

## Research Article

# The Existence of Solution to Fractional Boundary Value Problem with Riemann-Liouville Type History-State-Based Variable-Order Derivative

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**Abstract:** The paper is devoted to studying the solutions of boundary value problem for nonlinear fractional differential equation with Riemann-Liouville type history-state-based variable-order derivative. Using Schauder fixed point theorem and Banach fixed point theorem, we prove the existence and uniqueness of solutions in the Hölder space. Lastly, two examples are given to show the applicability of the existence theorems.

**Keywords:** fractional differential equations, Riemann-Liouville type variable-order derivative, boundary value problem, existence of solution, Hölder space

**MSC:** 34B15, 34A08

## 1. Introduction

The history of fractional calculus has more than three hundred years. There are many fractional derivatives, such as Riemann-Liouville, Caputo, Erdélyi-Kober, Hadamard fractional derivative and so on. The Riemann-Liouville derivative [1]  $D_{0+}^\eta$  ( $0 < \eta < 1$ ), as one of the most classical fractional derivative, is defined by

$$D_{0+}^\eta f(t) = \frac{d}{dt} I_{0+}^{1-\eta} f(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} f(s) ds, \quad t > 0. \quad (1)$$

It is easy to obtain that  $D_{0+}^\eta f(t)$  exists if the derivative of  $I_{0+}^{1-\eta} f(t)$  exists on interval  $[0, T]$  by the definition. That is to say, the existence conditions for Riemann-Liouville fractional derivative are weaker than other fractional derivatives such as Caputo derivative. For example, Riemann-Liouville fractional derivative of function  $t^{-\frac{1}{2}}$  is 0 on interval  $[0, 1]$ . But Caputo fractional derivative of function  $t^{-\frac{1}{2}}$  doesn't exist at point  $t = 0$ . Variable-Order (VO) fractional operators are the derivatives and integrals whose order are functions of some variables. VO fractional operators are conceived and mathematically formalized only in recent years and can be seen as a natural analytical extension of

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constant-order fractional operators. The first definition of VO fractional operator was given by Samko and Ross [2] in 1993. Subsequently, Lorenzo et al. [3] and Coimbra [4] studied VO calculus by discussing its possible applications in mechanics, which marked the starting point for applications of VO operators to the analysis of different complex physical problems. After that VO fractional differential equations are widely employed in mechanics and dynamics, viscoelasticity, the modelling of transport processes, control theory because VO fractional operators can describe accurately the memory and hereditary properties of many physical phenomena and processes depending on their non-stationary power-law kernel [5-10].

For Riemann-Liouville VO fractional integral, there are two forms [3, 11] as follows

$$I_{0+}^{\eta(t)} f(t) = \int_0^t \frac{(t-s)^{\eta(s)-1}}{\Gamma(\eta(s))} f(s) ds, \quad t > 0 \quad (2)$$

and

$$I_{0+}^{\eta(t)} f(t) = \frac{1}{\Gamma(\eta(t))} \int_0^t (t-s)^{\eta(t)-1} f(s) ds, \quad t > 0, \quad (3)$$

where  $0 < \eta(t) < 1$  is a given nonconstant function. Corresponding to (2) and (3), there are two types of Riemann-Liouville VO fractional derivative [3, 11]

$$D_{0+}^{\eta(t)} f(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\eta(s)}}{\Gamma(1-\eta(s))} f(s) ds \quad (4)$$

and

$$D_{0+}^{\eta(t)} f(t) = \frac{1}{\Gamma(1-\eta(t))} \frac{d}{dt} \int_0^t (t-s)^{-\eta(t)} f(s) ds. \quad (5)$$

In (3) and (5), the current state  $\eta(t)$  is used on entire interval  $[0, t]$ . For any fixed  $t$ , the convolution kernel can be integrated in a closed form as for the constant-order fractional derivative operator in (1), which greatly facilitates the analysis. In (2) and (4), the power  $\eta(s)$  assumes its historical state at the historical time instant  $s$ . Hence, the history-state-based fractional differential equations, which accounts for the influence of the quantity of interest at the historical time instant  $s$  with the historical state  $\eta(s)$ , is probably more physically relevant. However, the convolution kernel in (4) can't be integrated in a closed form in mathematics, which highly complicates the mathematical analysis. That is to say, it is difficult to obtain closed form solutions to history-state-based VO fractional differential equations because the law of exponents doesn't hold for VO fractional integral (refer to [2, 12]). Therefore, many authors [13-17] have made use of the numerical methods to solve VO fractional differential equations. As an important research topic of VO fractional differential equations, boundary value problems are concerned by many professors. Since the kernel of VO fractional operators has a variable exponent, there are many difficulties for us to obtain the existence of solutions to VO fractional boundary value problem. Few authors [18, 19] have attempted to consider the existence, uniqueness and stability of solutions to VO fractional boundary value problem using standard techniques in analysis.

