\tau^{\frac{1-\beta}{2}} \ln \tau\right| \tau^{\beta-p^*-1}$$

$$\le Qh\tau^{\beta-p^*-1}, \tag{18}$$

where $t < \zeta < t + h$ and $t - \tau < \xi < t - \tau + h$. In virtue of (15), (16), (17) and (18), we have

$$I_{11} \le Qh\int_0^t \tau^{\beta-p^*-1} d\tau \le Qht^{\beta-p^*} \le Qh, \tag{19}$$

$$I_{12} \le Qh^\beta \int_0^t \tau^{\beta-p^*-1} d\tau \le Qh^\beta t^{\beta-p^*} \le Qh^\beta, \tag{20}$$

$$I_{13} \le Qh\int_0^t \tau^{\beta-p^*-1} d\tau \le Qht^{\beta-p^*} \le Qh, \tag{21}$$

$$I_{15} \le Q\int_t^{t+h} \tau^{\beta-p^*-1} d\tau = \frac{Q}{\beta-p^*}\left[(t+h)^{\beta-p^*} - t^{\beta-p^*}\right] \le Qh^{\beta-p^*}. \tag{22}$$

According to (19), (20), (21) and (22), we obtain that when $h \to 0$, then

$$I_{11} \to 0,\ I_{12} \to 0,\ I_{13} \to 0,\ I_{15} \to 0. \tag{23}$$

For I_{14}, by $p'(t) \in C[0, T]$, we have

$$\left|p'(t+h-\tau) - p'(t-\tau)\right| \to 0\ (h \to 0),$$

$$\left|\frac{x(t)}{\Gamma(1-p(t))}\right|\left|\tau^{-p(t-\tau)}\ln\tau\right|\left|p'(t+h-\tau) - p'(t-\tau)\right| \le Q\tau^{\beta-p^*-1},$$

and

$$Q\int_0^t \tau^{\beta-p^*-1} d\tau \le Qt^{\beta-p^*}.$$

By Lebesgue dominated convergence theorem, we have

$$I_{14} \to 0\ (h \to 0). \tag{24}$$

From (23), (24) and (14), we have

$$\left|\psi_1(t+h) - \psi_1(t)\right| \to 0\ (h \to 0). \tag{25}$$

For $0 < t \le T$,

$$\psi_2(t) = \frac{x(t)}{\Gamma(1-p(t))}\int_0^t \tau^{-1-p(t-\tau)}[p(t-\tau) - p(t)]d\tau$$

$$= \frac{x(t)}{\Gamma(1-p(t))}\int_0^t \tau^{-p(t-\tau)}[-p'(t-\tau+\theta\tau)]d\tau,\ 0 < \theta < 1. \tag{26}$$

Form (26), we have

$$\begin{aligned}
&\left|\psi_2(t+h) - \psi_2(t)\right| \\
&\le \int_0^t \left|\frac{x(t+h)}{\Gamma(1-p(t+h))} - \frac{x(t+h)}{\Gamma(1-p(t))}\right|\left|\tau^{-p(t+h-\tau)}p'(t+h-\tau+\theta\tau)\right|d\tau \\
&+\int_0^t \left|\frac{x(t+h)}{\Gamma(1-p(t))} - \frac{x(t)}{\Gamma(1-p(t))}\right|\left|\tau^{-p(t+h-\tau)}p'(t+h-\tau+\theta\tau)\right|d\tau \\
&+\int_0^t \left|\frac{x(t)}{\Gamma(1-p(t))}\right|\left|p'(t+h-\tau+\theta\tau)\right|\left|\tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)}\right|d\tau \\
&+\int_0^t \left|\frac{x(t)}{\Gamma(1-p(t))}\right|\tau^{-p(t-\tau)}\left|p'(t+h-\tau+\theta\tau) - p'(t-\tau+\theta\tau)\right|d\tau
\end{aligned}$$

$$+\int_t^{t+h}\left|\frac{x(t+h)}{\Gamma(1-p(t+h))}\tau^{-p(t+h-\tau)}p'(t+h-\tau+\theta\tau)\right|d\tau.$$

In a similar fashion to $|\psi_1(t+h)-\psi_1(t)|$, we have

$$\left|\psi_2(t+h)-\psi_2(t)\right|\to 0\ (h\to 0). \tag{27}$$

For $0<t\le T$, we have

$$\begin{aligned}
&\left|\psi_3(t+h)-\psi_3(t)\right| \\
&=\left|\int_0^{t+h}\left[\frac{x(t+h-\tau)}{\Gamma(-p(t+h-\tau))}-\frac{x(t+h)}{\Gamma(-p(t+h))}\right]\tau^{-1-p(t+h-\tau)}d\tau\right. \\
&\left.-\int_0^t\left[\frac{x(t-\tau)}{\Gamma(-p(t-\tau))}-\frac{x(t)}{\Gamma(-p(t))}\right]\tau^{-1-p(t-\tau)}d\tau\right| \\
&=\left|\int_{-h}^t\left[\frac{x(t-s)}{\Gamma(-p(t-s))}-\frac{x(t+h)}{\Gamma(-p(t+h))}\right](s+h)^{-1-p(t-s)}ds\right. \\
&\left.-\int_0^t\left[\frac{x(t-s)}{\Gamma(-p(t-s))}-\frac{x(t)}{\Gamma(-p(t))}\right]s^{-1-p(t-s)}ds\right| \\
&=\left|\int_0^t\left[\frac{x(t-s)}{\Gamma(-p(t-s))}-\frac{x(t)}{\Gamma(-p(t))}\right]\left[(s+h)^{-1-p(t-s)}-s^{-1-p(t-s)}\right]ds\right. \\
&+\int_0^t\left[\frac{x(t)}{\Gamma(-p(t))}-\frac{x(t+h)}{\Gamma(-p(t+h))}\right](s+h)^{-1-p(t-s)}ds \\
&\left.+\int_{-h}^0\left[\frac{x(t-s)}{\Gamma(-p(t-s))}-\frac{x(t+h)}{\Gamma(-p(t+h))}\right](s+h)^{-1-p(t-s)}ds\right| \\
&=\left|I_{31}+I_{32}+I_{33}\right| \\
&\le |I_{31}|+|I_{32}|+|I_{33}|.
\end{aligned} \tag{28}$$