In [18], authors proved the existence of solutions and presented a generalized Lyapunov-type inequality to the boundary value problem for VO fractional differential equation

$$\begin{cases} D_{0+}^{\eta(t)} x(t) = f(t, x), \quad 0 < t < T, \quad 0 < T < +\infty, \\ x(0) = x(T) = 0, \end{cases} \quad (6)$$

where  $1 < q(t) < 2$  is a piecewise constant function,  $f : (0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and  $D_{0+}^{q(t)}$  is the Riemann-Liouville type VO fractional derivative defined by

$$D_{0+}^{q(t)} x(t) = \frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-q(s)}}{\Gamma(2-q(s))} x(s) ds, \quad t > 0.$$

In [19], authors proved the wellposedness and smoothing properties of the Dirichlet boundary value problem of one dimensional VO linear space-fractional diffusion equation

$$\begin{cases} -D^2 u - D \left[ c(x) \left( \zeta_0^l I_x^{\alpha(x)} + (1-\zeta)_x^r I_1^{\alpha(x)} \right) Du(x) \right] = f(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \quad (7)$$

where  $0 < \alpha(x) \leq 1$ ,  ${}_0^l I_x^{\alpha(x)}$  and  ${}_x^r I_1^{\alpha(x)}$  are defined by

$${}_0^l I_x^{\alpha(x)} u = \frac{1}{\Gamma(\alpha(x))} \int_0^x \frac{u(s)}{(x-s)^{1-\alpha(x)}} ds, \quad {}_x^r I_1^{\alpha(x)} u = \frac{1}{\Gamma(\alpha(x))} \int_x^1 \frac{u(s)}{(s-x)^{1-\alpha(x)}} ds.$$

The authors in [18] proved the existence of solutions to the problem (6) in the space of continuous functions and the variable order was a piecewise constant function. The authors in [19] proved the wellposedness of the problem (7) in the Hölder space and the order of derivative was two. It is well known that the Hölder space is the function space in which functions are Hölder continuous and it is included in the space of continuous functions. Many papers [20-25] considered the existence of solutions of Fredholm integral equations in the Hölder spaces using fixed point theorems. However, few papers [26, 27] considered the existence of solutions of boundary value problem for fractional differential equations, where the fractional order is constant order.

In [26], authors proved the existence and uniqueness of solutions for the following fractional boundary value problem in the Hölder space

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u(0) = \gamma I_{0+}^\rho u(\eta) = \gamma \int_0^\eta \frac{(\eta-s)^{\rho-1}}{\Gamma(\rho)} u(s) ds \end{cases} \quad (8)$$

where  ${}^C D_{0+}^\alpha$  denotes the Caputo fractional derivative [1],  $0 < \alpha \leq 1$ ,  $0 < \eta < 1$  and  $\lambda, \gamma, \rho \in \mathbb{R}$ .

In [27], authors studied the existence of positive solutions to the following fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D_{0+}^\alpha u(t) + f(t, u(t), (Hu)(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \quad (9)$$

where  $\alpha > 2$ ,  $n-1 < \alpha \leq n$ ,  $i \in [1, n-2]$  is a fixed integer,  $\alpha_j \geq 0$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{j-1} < \xi_j < \dots < 1$  ( $j = 1, 2, \dots$ ),  $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$ , where  $\Delta = (\alpha-1)(\alpha-2) \dots (\alpha-i)$ ,  $H$  is an operator applying  $C[0, 1]$  into itself satisfying certain assumptions and  $D_{0+}^\alpha$  denotes the Riemann-liouville fractional derivative.

To the best of our knowledge, few papers studied the existence of solutions to boundary value problems of history-state-based VO fractional differential equations in the Hölder space. Inspired by the above excellent work, in this paper,

we consider the existence of solution to the following boundary value problem for fractional differential equation involving Riemann-Liouville type history-state-based VO fractional derivative

$$\begin{cases} D_{0+}^\alpha x(t) + h(t)f(x, D_{0+}^{\alpha_1} x, D_{0+}^{p(t)} x), & 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases} \quad (10)$$

where  $0 < \alpha_1 < 1 < \alpha < 2$ ,  $0 < p(t) < 1$ ,  $p, h, f$  are all given real-valued functions,  $D_{0+}^{\alpha_1}$  denotes the constant-order Riemann-Liouville fractional derivative given by (1),  $D_{0+}^\alpha$  denotes the constant-order Riemann-Liouville fractional derivative [1]

$$D_{0+}^\alpha x(t) = \frac{d^2}{dt^2} I_{0+}^{2-\alpha} x(t) = \frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} x(s) ds, \quad (11)$$

and  $I_{0+}^{2-\alpha}, I_{0+}^{1-\alpha_1}$  are both the Riemann-Liouville fractional integrals (refer to [1]).  $D_{0+}^{p(t)}$  denotes the Riemann-Liouville type history-state-based VO fractional derivative given by (4).

The paper is organized as follows. In section 2, we present some basic notations, propositions and lemmas, which will play an important role in obtaining the existence of solution to the boundary value problem (10). In section 3, main results are stated based on our previous analysis, then two examples are given to substantiate the theoretical results.

## 2. Preliminary

In this section, we introduce some basic notations, propositions and lemmas which are used throughout this paper.

Let  $[0, b]$  be a finite interval.  $I_{0+}^\delta, I_{0+}^\gamma$  and  $D_{0+}^\delta$  denote the Riemann-Liouville fractional integral and derivative [1] respectively.  $I_{0+}^{1-p(t)}$  is the Riemann-Liouville type history-state-based VO fractional integral given by (2).

**Proposition 2.1** ([1] Lemma 2.9). If  $\gamma > 0, \delta > 0$ , then the equality  $I_{0+}^\gamma I_{0+}^\delta f(t) = I_{0+}^\delta I_{0+}^\gamma f(t) = I_{0+}^{\gamma+\delta} f(t)$  holds at any point of  $t \in [0, b]$  for  $f \in C[0, b]$ .

**Proposition 2.2** ([1] Lemma 2.9). If  $\gamma > 0, f \in C[0, b]$ , then the equality  $D_{0+}^\delta I_{0+}^\gamma f(t) = f(t)$  holds at any point of  $t \in [0, b]$ .