In (28), since

$$\begin{aligned}
&\left|(s+h)^{-1-p(t-s)}-s^{-1-p(t-s)}\right| \\
&=s^{-1-p(t-s)}-(s+h)^{-1-p(t-s)}
\end{aligned}$$

$$= s^{p^*-p(t-s)} s^{-1-p^*} - (s+h)^{p^*-p(t-s)} (s+h)^{-1-p^*}$$

$$\leq s^{p^*-p(t-s)} (s^{-1-p^*} - (s+h)^{-1-p^*})$$

$$\leq Q(s^{-1-p^*} - (s+h)^{-1-p^*})$$

and

$$Q\int_0^t s^\beta (s^{-1-p^*} - (s+h)^{-1-p^*}) ds$$

$$= Qh^{\beta-p^*} \int_0^{\frac{t}{h}} \tau^\beta (\tau^{-1-p^*} - (\tau+1)^{-1-p^*}) d\tau$$

$$\leq Qh^{\beta-p^*} \int_0^{\infty} \tau^\beta (\tau^{-1-p^*} - (\tau+1)^{-1-p^*}) d\tau$$

$$\leq Qh^{\beta-p^*},$$

where the last "≤" is obtained by Lemma 2.5, then

$$|I_{31}| \leq \int_0^t \left| \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right| \left| (s+h)^{-1-p(t-s)} - s^{-1-p(t-s)} \right| ds$$

$$\leq Q\int_0^t \left[\left| \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t-s))} \right| + \left| \frac{x(t)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right| \right]$$

$$(s^{-1-p^*} - (s+h)^{-1-p^*}) ds$$

$$\leq Q\int_0^t s^\beta (s^{-1-p^*} - (s+h)^{-1-p^*}) ds$$

$$\leq Qh^{\beta-p^*}, \tag{29}$$

$$|I_{32}| \leq \int_0^t \left[\left| \frac{x(t)}{\Gamma(-p(t))} - \frac{x(t+h)}{\Gamma(-p(t))} \right| + \left| \frac{x(t+h)}{\Gamma(-p(t))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right| \right]$$

$$(s+h)^{p^*-p(t-s)} (s+h)^{-p^*-1} ds$$

$$\leq Q\int_0^t (h^\beta + h)(s+h)^{-1-p^*} ds$$

$$\leq Qh^{\beta}\int_0^{\infty}(s+h)^{-1-p^*}ds$$

$$\leq Qh^{\beta-p^*} \tag{30}$$

and

$$|I_{33}|\leq\int_{-h}^{0}\left[\left|\frac{x(t-s)}{\Gamma(-p(t-s))}-\frac{x(t+h)}{\Gamma(-p(t-s))}\right|+\left|\frac{x(t+h)}{\Gamma(-p(t-s))}-\frac{x(t+h)}{\Gamma(-p(t+h))}\right|\right]$$

$$(s+h)^{p^*-p(t-s)}(s+h)^{-p^*-1}ds$$

$$\leq Q\int_{-h}^{0}\left[(h+s)^{\beta}+(h+s)\right](s+h)^{-1-p^*}ds$$

$$\leq Q\int_{-h}^{0}(s+h)^{\beta-p^*-1}ds$$

$$\leq Qh^{\beta-p^*}. \tag{31}$$

From (28), (29), (30) and (31), we have

$$\left|\psi_3(t+h)-\psi_3(t)\right|\to 0\ (h\to 0). \tag{32}$$

For $0<t\leq T$, we have

$$\left|\psi_4(t+h)-\psi_4(t)\right|$$

$$=\left|\frac{x(t+h)}{\Gamma(1-p(t+h))}(t+h)^{-p(0)}-\frac{x(t)}{\Gamma(1-p(t))}t^{-p(0)}\right|$$

$$\leq\left|\frac{x(t+h)-x(0)}{\Gamma(1-p(t+h))}-\frac{x(t+h)-x(0)}{\Gamma(1-p(t))}\right|(t+h)^{-p(0)}$$

$$+\left|\frac{x(t+h)}{\Gamma(1-p(t))}-\frac{x(t)}{\Gamma(1-p(t))}\right|(t+h)^{-p(0)}$$

$$+\left|\frac{x(t)-x(0)}{\Gamma(1-p(t))}\right|\left|(t+h)^{-p(0)}-t^{-p(0)}\right|$$

$$\leq Qh(t+h)^{\beta-p(0)}+Qh^{\beta}(t+h)^{-p(0)}+Qt^{\beta}\left|(t+h)^{-p(0)}-t^{-p(0)}\right|$$

$$\leq Qh + Qh^{\beta - p(0)} + Qt^{\beta}\left[t^{-p(0)} - (t+h)^{-p(0)}\right] \tag{33}$$

For $0 < t \leq h$, we have

$$t^{\beta}\left[t^{-p(0)} - (t+h)^{-p(0)}\right] \leq t^{\beta - p(0)} \leq h^{\beta - p(0)}. \tag{34}$$

For $t > h$, we have

$$\begin{aligned}
&t^{\beta}\left[t^{-p(0)} - (t+h)^{-p(0)}\right] \\
&\leq t^{\beta - p(0)}\left[1 - \left(1 + \frac{h}{t}\right)^{-p(0)}\right] \\
&\leq t^{\beta - p(0)}\left[1 - \left(1 + (-p(0))\frac{h}{t} + O\left(\frac{h^2}{t^2}\right)\right)\right] \\
&\leq p(0)t^{\beta - p(0) - 1}h \\
&\leq p(0)h^{\beta - p(0)}.
\end{aligned} \tag{35}$$

Form (33), (34) and (35), we have

$$\left|\psi_4(t+h) - \psi_4(t)\right| \to 0 \ (h \to 0). \tag{36}$$

By (13), (25), (27), (32) and (36), we conclude that $\psi(t) \in C(0, T]$, that is to say $I_{0+}^{1-p(t)}x(t) \in C^1(0, T]$. □

Remark 2.1 In Lemma 2.6, we have

$$\lim_{t \to 0^+} \psi(t) = \lim_{t \to 0^+} \frac{d}{dt} I_{0+}^{1-p(t)}x(t) = 0. \tag{37}$$

That is to say $t = 0$ is a removable discontinuity point of function $\frac{d}{dt} I_{0+}^{1-p(t)}x(t)$.

Proof. The proof of Remark 2.1 is similar to Lemma 2.6. □

Remark 2.2 In Lemma 2.6, if $p(t) \equiv \alpha_1$ for $t \in [0, T]$, $x(t) \in H_{\beta}[0, T]$ ($\alpha_1 < \beta < 1$) with $x(0) = 0$, then $I_{0+}^{1-\alpha_1} x(t) \in C^1(0, T] \cap C[0, T]$ and

$$\frac{d}{dt} I_{0+}^{1-\alpha_1} x(t) = \int_0^t \frac{x(s) - x(t)}{\Gamma(-\alpha_1)}(t-s)^{-\alpha_1 - 1} ds + \frac{x(t)}{\Gamma(1-\alpha_1)} t^{-\alpha_1}.$$

3. Main results

In this section we devote to dealing with the existence and uniqueness of solution to the boundary value problem (10).