**Proposition 2.3** ([1] Lemma 2.9). If  $f \in C[0, b]$  and  $D_{0+}^\delta f \in C[0, b]$ , then

$$I_{0+}^\delta D_{0+}^\delta f(t) = f(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n},$$

where  $n-1 < \delta \leq n$ ,  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ .

Let  $H_\eta[0, T]$  ( $0 < \eta < 1$ ) be the space of functions  $x(t)$  such that  $x(t)$  is Hölder continuous with index  $\eta$  on the interval  $[0, T]$  with respect to the norm

$$\|x\|_\eta = |x(0)| + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\eta}, \quad x \in H_\eta[0, T].$$

In [28],  $(H_\eta[0, T], \|\cdot\|_\eta)$  is a Banach space.

**Lemma 2.1** ([28]). For  $x \in H_\eta[0, T]$ , the following inequality is satisfied

$$\max_{0 \leq t \leq T} |x(t)| \leq \max\{1, T^\eta\} \|x\|_\eta.$$

**Lemma 2.2** ([28]). For  $0 < \gamma < \eta < 1$ , we have

$$H_\eta[0, T] \subset H_\gamma[0, T] \subset C[0, T].$$

Moreover, for  $x \in H_\gamma[0, T]$ , the following inequality is satisfied

$$\|x\|_\gamma \leq \max\{1, T^{\eta-\gamma}\} \|x\|_\eta.$$

**Lemma 2.3** ([27]). Suppose that  $0 < \gamma < \eta < 1$  and that  $\Omega_\eta$  is a bounded subset in  $H_\eta[0, T]$  (this means that  $\|x\|_\eta \leq M$  for certain constant  $M > 0$ , for any  $x \in \Omega_\eta$ ), then  $\Omega_\eta$  is a relatively compact subset of  $H_\gamma[0, T]$ .

We assume that

$$(A_1) \quad 0 < \alpha_1 < \alpha - 1 < 1 < \alpha < 2, \quad p \in C^1[0, T], \quad p_* = \min_{0 \leq t \leq T} p(t), \quad p^* = \max_{0 \leq t \leq T} p(t) \text{ and } 0 < p_* \leq p^* < \alpha_1.$$

**Lemma 2.4** ([29]). If  $0 \leq \gamma \leq 1$  and  $0 < a < b$ , then

$$b^\gamma - a^\gamma \leq (b-a)^\gamma.$$

**Lemma 2.5** ([30, 31]). If  $0 < \beta^* < \eta < 1$ , then

$$\int_0^\infty t^\eta \left[ t^{-1-\beta^*} - (t+1)^{-1-\beta^*} \right] dt < \infty.$$

**Lemma 2.6** Assume that  $(A_1)$  holds and  $x \in H_\beta[0, T]$  ( $p^* < \beta < 1$ ) with  $x(0) = 0$ , then  $I_{0+}^{1-p(t)} x(t) \in C^1(0, T] \cap C[0, T]$  and

$$\begin{aligned} \frac{d}{dt} I_{0+}^{1-p(t)} x(t) &= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} \left[ -p'(s) \ln(t-s) + \frac{p(s)-p(t)}{t-s} \right] ds \\ &+ \int_0^t \left[ \frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right] (t-s)^{-p(s)-1} ds + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}. \end{aligned} \quad (12)$$

**Proof.** We take  $\alpha(t) = 1 - p(t)$ . Since  $\min_{0 \leq t \leq T} (1 - p(t)) = 1 - p^*$ , then  $\beta + 1 - p^* > 1$  from  $\beta > p^*$ . According to Lemma 2.2 in [32],  $I_{0+}^{1-p(t)} x(t) \in C[0, T]$ . According to Theorem 3.3 in [29], we obtain that (12) holds. But in [29], the detailed proof of  $\frac{d}{dt} I_{0+}^{1-p(t)} x(t) \in C(0, T]$  is not presented.

Next, we prove  $\frac{d}{dt} I_{0+}^{1-p(t)} x(t) \in C(0, T]$ .

For  $0 < t \leq T$ , we write

$$\begin{aligned} \psi(t) &= \frac{d}{dt} I_{0+}^{1-p(t)} x(t) \\ &= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} (-p'(s)) \ln(t-s) ds \\ &+ \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-1-p(s)} (p(s)-p(t)) ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left[ \frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right] (t-s)^{-1-p(s)} ds + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)} \\
& = \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-p(t-\tau)} (-p'(t-\tau)) \ln \tau d\tau \\
& + \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-1-p(t-\tau)} [p(t-\tau) - p(t)] d\tau \\
& + \int_0^t \left[ \frac{x(t-\tau)}{\Gamma(-p(t-\tau))} - \frac{x(t)}{\Gamma(-p(t))} \right] \tau^{-1-p(t-\tau)} d\tau + \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)} \\
& = \psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t). \tag{13}
\end{aligned}$$

Since  $1 - \beta > 0$ , there exists  $Q_1 > 0$  such that

$$|t^{1-\beta} \ln t| \leq Q_1, \quad 0 < t \leq T.$$

We write  $Q_2 = \max\{1, T\}$ ,  $Q_3 = \max_{0 \leq t \leq T} \left| \frac{1}{\Gamma(1-p(t))} \right|$ ,  $Q_4 = \max_{0 \leq t \leq T} |p'(t)|$ ,  $Q_5 = \max_{1-p^* \leq \omega \leq 1-p} \left| \left( \frac{1}{\Gamma(\omega)} \right)' \right|$ ,  $Q_6 = \max_{0 \leq t \leq T} \left| \frac{1}{\Gamma(-p(t))} \right|$ ,  $Q_7 = \max_{p^* \leq g \leq p} \left| \left( \frac{1}{\Gamma(-g)} \right)' \right|$ ,  $Q_8 = \int_0^\infty t^\beta \left[ t^{-1-p^*} - (t+1)^{-1-p^*} \right] dt$ ,  $\mathcal{Q} = \mathcal{Q}(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \|x\|_\beta, \beta, p^*)$ . Here  $\mathcal{Q}$  is a varying constant and is related to  $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, \|x\|_\beta, \beta$  and  $p^*$ .