We assume that

(A_2) $h \in L(0, T)$;

(A_3) h satisfies $\sup_{0\le t\le T} I_{0+}^{\alpha-1} | t^{-\alpha_1} h(t) | < +\infty$;

(A_4) $f: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function;

(A_5) there exists positive constants L_i $(i = 1, 2, 3)$ such that

$$| f(x_1(t), y_1(t), z_1(t)) - f(x_2(t), y_2(t), z_2(t)) |$$

$$\le L_1 | x_1 - x_2 | + L_2 | y_1 - y_2 | + L_3 | z_1 - z_2 |,\ t \in [0,T],\ x_j(t),\ y_j(t),\ z_j(t) \in \mathbb{R},\ j = 1,2; \tag{38}$$

(A_6) there exists constants $M_i > 0$, $0 < \mu_i < 1$ $(i = 1, 2, 3)$ such that

$$| f(x(t), y(t), z(t)) | \le M_1 | x(t) |^{\mu_1} + M_2 | y(t) |^{\mu_2} + M_3 | z(t) |^{\mu_3},\ x(t),\ y(t),\ z(t) \in \mathbb{R}. \tag{39}$$

Lemma 3.1 Assume that (A_1), (A_2) and (A_4) hold. If $x \in H_\beta[0, T]$ with $\alpha_1 < \beta < 1$ is a solution of boundary value problem (10), then if and only if $x(t)$ must be a solution of the integral equation

$$x(t) = \int_0^T G(t,s) h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3} \omega_i(x(s)) \right) ds,\ 0 \le t \le T, \tag{40}$$

where

$$G(t,s) = \begin{cases} \dfrac{T^{1-\alpha} t^{\alpha-1} (T-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le T, \\ \dfrac{T^{1-\alpha} t^{\alpha-1} (T-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le T, \end{cases} \tag{41}$$

$$\omega_1(x(t)) = \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} \left[-p'(s) \ln(t-s) + \frac{p(s)-p(t)}{t-s} \right] ds, \tag{42}$$

$$\omega_2(x(t)) = \int_0^t \left[\frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right] (t-s)^{-p(s)-1} ds, \tag{43}$$

$$\omega_3(x(t)) = \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}, \tag{44}$$

$$g_1(x(t)) = \int_0^t \frac{x(s)-x(t)}{\Gamma(-\alpha_1)} (t-s)^{-\alpha_1-1} ds,\ g_2(x(t)) = \frac{x(t)}{\Gamma(1-\alpha_1)} t^{-\alpha_1}. \tag{45}$$

Proof. If $x \in H_\beta[0, T]$ is a solution of boundary value problem (10), then $x(t)$ satisfies $x(0) = x(T) = 0$ and

$$D_{0+}^{\alpha} x(t) + h(t) f\left(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t) \right) = 0,\ 0 < t < T. \tag{46}$$

Applying operator I_{0+}^{α} to both sides of equation (46), from Proposition 2.3, we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds.$$

As a result of $x \in H_\beta[0, T] \subset C[0, \mathrm{T}]$ and $x(0) = x(T) = 0$, we get that $c_2 = 0$ and

$$c_1 = \frac{T^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds.$$

Then we obtain that

$$x(t) = \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds,$$

$$= \int_0^T G(t,s) h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds, \ 0 \le t \le T,$$

where $G(t, s)$ is defined by (41). From Lemma 2.6 and Remark 2.2, it holds that

$$D_{0+}^{\alpha_1} x(s) = g_1(x(s)) + g_2(x(s)), \ D_{0+}^{p(s)} x(s) = \sum_{i=1}^{3} \omega_i(x(s)), \ 0 \le s \le T.$$

Therefore, we have

$$x(t) = \int_0^T G(t,s) h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3} \omega_i(x(s)) \right) ds, \ 0 \le t \le T,$$

which implies that $x(t)$ is a solution of integral equation (40).

Conversely, if $x \in H_\beta[0, T]$ is a solution of integral equation (40), then

$$x(t) = \int_0^T G(t,s) h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3} \omega_i(x(s)) \right) ds, \ 0 \le t \le T.$$

Obviously, $x(0) = 0$ and $x(T) = 0$. From Lemma 2.6 and Remark 2.2, we have

$$g_1(x(s)) + g_2(x(s)) = D_{0+}^{\alpha_1} x(s), \ \sum_{i=1}^{3} \omega_i(x(s)) = D_{0+}^{p(s)} x(s), \ 0 \le s \le T.$$

Therefore

$$x(t) = \int_0^T G(t,s) h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds$$

$$= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds$$

$$-I_{0+}^{\alpha}\left(h(t) f\left(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t)\right)\right), \ 0 \le t \le T. \tag{47}$$

Applying D_{0+}^{α} to both sides of (47), by Proposition 2.2, we get

$$D_{0+}^{\alpha} x(t) + h(t) f\left(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t)\right) = 0, \ 0 < t < T,$$

together with $x(0) = x(T) = 0$, which implies $x \in H_\beta[0, T]$ is a solution of boundary value problem (10). □

In the following analysis, for convenience, we let

$$a_1 = \max_{0\le t\le T} \left| \frac{1}{\Gamma(1-p(t))} \right|, \ a_2 = \max_{0\le t\le T} | p'(t) |, \ a_3 = \max_{0\le t\le T} \left| \frac{1}{\Gamma(-p(t))} \right|, \tag{48}$$

$$a_4 = \max_{-p^* \le w \le -p_*} \left| \left(\frac{1}{\Gamma(w)} \right)' \right|, \ a_5 = \max\{1, T^{\beta}\}, \tag{49}$$

$$B_1 = a_1 a_2 a_5 \max\{1, T^{p^*-p_*}\} \left(\frac{KT^{1-\beta}}{1-\beta} + \frac{T^{1-p^*}}{1-p^*} \right), \tag{50}$$

$$B_2 = \max\{1, T^{p^*-p_*}\} \left[\frac{a_3 T^{\beta-p^*}}{\beta - p^*} + \frac{a_2 a_4 a_5 T^{1-p^*}}{1-p^*} \right], \tag{51}$$

$$L = \max\left\{ L_1 a_5 + \frac{L_2 T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|} + L_3(B_1 + B_2), \ \frac{L_2 a_5}{\Gamma(1-\alpha_1)}, \ L_3 a_1 a_5 \right\}, \tag{52}$$

$$L_{1,h} = \max\left\{ \frac{\int_0^T |h(s)| ds}{\Gamma(\alpha)}, \ \frac{\int_0^T s^{-\alpha_1} | h(s) | ds}{\Gamma(\alpha)}, \ \frac{\int_0^T s^{-p(0)} | h(s) | ds}{\Gamma(\alpha)} \right\}, \tag{53}$$

$$L_{2,h} = \max\left\{ \sup_{0\le t\le T} I_{0+}^{\alpha-1} | h(t) |, \ \sup_{0\le t\le T} I_{0+}^{\alpha-1} \left| t^{-\alpha_1} h(t) \right|, \ \sup_{0\le t\le T} I_{0+}^{\alpha-1} \left| t^{-p(0)} h(t) \right| \right\}, \tag{54}$$

where $p^* < \alpha_1 < \beta < 1$, $K > 0$ is a constant satisfying

$$| t^{\beta-p^*} \ln t | < K, \ 0 < t \le T. \tag{55}$$

Theorem 3.1 Suppose (A_1), (A_2), (A_3), (A_4) and (A_5) hold. Then the boundary value problem (10) exists a unique solution $x \in H_\beta[0, T]$ with $\alpha_1 < \beta < \alpha - 1$ provided if

$$3LL_{1,h}T^{\alpha-1-\beta}+3LL_{2,h}T^{1-\beta}<1. \tag{56}$$

Proof. According to Lemma 3.1, it is sufficient to consider the existence of fixed point of operator A defined by

$$Ax(t)=\int_0^T G(t,s)h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds,\ 0\le t\le T. \tag{57}$$

Firstly, we show that $A:H_\beta[0,T]\to H_\beta[0,T]$ with $\alpha_1<\beta<\alpha-1$. In fact, for $x\in H_\beta[0,T]$, from (48)-(51) and (55), we have

$$|\omega_1(x(t))|\le\frac{\max\limits_{0\le t\le T}|x(t)|}{\Gamma(1-p(t))}\int_0^t\Big[(t-s)^{-(\beta-p^*)-p(s)}|p'(s)||(t-s)^{\beta-p^*}\ln(t-s)|$$

$$+(t-s)^{-p(s)}|p'(\xi)|\Big]ds$$

$$\le a_1a_2a_5\|x\|_\beta\max\{1,T^{p^*-p*}\}\left[K\int_0^t(t-s)^{-\beta}ds+\int_0^t(t-s)^{-p^*}ds\right]$$

$$=a_1a_2a_5\|x\|_\beta\max\{1,T^{p^*-p*}\}\left(\frac{Kt^{1-\beta}}{1-\beta}+\frac{t^{1-p^*}}{1-p^*}\right)$$

$$\le a_1a_2a_5\|x\|_\beta\max\{1,T^{p^*-p*}\}\left(\frac{KT^{1-\beta}}{1-\beta}+\frac{T^{1-p^*}}{1-p^*}\right)$$

$$=B_1\|x\|_\beta,\ 0\le s<\xi<t\le T, \tag{58}$$

$$|\omega_2(x(t))|\le\int_0^t\left|\frac{x(s)}{\Gamma(-p(s))}-\frac{x(t)}{\Gamma(-p(t))}\right|(t-s)^{-p(s)-1}ds$$

$$\le\int_0^t\left[\left|\frac{x(s)}{\Gamma(-p(s))}-\frac{x(t)}{\Gamma(-p(s))}\right|+\left|\frac{x(t)}{\Gamma(-p(s))}-\frac{x(t)}{\Gamma(-p(t))}\right|\right](t-s)^{-p(s)-1}ds$$

$$\le a_3\|x\|_\beta\int_0^t(t-s)^{\beta-p(s)-1}ds+a_5\|x\|_\beta\int_0^t(t-s)^{-p(s)}\left|\left(\frac{1}{\Gamma(\omega)}\right)'_\omega\right||p'(\theta)|ds$$

$$\le a_3\|x\|_\beta\int_0^t(t-s)^{\beta-p(s)-1}ds+a_2a_4a_5\|x\|_\beta\int_0^t(t-s)^{-p(s)}ds$$

$$=a_3\|x\|_\beta\int_0^t(t-s)^{p^*-p(s)}(t-s)^{\beta-p^*-1}ds+a_2a_4a_5\|x\|_\beta\int_0^t(t-s)^{p^*-p(s)}(t-s)^{-p^*}ds$$

$$\leq \| x\|_\beta \max\{1, T^{p^*-p_*}\}\left[a_3\int_0^t (t-s)^{\beta-p^*-1}ds + a_2a_4a_5\int_0^t (t-s)^{-p^*}ds\right]$$

$$\leq \| x\|_\beta \max\{1, T^{p^*-p_*}\}\left[\frac{a_3T^{\beta-p^*}}{\beta-p^*} + \frac{a_2a_4a_5T^{1-p^*}}{1-p^*}\right]$$

$$= B_2\| x\|_\beta, \ -p^* \leq \omega \leq -p_*, \ 0 \leq s < \theta < t \leq T, \tag{59}$$

$$|\omega_3(x(t))| \leq \frac{|x(t)|}{\Gamma(1-p(t))}t^{-p(0)} \leq a_1a_5\| x\|_\beta t^{-p(0)}, \tag{60}$$

$$|g_1(x(t))| \leq \int_0^t \frac{|x(s)-x(t)|}{|\Gamma(-\alpha_1)|}(t-s)^{-\alpha_1-1}ds$$

$$\leq \frac{\| x\|_\beta}{|\Gamma(-\alpha_1)|}\int_0^t (t-s)^{\beta-\alpha_1-1}ds$$

$$\leq \frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|}\| x\|_\beta, \tag{61}$$

and

$$|g_2(x(t))| \leq \frac{a_5t^{-\alpha_1}}{\Gamma(1-\alpha_1)}\| x\|_\beta. \tag{62}$$

From (38), (52) and (58)-(62), we have

$$\left|f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^{3}\omega_i(x(t))\right)\right|$$

$$= \left|f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^{3}\omega_i(x(t))\right) - f(0,0,0) + f(0,0,0)\right|$$

$$\leq L_1|x(t)| + L_2|g_1(x(t)) + g_2(x(t))| + L_3\left|\sum_{i=1}^{3}\omega_i(x(t))\right| + |f(0,0,0)|$$

$$\leq L_1a_5\| x\|_\beta + \frac{L_2T^{\beta-\alpha_1}\| x\|_\beta}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|} + \frac{L_2a_5t^{-\alpha_1}\| x\|_\beta}{\Gamma(1-\alpha_1)} + L_3(B_1+B_2)\| x\|_\beta$$

$$+L_3a_1a_5t^{-p(0)}\| x\|_\beta + |f(0,0,0)|$$

$$\leq (L + Lt^{-\alpha_1} + Lt^{-p(0)})\| x\|_\beta + |f(0,0,0)|. \tag{63}$$

From (41), (57) and Proposition 2.1 , we have

$$Ax(t)=\frac{T^{1-\alpha}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds$$

$$-I_{0+}^{\alpha}\left[h(t)f\left(x(t),g_1(x(t))+g_2(x(t)),\sum_{i=1}^{3}\omega_i(x(t))\right)\right]$$

$$=\frac{T^{1-\alpha}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds$$

$$-I_{0+}^{1}I_{0+}^{\alpha-1}\left[h(t)f\left(x(t),g_1(x(t))+g_2(x(t)),\sum_{i=1}^{3}\omega_i(x(t))\right)\right]$$

$$=\frac{T^{1-\alpha}t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds$$

$$-\int_0^t I_{0+}^{\alpha-1}\left[h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)\right]ds. \tag{64}$$