For any  $h \in \mathbb{R}$  such that  $t+h \in (0, T]$ , without loss of generality, we assume that  $h > 0$ , then

$$\begin{aligned}
& |\psi_1(t+h) - \psi_1(t)| \\
& = \left| \int_0^t \left[ \frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) - \frac{x(t)}{\Gamma(1-p(t))} \tau^{-p(t-\tau)} p'(t-\tau) \right] \ln \tau d\tau \right. \\
& \quad \left. + \int_t^{t+h} \frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau d\tau \right| \\
& \leq \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} - \frac{x(t+h)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau \right| d\tau \\
& \quad + \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t))} - \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau \right| d\tau \\
& \quad + \int_0^t \left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| p'(t+h-\tau) \ln \tau \right| \left| \tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)} \right| d\tau
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t-\tau)} \ln \tau \right| \left| p'(t+h-\tau) - p'(t-\tau) \right| d\tau \\
& + \int_t^{t+h} \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau) \ln \tau \right| d\tau \\
& = I_{11} + I_{12} + I_{13} + I_{14} + I_{15}. \tag{14}
\end{aligned}$$

In (14), we have

$$\left| \tau^{-p(t+h-\tau)} \ln \tau \right| = \left| \tau^{p^* - p(t+h-\tau)} \tau^{\beta - p^* - 1} \tau^{1-\beta} \ln \tau \right| \leq Q \tau^{\beta - p^* - 1}, \quad 0 < \tau \leq t, \tag{15}$$

$$\left| \frac{1}{\Gamma(1-p(t+h))} - \frac{1}{\Gamma(1-p(t))} \right| = \left| \left( \frac{1}{\Gamma(\omega)} \right)' \right|_{\omega=1-p(\zeta)} \left| p'(\zeta) \right| h \leq Qh, \tag{16}$$

$$|x(t+h) - x(t)| \leq \|x\|_\beta h^\beta \leq Qh^\beta \tag{17}$$

and

$$\begin{aligned}
& \left| (\tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)}) \ln \tau \right| = \left| \tau^{p^* - p(t+h-\tau)} - \tau^{p^* - p(t-\tau)} \right| \left| \tau^{1-\beta} \ln \tau \right| \tau^{\beta - p^* - 1} \\
& = h \left| \tau^{p^* - p(\xi)} p'(\xi) \tau^{\frac{1-\beta}{2}} \ln \tau \right| \left| \tau^{\frac{1-\beta}{2}} \ln \tau \right| \tau^{\beta - p^* - 1} \\
& \leq Qh \tau^{\beta - p^* - 1}, \tag{18}
\end{aligned}$$

where  $t < \zeta < t+h$  and  $t-\tau < \xi < t-\tau+h$ . In virtue of (15), (16), (17) and (18), we have

$$I_{11} \leq Qh \int_0^t \tau^{\beta - p^* - 1} d\tau \leq Qht^{\beta - p^*} \leq Qh, \tag{19}$$

$$I_{12} \leq Qh^\beta \int_0^t \tau^{\beta - p^* - 1} d\tau \leq Qh^\beta t^{\beta - p^*} \leq Qh^\beta, \tag{20}$$

$$I_{13} \leq Qh \int_0^t \tau^{\beta - p^* - 1} d\tau \leq Qht^{\beta - p^*} \leq Qh, \tag{21}$$

$$I_{15} \leq Q \int_t^{t+h} \tau^{\beta - p^* - 1} d\tau = \frac{Q}{\beta - p^*} \left[ (t+h)^{\beta - p^*} - t^{\beta - p^*} \right] \leq Qh^{\beta - p^*}. \tag{22}$$

According to (19), (20), (21) and (22), we obtain that when  $h \rightarrow 0$ , then

$$I_{11} \rightarrow 0, I_{12} \rightarrow 0, I_{13} \rightarrow 0, I_{15} \rightarrow 0. \quad (23)$$

For  $I_{14}$ , by  $p'(t) \in C[0, T]$ , we have

$$|p'(t+h-\tau) - p'(t-\tau)| \rightarrow 0 \quad (h \rightarrow 0),$$

$$\left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t-\tau)} \ln \tau \right| |p'(t+h-\tau) - p'(t-\tau)| \leq Q \tau^{\beta-p^* - 1},$$

and

$$Q \int_0^t \tau^{\beta-p^* - 1} d\tau \leq Q t^{\beta-p^*}.$$

By Lebesgue dominated convergence theorem, we have

$$I_{14} \rightarrow 0 \quad (h \rightarrow 0). \quad (24)$$

From (23), (24) and (14), we have

$$|\psi_1(t+h) - \psi_1(t)| \rightarrow 0 \quad (h \rightarrow 0). \quad (25)$$

For  $0 < t \leq T$ ,

$$\begin{aligned} \psi_2(t) &= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-1-p(t-\tau)} [p(t-\tau) - p(t)] d\tau \\ &= \frac{x(t)}{\Gamma(1-p(t))} \int_0^t \tau^{-p(t-\tau)} [-p'(t-\tau + \theta\tau)] d\tau, \quad 0 < \theta < 1. \end{aligned} \quad (26)$$

From (26), we have

$$\begin{aligned} &|\psi_2(t+h) - \psi_2(t)| \\ &\leq \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} - \frac{x(t+h)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau + \theta\tau) \right| d\tau \\ &+ \int_0^t \left| \frac{x(t+h)}{\Gamma(1-p(t))} - \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t+h-\tau)} p'(t+h-\tau + \theta\tau) \right| d\tau \\ &+ \int_0^t \left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| p'(t+h-\tau + \theta\tau) \right| \left| \tau^{-p(t+h-\tau)} - \tau^{-p(t-\tau)} \right| d\tau \\ &+ \int_0^t \left| \frac{x(t)}{\Gamma(1-p(t))} \right| \left| \tau^{-p(t-\tau)} \right| \left| p'(t+h-\tau + \theta\tau) - p'(t-\tau + \theta\tau) \right| d\tau \end{aligned}$$

$$+\int_t^{t+h} \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} \tau^{-p(t+h-\tau)} p'(t+h-\tau+\theta\tau) \right| d\tau.$$

In a similar fashion to  $|\psi_1(t+h) - \psi_1(t)|$ , we have

$$|\psi_2(t+h) - \psi_2(t)| \rightarrow 0 \ (h \rightarrow 0). \quad (27)$$

For  $0 < t \leq T$ , we have

$$\begin{aligned} & |\psi_3(t+h) - \psi_3(t)| \\ &= \left| \int_0^{t+h} \left[ \frac{x(t+h-\tau)}{\Gamma(-p(t+h-\tau))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right] \tau^{-1-p(t+h-\tau)} d\tau \right. \\ &\quad \left. - \int_0^t \left[ \frac{x(t-\tau)}{\Gamma(-p(t-\tau))} - \frac{x(t)}{\Gamma(-p(t))} \right] \tau^{-1-p(t-\tau)} d\tau \right| \\ &= \left| \int_{-h}^t \left[ \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right] (s+h)^{-1-p(t-s)} ds \right. \\ &\quad \left. - \int_0^t \left[ \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right] s^{-1-p(t-s)} ds \right| \\ &= \left| \int_0^t \left[ \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right] \left[ (s+h)^{-1-p(t-s)} - s^{-1-p(t-s)} \right] ds \right. \\ &\quad \left. + \int_0^t \left[ \frac{x(t)}{\Gamma(-p(t))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right] (s+h)^{-1-p(t-s)} ds \right. \\ &\quad \left. + \int_{-h}^0 \left[ \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right] (s+h)^{-1-p(t-s)} ds \right| \\ &= |I_{31} + I_{32} + I_{33}| \\ &\leq |I_{31}| + |I_{32}| + |I_{33}|. \end{aligned} \quad (28)$$

In (28), since

$$|(s+h)^{-1-p(t-s)} - s^{-1-p(t-s)}|$$

$$= s^{-1-p(t-s)} - (s+h)^{-1-p(t-s)}$$

$$\begin{aligned}
&= s^{p^* - p(t-s)} s^{-1-p^*} - (s+h)^{p^* - p(t-s)} (s+h)^{-1-p^*} \\
&\leq s^{p^* - p(t-s)} (s^{-1-p^*} - (s+h)^{-1-p^*}) \\
&\leq Q(s^{-1-p^*} - (s+h)^{-1-p^*})
\end{aligned}$$

and

$$\begin{aligned}
&Q \int_0^t s^\beta (s^{-1-p^*} - (s+h)^{-1-p^*}) ds \\
&= Q h^{\beta-p^*} \int_0^t \frac{1}{h} \tau^\beta (\tau^{-1-p^*} - (\tau+1)^{-1-p^*}) d\tau \\
&\leq Q h^{\beta-p^*} \int_0^\infty \tau^\beta (\tau^{-1-p^*} - (\tau+1)^{-1-p^*}) d\tau \\
&\leq Q h^{\beta-p^*},
\end{aligned}$$

where the last “ $\leq$ ” is obtained by Lemma 2.5, then

$$\begin{aligned}
|I_{31}| &\leq \int_0^t \left| \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right| \left| (s+h)^{-1-p(t-s)} - s^{-1-p(t-s)} \right| ds \\
&\leq Q \int_0^t \left[ \left| \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t-s))} \right| + \left| \frac{x(t)}{\Gamma(-p(t-s))} - \frac{x(t)}{\Gamma(-p(t))} \right| \right] \\
&\quad (s^{-1-p^*} - (s+h)^{-1-p^*}) ds \\
&\leq Q \int_0^t s^\beta (s^{-1-p^*} - (s+h)^{-1-p^*}) ds \\
&\leq Q h^{\beta-p^*}, \tag{29}
\end{aligned}$$

$$\begin{aligned}
|I_{32}| &\leq \int_0^t \left[ \left| \frac{x(t)}{\Gamma(-p(t))} - \frac{x(t+h)}{\Gamma(-p(t))} \right| + \left| \frac{x(t+h)}{\Gamma(-p(t))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right| \right] \\
&\quad (s+h)^{p^* - p(t-s)} (s+h)^{-p^* - 1} ds \\
&\leq Q \int_0^t (h^\beta + h) (s+h)^{-1-p^*} ds
\end{aligned}$$