For $x \in H_\beta[0, T]$, $t_1, t_2 \in [0, T]$ with $t_1 \neq t_2$ (without loss of generality, we assume that $0 \leq t_1 < t_2 \leq T$), in view of (53), (54), (63) and (64) together with $t_2^\delta - t_1^\delta \leq (t_2 - t_1)^\delta$ for $t_2 > t_1 \geq 0$ and $0 < \delta \leq 1$, we have

$$\left|Ax(t_2)-Ax(t_1)\right|$$

$$=\left|\frac{T^{1-\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds\right.$$

$$\left.-\int_{t_1}^{t_2} I_{0+}^{\alpha-1}\left[h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)\right]ds\right|$$

$$\leq\frac{T^{1-\alpha}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\,|\,h(s)\,|\left|f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)\right|ds$$

$$+\int_{t_1}^{t_2}\left|I_{0+}^{\alpha-1}\left[h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)\right]\right|ds$$

$$\leq\frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^T |h(s)|\left[(L+Ls^{-\alpha_1}+Ls^{-p(0)})\|x\|_\beta+|f(0,0,0)|\right]ds$$

$$+\int_{t_1}^{t_2} I_{0+}^{\alpha-1}\left[|h(s)|\left((L+Ls^{-\alpha_1}+Ls^{-p(0)})\|x\|_\beta+|f(0,0,0)|\right)\right]ds$$

$$\leq (3LL_{1,h}\| x\|_\beta + | f(0,0,0)| L_{1,h})(t_2 - t_1)^{\alpha-1} + (3LL_{2,h}\| x\|_\beta + | f(0,0,0)| L_{2,h})(t_2 - t_1)$$

$$\leq [(3LL_{1,h}T^{\alpha-1-\beta} + 3LL_{2,h}T^{1-\beta})\| x\|_\beta + (L_{1,h}T^{\alpha-1-\beta} + L_{2,h}T^{1-\beta})| f(0,0,0)|](t_2 - t_1)^\beta. \tag{65}$$

The inequality (65) implies that $Ax(t) \in H_\beta[0, T]$. Then, $A : H_\beta[0, T] \to H_\beta[0, T]$.

Secondly, we prove that $A : H_\beta[0, T] \to H_\beta[0, T]$ is a contractive operator. For $x, y \in H_\beta[0, T]$, by the similar way as the first step, we get

$$| \omega_1(x(t)) - \omega_1(y(t))| \leq B_1\| x - y\|_\beta,$$

$$| \omega_2(x(t)) - \omega_2(y(t))\| \leq B_2\| x - y\|_\beta,$$

$$| \omega_3(x(t)) - \omega_3(y(t))| \leq a_1a_5t^{-p(0)}\| x - y\|_\beta,$$

$$| g_1(x(t)) - g_1(y(t))| \leq \frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|}\| x - y\|_\beta$$

and

$$| g_2(x(t)) - g_2(y(t))| \leq \frac{a_5t^{-\alpha_1}}{\Gamma(1-\alpha_1)}\| x - y\|_\beta.$$

From (38) and the obtained above inequalities, we obtain

$$\left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^{3}\omega_i(x(t))\right) - f\left(y(t), g_1(y(t)) + g_2(y(t)), \sum_{i=1}^{3}\omega_i(y(t))\right)\right|$$

$$\leq L_1 | x(t) - y(t)| + L_2 | g_1(x(t)) - g_1(y(t)) + g_2(x(t)) - g_2(y(t))|$$

$$+ L_3\left|\sum_{i=1}^{3}\omega_i(x(t)) - \sum_{i=1}^{3}\omega_i(y(t))\right|$$

$$\leq (L + Lt^{-\alpha_1} + Lt^{-p(0)})\| x - y\|_\beta. \tag{66}$$

From (66), we have

$$| (Ax - Ay)(t_2) - (Ax - Ay)(t_1)|$$

$$\leq \frac{(t_2 - t_1)^{\alpha-1}\| x - y\|_\beta}{\Gamma(\alpha)}\int_0^T |h(s)| (L + Ls^{-\alpha_1} + Ls^{-p(0)})ds$$

$$+ \| x - y\|_\beta\int_{t_1}^{t_2} I_0^{\alpha-1}\Big[| h(s)| (L + Ls^{-\alpha_1} + Ls^{-p(0)})\Big]ds$$

$$\le\left(3LL_{1,h}T^{\alpha-1-\beta}+3LL_{2,h}T^{1-\beta}\right)\| x-y\|_{\beta}(t_2-t_1)^{\beta}. \tag{67}$$

Obviously, $Ax(0) = 0$ and $Ay(0) = 0$. Therefore, from (67), we have

$$\| Ax-Ay\|_{\beta}=|(Ax-Ay)(0)|+\sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}}\frac{|(Ax-Ay)(t_2)-(Ax-Ay)(t_1)|}{|t_1-t_2|^{\beta}}$$

$$\le\left(3LL_{1,h}T^{\alpha-1-\beta}+3LL_{2,h}T^{1-\beta}\right)\| x-y\|_{\beta}.$$

We obtain that $A : H_{\beta}[0, T] \to H_{\beta}[0, T]$ is a contractive operator together with (56). Then A exists a unique fixed point $x^* \in H_{\beta}[0, T]$ according to Banach fixed point theorem. Lemma 3.1 indicates that the obtained unique fixed pointed $x \in H_{\beta}[0, T]$ is the solution of the boundary value problem (10). □

Theorem 3.2 Suppose that (A_1), (A_2), (A_3), (A_4) and (A_6) hold. Then the boundary value problem (10) exists a solution $x \in H_{\gamma}[0, T]$ with $\alpha_1 < \gamma < \beta < \alpha - 1$.