$$\begin{aligned}
&\leq Qh^\beta \int_0^\infty (s+h)^{-1-p^*} ds \\
&\leq Qh^{\beta-p^*}
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
|I_{33}| &\leq \int_{-h}^0 \left[ \left| \frac{x(t-s)}{\Gamma(-p(t-s))} - \frac{x(t+h)}{\Gamma(-p(t-s))} \right| + \left| \frac{x(t+h)}{\Gamma(-p(t-s))} - \frac{x(t+h)}{\Gamma(-p(t+h))} \right| \right] \\
&\quad (s+h)^{p^*-p(t-s)} (s+h)^{-p^*-1} ds \\
&\leq Q \int_{-h}^0 [(h+s)^\beta + (h+s)] (s+h)^{-1-p^*} ds \\
&\leq Q \int_{-h}^0 (s+h)^{\beta-p^*-1} ds \\
&\leq Qh^{\beta-p^*}.
\end{aligned} \tag{31}$$

From (28), (29), (30) and (31), we have

$$|\psi_3(t+h) - \psi_3(t)| \rightarrow 0 \ (h \rightarrow 0). \tag{32}$$

For  $0 < t \leq T$ , we have

$$\begin{aligned}
&|\psi_4(t+h) - \psi_4(t)| \\
&= \left| \frac{x(t+h)}{\Gamma(1-p(t+h))} (t+h)^{-p(0)} - \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)} \right| \\
&\leq \left| \frac{x(t+h) - x(0)}{\Gamma(1-p(t+h))} - \frac{x(t+h) - x(0)}{\Gamma(1-p(t))} \right| (t+h)^{-p(0)} \\
&\quad + \left| \frac{x(t+h)}{\Gamma(1-p(t))} - \frac{x(t)}{\Gamma(1-p(t))} \right| (t+h)^{-p(0)} \\
&\quad + \left| \frac{x(t) - x(0)}{\Gamma(1-p(t))} \right| \left| (t+h)^{-p(0)} - t^{-p(0)} \right| \\
&\leq Qh(t+h)^{\beta-p(0)} + Qh^\beta (t+h)^{-p(0)} + Qt^\beta \left| (t+h)^{-p(0)} - t^{-p(0)} \right|
\end{aligned}$$

$$\leq Qh + Qh^{\beta-p(0)} + Qt^\beta \left[ t^{-p(0)} - (t+h)^{-p(0)} \right] \quad (33)$$

For  $0 < t \leq h$ , we have

$$t^\beta \left[ t^{-p(0)} - (t+h)^{-p(0)} \right] \leq t^{\beta-p(0)} \leq h^{\beta-p(0)}. \quad (34)$$

For  $t > h$ , we have

$$\begin{aligned} & t^\beta \left[ t^{-p(0)} - (t+h)^{-p(0)} \right] \\ & \leq t^{\beta-p(0)} \left[ 1 - \left( 1 + \frac{h}{t} \right)^{-p(0)} \right] \\ & \leq t^{\beta-p(0)} \left[ 1 - \left( 1 + (-p(0)) \frac{h}{t} + O\left(\frac{h^2}{t^2}\right) \right) \right] \\ & \leq p(0)t^{\beta-p(0)-1}h \\ & \leq p(0)h^{\beta-p(0)}. \end{aligned} \quad (35)$$

From (33), (34) and (35), we have

$$|\psi_4(t+h) - \psi_4(t)| \rightarrow 0 \quad (h \rightarrow 0). \quad (36)$$

By (13), (25), (27), (32) and (36), we conclude that  $\psi(t) \in C(0, T]$ , that is to say  $I_{0+}^{1-p(t)}x(t) \in C^1(0, T]$ .  $\square$

**Remark 2.1** In Lemma 2.6, we have

$$\lim_{t \rightarrow 0^+} \psi(t) = \lim_{t \rightarrow 0^+} \frac{d}{dt} I_{0+}^{1-p(t)} x(t) = 0. \quad (37)$$

That is to say  $t = 0$  is a removable discontinuity point of function  $\frac{d}{dt} I_{0+}^{1-p(t)} x(t)$ .

**Proof.** The proof of Remark 2.1 is similar to Lemma 2.6.  $\square$

**Remark 2.2** In Lemma 2.6, if  $p(t) \equiv \alpha_1$  for  $t \in [0, T]$ ,  $x(t) \in H_\beta[0, T]$  ( $\alpha_1 < \beta < 1$ ) with  $x(0) = 0$ , then  $I_{0+}^{1-\alpha_1} x(t) \in C^1(0, T] \cap C[0, T]$  and

$$\frac{d}{dt} I_{0+}^{1-\alpha_1} x(t) = \int_0^t \frac{x(s) - x(t)}{\Gamma(-\alpha_1)} (t-s)^{-\alpha_1-1} ds + \frac{x(t)}{\Gamma(1-\alpha_1)} t^{-\alpha_1}.$$

### 3. Main results

In this section we devote to dealing with the existence and uniqueness of solution to the boundary value problem (10).

We assume that

(A<sub>2</sub>)  $h \in L(0, T)$ ;

(A<sub>3</sub>)  $h$  satisfies  $\sup_{0 \leq t \leq T} I_{0+}^{\alpha-1} |t^{-\alpha_1} h(t)| < +\infty$ ;

(A<sub>4</sub>)  $f: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function;

(A<sub>5</sub>) there exists positive constants  $L_i$  ( $i = 1, 2, 3$ ) such that

$$\begin{aligned} & |f(x_1(t), y_1(t), z_1(t)) - f(x_2(t), y_2(t), z_2(t))| \\ & \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| + L_3 |z_1 - z_2|, \quad t \in [0, T], \quad x_j(t), y_j(t), z_j(t) \in \mathbb{R}, \quad j = 1, 2; \end{aligned} \quad (38)$$

(A<sub>6</sub>) there exists constants  $M_i > 0$ ,  $0 < \mu_i < 1$  ( $i = 1, 2, 3$ ) such that

$$|f(x(t), y(t), z(t))| \leq M_1 |x(t)|^{\mu_1} + M_2 |y(t)|^{\mu_2} + M_3 |z(t)|^{\mu_3}, \quad x(t), y(t), z(t) \in \mathbb{R}. \quad (39)$$

**Lemma 3.1** Assume that (A<sub>1</sub>), (A<sub>2</sub>) and (A<sub>4</sub>) hold. If  $x \in H_\beta[0, T]$  with  $\alpha_1 < \beta < 1$  is a solution of boundary value problem (10), then if and only if  $x(t)$  must be a solution of the integral equation

$$x(t) = \int_0^T G(t, s)h(s)f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right)ds, \quad 0 \leq t \leq T, \quad (40)$$

where

$$G(t, s) = \begin{cases} \frac{T^{1-\alpha} t^{\alpha-1} (T-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq T, \\ \frac{T^{1-\alpha} t^{\alpha-1} (T-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq T, \end{cases} \quad (41)$$

$$\omega_1(x(t)) = \frac{x(t)}{\Gamma(1-p(t))} \int_0^t (t-s)^{-p(s)} \left[ -p'(s) \ln(t-s) + \frac{p(s) - p(t)}{t-s} \right] ds, \quad (42)$$

$$\omega_2(x(t)) = \int_0^t \left[ \frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right] (t-s)^{-p(s)-1} ds, \quad (43)$$

$$\omega_3(x(t)) = \frac{x(t)}{\Gamma(1-p(t))} t^{-p(0)}, \quad (44)$$

$$g_1(x(t)) = \int_0^t \frac{x(s) - x(t)}{\Gamma(-\alpha_1)} (t-s)^{-\alpha_1-1} ds, \quad g_2(x(t)) = \frac{x(t)}{\Gamma(1-\alpha_1)} t^{-\alpha_1}. \quad (45)$$

**Proof.** If  $x \in H_\beta[0, T]$  is a solution of boundary value problem (10), then  $x(t)$  satisfies  $x(0) = x(T) = 0$  and

$$D_{0+}^\alpha x(t) + h(t)f\left(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t)\right) = 0, \quad 0 < t < T. \quad (46)$$

Applying operator  $I_{0+}^\alpha$  to both sides of equation (46), from Proposition 2.3, we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds.$$

As a result of  $x \in H_\beta[0, T] \subset C[0, T]$  and  $x(0) = x(T) = 0$ , we get that  $c_2 = 0$  and

$$c_1 = \frac{T^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds.$$

Then we obtain that

$$\begin{aligned} x(t) &= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds, \\ &= \int_0^T G(t, s) h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds, \quad 0 \leq t \leq T, \end{aligned}$$

where  $G(t, s)$  is defined by (41). From Lemma 2.6 and Remark 2.2, it holds that

$$D_{0+}^{\alpha_1} x(s) = g_1(x(s)) + g_2(x(s)), \quad D_{0+}^{p(s)} x(s) = \sum_{i=1}^3 \omega_i(x(s)), \quad 0 \leq s \leq T.$$

Therefore, we have

$$x(t) = \int_0^T G(t, s) h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds, \quad 0 \leq t \leq T,$$

which implies that  $x(t)$  is a solution of integral equation (40).

Conversely, if  $x \in H_\beta[0, T]$  is a solution of integral equation (40), then

$$x(t) = \int_0^T G(t, s) h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds, \quad 0 \leq t \leq T.$$

Obviously,  $x(0) = 0$  and  $x(T) = 0$ . From Lemma 2.6 and Remark 2.2, we have

$$g_1(x(s)) + g_2(x(s)) = D_{0+}^{\alpha_1} x(s), \quad \sum_{i=1}^3 \omega_i(x(s)) = D_{0+}^{p(s)} x(s), \quad 0 \leq s \leq T.$$

Therefore

$$x(t) = \int_0^T G(t, s) h(s) f\left(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)\right) ds$$

$$\begin{aligned}
&= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f(x(s), D_{0+}^{\alpha_1} x(s), D_{0+}^{p(s)} x(s)) ds \\
&\quad - I_{0+}^{\alpha} \left( h(t) f(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t)) \right), \quad 0 \leq t \leq T.
\end{aligned} \tag{47}$$

Applying  $D_{0+}^{\alpha}$  to both sides of (47), by Proposition 2.2, we get

$$D_{0+}^{\alpha} x(t) + h(t) f(x(t), D_{0+}^{\alpha_1} x(t), D_{0+}^{p(t)} x(t)) = 0, \quad 0 < t < T,$$

together with  $x(0) = x(T) = 0$ , which implies  $x \in H_{\beta}[0, T]$  is a solution of boundary value problem (10).  $\square$