Proof. According to Lemma 3.1, it is sufficient to consider the existence of fixed point of operator A defined by

$$Ax(t)=\int_0^T G(t,s)h(s)f\left(x(s),g_1(x(s))+g_2(x(s)),\sum_{i=1}^{3}\omega_i(x(s))\right)ds,\ 0\le t\le T.$$

In the next analysis, we let

$$\bar{B}_2=\max\{1,T^{p^*-p_*}\}\left[\max\left\{\frac{a_3T^{\beta-p^*}}{\beta-p^*},\ \frac{a_3T^{\gamma-p^*}}{\gamma-p^*}\right\}+\frac{a_2a_4a_5T^{1-p^*}}{1-p^*}\right].$$

$$\bar{M}=\max\left\{M_1a_5^{\mu_1},\ M_2\left(\frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|}\right)^{\mu_2},\ M_2\left(\frac{T^{\gamma-\alpha_1}}{(\gamma-\alpha_1)|\Gamma(-\alpha_1)|}\right)^{\mu_2},\right.$$

$$\left.\frac{M_2a_5^{\mu_2}}{[\Gamma(1-\alpha_1)]^{\mu_2}},\ M_3(B_1+\bar{B}_2)^{\mu_3},\ M_3a_1^{\mu_3}a_5^{\mu_3}\right\}, \tag{68}$$

$$M_{1,h}=\max\left\{\frac{\int_0^T|h(s)|ds}{\Gamma(\alpha)},\ \frac{\int_0^T s^{-\alpha_1\mu_2}|h(s)|ds}{\Gamma(\alpha)},\ \frac{\int_0^T s^{-p(0)\mu_3}|h(s)|ds}{\Gamma(\alpha)}\right\}, \tag{69}$$

$$M_{2,h}=\max\left\{\sup_{0\le t\le T}I_0^{\alpha-1}|h(t)|,\ \sup_{0\le t\le T}I_0^{\alpha-1}|t^{-\alpha_1\mu_2}h(t)|,\ \sup_{0\le t\le T}I_0^{\alpha-1}|t^{-p(0)\mu_3}h(t)|\right\}, \tag{70}$$

$$M=\max\left\{\bar{M}(T^{\alpha-1-\beta}M_{1,h}+T^{1-\beta}M_{2,h}),\ \bar{M}(T^{\alpha-1-\gamma}M_{1,h}+T^{1-\gamma}M_{2,h})\right\}, \tag{71}$$

where a_1, a_5 and B_1 are the constants given by (48)-(50).

Firstly, we show that $A : H_{\beta}[0, T] \to H_{\beta}[0, T]$ with $\alpha_1 < \gamma < \beta < \alpha - 1$. In fact, for $x \in H_{\beta}[0, T]$, we obtain the same estimation of $\omega_1(x(t))$, $\omega_2(x(t))$, $\omega_3(x(t))$, $g_1(x(t))$ and $g_2(x(t))$ as the Theorem 3.1.

From (58)-(62), (39) and (68) together with the inequality $(a + b)^\delta \le a^\delta + b^\delta$ for $0 < \delta < 1$, $a, b \ge 0$, we have

$$\left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^{3} \omega_i(x(t)) \right) \right|$$

$$\le M_1 |x(t)|^{\mu_1} + M_2 | g_1(x(t)) + g_2(x(t)) |^{\mu_2} + M_3 \left| \sum_{i=1}^{3} \omega_i(x(t)) \right|^{\mu_3}$$

$$\le M_1 a_5^{\mu_1} \| x\|_\beta^{\mu_1} + M_2 \left(| g_1(x(t)) |^{\mu_2} + g_2 | (x(t)) |^{\mu_2} \right)$$

$$+ M_3 \left(| \omega_1(x(t)) |^{\mu_3} + \omega_2 | (x(t)) |^{\mu_3} + \omega_3 | (x(t)) |^{\mu_3} \right)$$

$$\le M_1 a_5^{\mu_1} \| x\|_\beta^{\mu_1} + M_2 \left(\frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1) | \Gamma(-\alpha_1) |} \right)^{\mu_2} \| x\|_\beta^{\mu_2} + \frac{M_2 a_5^{\mu_2} t^{-\alpha_1 \mu_2}}{[\Gamma(1-\alpha_1)]^{\mu_2}} \| x\|_\beta^{\mu_2}$$

$$+ M_3 (B_1 + \bar{B}_2)^{\mu_3} \| x\|_\beta^{\mu_3} + M_3 a_1^{\mu_3} a_5^{\mu_3} t^{-p(0)\mu_3} \| x\|_\beta^{\mu_3}$$

$$\le \bar{M} (\| x\|_\beta^{\mu_1} + \| x\|_\beta^{\mu_2} + \| x\|_\beta^{\mu_3} + t^{-\alpha_1 \mu_2} \| x\|_\beta^{\mu_2} + t^{-p(0)\mu_3} \| x\|_\beta^{\mu_3}). \tag{72}$$

For $x \in H_\beta[0, T]$, $t_1, t_2 \in [0, T]$ with $t_1 \ne t_2$ (without loss of generality, we assume that $0 \le t_1 < t_2 \le T$), in a similar fashion to (65) and together with (68)-(72), we have

$$| Ax(t_2) - Ax(t_1) |$$

$$\le \frac{T^{1-\alpha} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} | h(s) | \left| f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3} \omega_i(x(s)) \right) \right| ds$$

$$+ \int_{t_1}^{t_2} \left| I_{0+}^{\alpha-1} \left[h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3} \omega_i(x(s)) \right) \right] \right| ds$$

$$\le \frac{\bar{M} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T | h(s) | (\| x\|_\beta^{\mu_1} + \| x\|_\beta^{\mu_2} + \| x\|_\beta^{\mu_3} + s^{-\alpha_1 \mu_2} \| x\|_\beta^{\mu_2} + s^{-p(0)\mu_3}) \| x\|_\beta^{\mu_3}) ds$$

$$+ \bar{M} \int_{t_1}^{t_2} I_{0+}^{\alpha-1} \left[| h(s) | (\| x\|_\beta^{\mu_1} + \| x\|_\beta^{\mu_2} + \| x\|_\beta^{\mu_3} + s^{-\alpha_1 \mu_2} \| x\|_\beta^{\mu_2} + s^{-p(0)\mu_3}) \| x\|_\beta^{\mu_3}) \right] ds$$

$$\le \bar{M} M_{1,h} (\| x\|_\beta^{\mu_1} + 2\| x\|_\beta^{\mu_2} + 2\| x\|_\beta^{\mu_3}) T^{\alpha-1-\beta} (t_2 - t_1)^\beta$$

$$+ \bar{M} M_{2,h} (\| x\|_\beta^{\mu_1} + 2\| x\|_\beta^{\mu_2} + 2\| x\|_\beta^{\mu_3}) T^{1-\beta} (t_2 - t_1)^\beta$$

$$\le M (\| x\|_\beta^{\mu_1} + 2\| x\|_\beta^{\mu_2} + 2\| x\|_\beta^{\mu_3}) (t_2 - t_1)^\beta. \tag{73}$$

The inequality (73) implies that $Ax(t) \in H_\beta[0, T]$. Then, $A : H_\beta[0, T] \to H_\beta[0, T]$.

Let the ball Ω_ρ in $H_\beta[0, T]$ be $\Omega\rho = \{x \in H_\beta; \|x\|_\beta \le \rho\}$, where

$$\rho \ge \max\left\{(3M)^{\frac{1}{1-\mu_1}}, (6M)^{\frac{1}{1-\mu_2}}, (6M)^{\frac{1}{1-\mu_3}}, (3Ma_5^{\mu_1})^{\frac{1}{1-\mu_1}}, (6Ma_5^{\mu_2})^{\frac{1}{1-\mu_2}}, (6Ma_5^{\mu_3})^{\frac{1}{1-\mu_3}}\right\}. \tag{74}$$

Secondly, we prove $A : \Omega_\rho \to \Omega_\rho$ in the space $H_\beta[0, T]$. For $x \in \Omega_\rho$, by (73) together with $Ax(0) = 0$, we have

$$\begin{aligned}\| Ax\|_\beta &= | Ax(0) | + \sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}} \frac{| Ax(t_1) - Ax(t_2) |}{| t_1 - t_2 |^\beta} \\ &\le M(\| x\|_\beta^{\mu_1} + 2\| x\|_\beta^{\mu_2} + 2\| x\|_\beta^{\mu_3}) \\ &\le M\rho^{\mu_1} + 2M\rho^{\mu_2} + 2M\rho^{\mu_3} \\ &\le \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3} \le \rho. \end{aligned} \tag{75}$$

Hence $A(\Omega_\rho)$ is a bounded subset in $H_\beta[0, T]$. Then we obtain that $A(\Omega_\rho)$ is relatively compact in $H_\gamma[0, T]$ by Lemma 2.3 for $\alpha_1 < \gamma < \beta$. According to Lemma 2.2, the ball Ω_ρ is contained in the space $H_\gamma[0, T]$ for $\alpha_1 < \gamma < \beta$.

Thirdly, we prove the operator $A : \Omega_\rho \to \Omega_\rho$ according to the norm $\| \cdot \|_\gamma$ in the space $H_\gamma[0, T]$. For $x \in \Omega_\rho \subset H_\gamma[0, T]$, in a similar fashion to Theorem 3.1, we have

$$| \omega_1(x(t)) | \le B_1\| x\|_\gamma,\ | \omega_2(x(t)) | \le \overline{B}_2\| x\|_\gamma,\ | \omega_3(x(t)) | \le a_1 a_5\| x\|_\gamma t^{-p(0)} \tag{76}$$

and

$$| g_1(x(t)) | \le \frac{T^{\gamma-\alpha_1}}{(\gamma-\alpha_1)| \Gamma(-\alpha_1) |}\| x\|_\gamma,\ \ | g_2(x(t)) | \le \frac{a_5 t^{-\alpha_1}}{\Gamma(1-\alpha_1)}\| x\|_\gamma. \tag{77}$$

Form (76), (77) and (68), in similar fashion to (72), we have

$$\begin{aligned}&\left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^{3}\omega_i(x(t))\right)\right| \\ &\le \overline{M}(\| x\|_\gamma^{\mu_1} + \| x\|_\gamma^{\mu_2} + \| x\|_\gamma^{\mu_3} + t^{-\alpha_1\mu_2}\| x\|_\gamma^{\mu_2} + t^{-p(0)\mu_3}\| x\|_\gamma^{\mu_3}).\end{aligned} \tag{78}$$

Form (78) and (71), in similar fashion to (73), we have

$$| Ax(t_2) - Ax(t_1) | \le M(\| x\|_\gamma^{\mu_1} + 2\| x\|_\gamma^{\mu_2} + 2\| x\|_\gamma^{\mu_3})(t_2 - t_1)^\gamma. \tag{79}$$

Form (79) and (74) together with $Ax(0) = 0$, in similar fashion to (75), we have

$$\| Ax\|_\gamma = |Ax(0)| + \sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}} \frac{|Ax(t_1)-Ax(t_2)|}{|t_1-t_2|^\gamma}$$

$$\leq M(\|x\|_\gamma^{\mu_1} + 2\|x\|_\gamma^{\mu_2} + 2\|x\|_\gamma^{\mu_3})$$

$$\leq M(a_5^{\mu_1}\|x\|_\beta^{\mu_1} + 2a_5^{\mu_2}\|x\|_\beta^{\mu_2} + 2a_5^{\mu_3}\|x\|_\beta^{\mu_3})$$

$$\leq M(a_5^{\mu_1}\rho^{\mu_1} + 2a_5^{\mu_2}\rho^{\mu_2} + 2a_5^{\mu_3}\rho^{\mu_3})$$

$$\leq \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3} \leq \rho, \tag{80}$$

which indicates that the operator $A : \Omega_\rho \to \Omega_\rho$ according to the norm $\|\cdot\|_\gamma$. Hence, as a result, we obtain that the operator A maps Ω_ρ into itself for the norm $\|\cdot\|_\gamma$ and $A(\Omega_\rho)$ is relatively compact in $H_\gamma[0, T]$.

In the end, we prove that the operator $A : \Omega_\rho \to \Omega_\rho$ is continuous according to the norm $\|\cdot\|_\gamma$. Consider any $x_n \in \Omega_\rho$ such that $x_n \to x$ in Ω_ρ. By virtue of the continuity of function $f(x, y, z)$, then $\forall \varepsilon > 0$, $\exists N > 0$, for $n > N$, we have

$$\left| f\left(x_n(s), g_1(x_n(s)) + g_2(x_n(s)), \sum_{i=1}^{3}\omega_i(x_n(s))\right) - f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^{3}\omega_i(x(s))\right)\right| < \varepsilon. \tag{81}$$

From (81), in a similar fashion to (73), for $n > N$, we have

$$|(Ax_n - Ax)(t_2) - (Ax_n - Ax)(t_1)|$$

$$\leq \frac{\varepsilon(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^T |h(s)|\,ds + \varepsilon\int_{t_1}^{t_2} I_{0+}^{\alpha-1}|h(s)|\,ds$$

$$\leq \varepsilon(M_{1,h}T^{\alpha-1-\gamma} + M_{2,h}T^{1-\gamma})(t_2-t_1)^\gamma,\ 0 \leq t_1 < t_2 \leq T. \tag{82}$$

Since $(Ax_n - Ax)(0) = 0$, by (82), for $n > N$, we have

$$\|Ax_n(t) - Ax(t)\|_\gamma = |(Ax_n - Ax)(0)| + \sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}} \frac{|(Ax_n - Ax)(t_2) - (Ax_n - Ax)(t_1)|}{|t_2-t_1|^\gamma}$$

$$\leq \varepsilon(M_{1,h}T^{\alpha-1-\gamma} + M_{2,h}T^{1-\gamma}), \tag{83}$$

which indicates that A is continuous at the any point $x \in \Omega_\rho$ with the norm $\|\cdot\|_\gamma$.

Hence, $A : \Omega_\rho \to \Omega_\rho$ is completely continuous in $H_\gamma[0, T]$. Then we deduce that the boundary value problem (10) exists a solution $x \in H_\gamma[0, T]$ $(\alpha_1 < \gamma < \beta < \alpha - 1)$ according to Schauder fixed point theorem. □

Example 3.1 We consider the following boundary value problem

$$\begin{cases} D_{0+}^{1.9}x(t)+t^{-0.1}\left(\dfrac{1}{400(1+x^2)}+\dfrac{(D_{0+}^{0.7}x)^2}{600(1+(D_{0+}^{0.7}x)^2)}+\dfrac{(D_{0+}^{0.6-0.3t}x)^2}{600(1+(D_{0+}^{0.6-0.3t}x)^2)}\right)=0,\ 0<t<1, \\ x(0)=x(1)=0, \end{cases} \tag{84}$$

then we claim that the boundary value problem (84) exists a unique nontrivial solution $x \in H_{0.8}[0, 1]$.

Proof. Let $\alpha = 1.9$, $\alpha_1 = 0.7$, $p(t) = 0.6 - 0.3t$, $\beta = 0.8$, $h(t) = t^{-0.1}$, $f(x,y,z)=\dfrac{1}{400(1+x^2)}+\dfrac{y^2}{600(1+y^2)}+\dfrac{z^2}{600(1+z^2)}$. Obviously, $p^* = 0.6$, $p_* = 0.3$ and $p^* < \alpha_1 < \beta < \alpha - 1$. Since $h(t) \in L(0, 1)$,

$$\frac{\int_0^1 |h(s)|\,ds}{\Gamma(\alpha)}=\frac{1}{0.9\Gamma(1.9)},\quad \frac{\int_0^1 s^{-\alpha_1}|h(s)|\,ds}{\Gamma(\alpha)}=\frac{1}{0.2\Gamma(1.9)},\quad \frac{\int_0^1 s^{-p(0)}|h(s)|\,ds}{\Gamma(\alpha)}=\frac{1}{0.3\Gamma(1.9)}$$

and

$$\sup_{0\le t\le 1} I_{0+}^{\alpha-1}|h(t)|=\frac{\Gamma(0.9)}{\Gamma(1.8)}\approx 1.1474,\ \sup_{0\le t\le 1} I_{0+}^{\alpha-1}|t^{-0.7}h(t)|=\frac{\Gamma(0.2)}{\Gamma(1.1)}\approx 4.8256,$$

$$\sup_{0\le t\le 1} I_{0+}^{\alpha-1}|t^{-0.6}h(t)|=\frac{\Gamma(0.3)}{\Gamma(1.2)}\approx 3.2582,$$

then $p(t)$, α, α_1, $h(t)$ and f satisfy the assumptions (A_1), (A_2), (A_3) and (A_4).

Since

$$\begin{aligned} &\left|\frac{1}{400(1+x_1^2)}-\frac{1}{400(1+x_2^2)}\right| \\ &=\frac{|x_1+x_2||x_1-x_2|}{400(1+x_1^2)(1+x_2^2)} \\ &\le\left(\frac{|x_1|}{400(1+x_1^2)}+\frac{|x_2|}{400(1+x_2^2)}\right)|x_1-x_2| \\ &\le\frac{|x_1-x_2|}{200}, \end{aligned}$$

by the same token, we have

$$\left|\frac{y_1^2}{600(1+y_1^2)}-\frac{y_2^2}{600(1+y_2^2)}\right|\le\frac{|y_1-y_2|}{300}.$$

Then, we have

$$|f(x_1,y_1,z_1)-f(x_2,y_2,z_2)|\le\frac{|x_1-x_2|}{200}+\frac{|y_1-y_2|}{300}+\frac{|z_1-z_2|}{300},\ x,y,z\in\mathbb{R},$$

which implies that f satisfies the assumption (A_5) of Theorem 3.1 with $L_1 = \frac{1}{200}$, $L_2 = \frac{1}{300}$ and $L_3 = \frac{1}{300}$. By calculation, we have

$$a_1 = \max_{0 \le t \le 1} \left| \frac{1}{\Gamma(0.4 + 0.3t)} \right| \approx 0.7704,\ a_2 = 0.3,\ a_3 = \max_{0 \le t \le 1} \left| \frac{1}{\Gamma(-0.6 + 0.3t)} \right| \approx 0.2821,$$

$$a_4 = \max_{-0.6 \le t \le -0.3} \left| \left(\frac{1}{\Gamma(t)} \right)' \right| \approx 0.4884,\ a_5 = 1,\ K = \frac{5}{e},\ B_1 \approx 2.7034,\ B_2 \approx 1.7768,$$

$$L \approx 0.0277,\ L_{1,h} = \frac{1}{0.2\Gamma(1.9)} \approx 5.1988,\ L_{2,h} \approx 4.8256,$$

then

$$3L(L_{1,h} + L_{2,h}) \approx 0.833 < 1,$$

which implies that the condition (56) is satisfied. According to Theorem 3.1, the boundary value problem (84) exists a unique nontrivial solution in $H_{0.8}[0, 1]$. □

Example 3.2 We consider the following boundary value problem

$$\begin{cases} D_{0+}^{1.8} x(t) + t^{-0.2} \left(\dfrac{(\sin x)^{\frac{4}{3}}}{x} + \dfrac{(D_{0+}^{0.5} x)^{\frac{1}{3}}}{1 + x^2} + \dfrac{(D_{0+}^{0.4-0.2t} x)^{\frac{2}{3}}}{1 + (D_{0+}^{0.5} x)^2 + (D_{0+}^{0.4-0.2t} x)^2} \right),\ 0 < t < 1, \\ x(0) = x(1) = 0. \end{cases} \tag{85}$$

Then we claim that the boundary value problem (85) exists a nontrivial solution $x \in H_{0.6}[0, 1]$.

Proof. Let $\alpha = 1.8$, $p(t) = 0.4 - 0.2t$, $\alpha_1 = 0.5$, $\beta = 0.7$, $\gamma = 0.6$, $f(x, y, z) = \frac{(\sin x)^{\frac{4}{3}}}{x} + \frac{y^{\frac{1}{3}}}{1 + x^2} + \frac{z^{\frac{2}{3}}}{1 + y^2 + z^2}$. Obviously, $p^* = 0.4$, $p_* = 0.2$ and $p^* < \alpha_1 < \gamma < \beta < \alpha - 1$. Then, $p(t)$, α, α_1, $h(t)$ and f satisfy the assumptions (A_1), (A_2), (A_3) and (A_4).

Since

$$\left| \frac{(\sin x)^{\frac{4}{3}}}{x} \right| \le \frac{|x|^{\frac{4}{3}}}{|x|} = |x|^{\frac{1}{3}},$$

then

$$|f(x, y, z)| \le |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{2}{3}},\ x, y, z \in \mathbb{R},$$

which implies that f satisfies the assumption (A_6) of Theorem 3.2 with $M_1 = M_2 = M_3 = 1$, $\mu_1 = \frac{1}{3}$, $\mu_2 = \frac{1}{3}$ and $\mu_3 = \frac{2}{3}$.

According to Theorem 3.2, we obtain that the boundary value problem (85) exists a nontrivial solution $x \in H_{0.6}[0, 1]$. □

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Conflict of interest

The authors declare that they have no competing interests.

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