In the following analysis, for convenience, we let

$$a_1 = \max_{0 \leq t \leq T} \left| \frac{1}{\Gamma(1-p(t))} \right|, \quad a_2 = \max_{0 \leq t \leq T} |p'(t)|, \quad a_3 = \max_{0 \leq t \leq T} \left| \frac{1}{\Gamma(-p(t))} \right|, \tag{48}$$

$$a_4 = \max_{-p^* \leq w \leq -p^*} \left| \left( \frac{1}{\Gamma(w)} \right)' \right|, \quad a_5 = \max \{1, T^{\beta}\}, \tag{49}$$

$$B_1 = a_1 a_2 a_5 \max \{1, T^{p^* - p^*}\} \left( \frac{KT^{1-\beta}}{1-\beta} + \frac{T^{1-p^*}}{1-p^*} \right), \tag{50}$$

$$B_2 = \max \{1, T^{p^* - p^*}\} \left[ \frac{a_3 T^{\beta-p^*}}{\beta-p^*} + \frac{a_2 a_4 a_5 T^{1-p^*}}{1-p^*} \right], \tag{51}$$

$$L = \max \left\{ L_1 a_5 + \frac{L_2 T^{\beta-\alpha_1}}{(\beta-\alpha_1) |\Gamma(-\alpha_1)|} + L_3 (B_1 + B_2), \frac{L_2 a_5}{\Gamma(1-\alpha_1)}, L_3 a_1 a_5 \right\}, \tag{52}$$

$$L_{1,h} = \max \left\{ \frac{\int_0^T |h(s)| ds}{\Gamma(\alpha)}, \frac{\int_0^T s^{-\alpha_1} |h(s)| ds}{\Gamma(\alpha)}, \frac{\int_0^T s^{-p(0)} |h(s)| ds}{\Gamma(\alpha)} \right\}, \tag{53}$$

$$L_{2,h} = \max \left\{ \sup_{0 \leq t \leq T} I_{0+}^{\alpha-1} |h(t)|, \sup_{0 \leq t \leq T} I_{0+}^{\alpha-1} |t^{-\alpha_1} h(t)|, \sup_{0 \leq t \leq T} I_{0+}^{\alpha-1} |t^{-p(0)} h(t)| \right\}, \tag{54}$$

where  $p^* < \alpha_1 < \beta < 1$ ,  $K > 0$  is a constant satisfying

$$|t^{\beta-p^*} \ln t| < K, \quad 0 < t \leq T. \tag{55}$$

**Theorem 3.1** Suppose  $(A_1)$ ,  $(A_2)$ ,  $(A_3)$ ,  $(A_4)$  and  $(A_5)$  hold. Then the boundary value problem (10) exists a unique solution  $x \in H_{\beta}[0, T]$  with  $\alpha_1 < \beta < \alpha - 1$  provided if

$$3LL_{1,h}T^{\alpha-1-\beta} + 3LL_{2,h}T^{1-\beta} < 1. \quad (56)$$

**Proof.** According to Lemma 3.1, it is sufficient to consider the existence of fixed point of operator  $A$  defined by

$$Ax(t) = \int_0^T G(t,s)h(s)f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right)ds, \quad 0 \leq t \leq T. \quad (57)$$

Firstly, we show that  $A : H_\beta[0, T] \rightarrow H_\beta[0, T]$  with  $\alpha_1 < \beta < \alpha - 1$ . In fact, for  $x \in H_\beta[0, T]$ , from (48)-(51) and (55), we have

$$\begin{aligned} |\omega_1(x(t))| &\leq \frac{\max_{0 \leq t \leq T} |x(t)|}{\Gamma(1-p(t))} \int_0^t \left[ (t-s)^{-(\beta-p^*)-p(s)} |p'(s)| (t-s)^{\beta-p^*} \ln(t-s) \right. \\ &\quad \left. + (t-s)^{-p(s)} |p'(\xi)| \right] ds \\ &\leq a_1 a_2 a_5 \|x\|_\beta \max\{1, T^{p^*-p_*}\} \left[ K \int_0^t (t-s)^{-\beta} ds + \int_0^t (t-s)^{-p^*} ds \right] \\ &= a_1 a_2 a_5 \|x\|_\beta \max\{1, T^{p^*-p_*}\} \left( \frac{Kt^{1-\beta}}{1-\beta} + \frac{t^{1-p^*}}{1-p^*} \right) \\ &\leq a_1 a_2 a_5 \|x\|_\beta \max\{1, T^{p^*-p_*}\} \left( \frac{KT^{1-\beta}}{1-\beta} + \frac{T^{1-p^*}}{1-p^*} \right) \\ &= B_1 \|x\|_\beta, \quad 0 \leq s < \xi < t \leq T, \end{aligned} \quad (58)$$

$$\begin{aligned} |\omega_2(x(t))| &\leq \int_0^t \left| \frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right| (t-s)^{-p(s)-1} ds \\ &\leq \int_0^t \left[ \left| \frac{x(s)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(s))} \right| + \left| \frac{x(t)}{\Gamma(-p(s))} - \frac{x(t)}{\Gamma(-p(t))} \right| \right] (t-s)^{-p(s)-1} ds \\ &\leq a_3 \|x\|_\beta \int_0^t (t-s)^{\beta-p(s)-1} ds + a_5 \|x\|_\beta \int_0^t (t-s)^{-p(s)} \left| \left( \frac{1}{\Gamma(\omega)} \right)'_\omega \right| |p'(\theta)| ds \\ &\leq a_3 \|x\|_\beta \int_0^t (t-s)^{\beta-p(s)-1} ds + a_2 a_4 a_5 \|x\|_\beta \int_0^t (t-s)^{-p(s)} ds \\ &= a_3 \|x\|_\beta \int_0^t (t-s)^{p^*-p(s)} (t-s)^{\beta-p^*-1} ds + a_2 a_4 a_5 \|x\|_\beta \int_0^t (t-s)^{p^*-p(s)} (t-s)^{-p^*} ds \end{aligned}$$

$$\begin{aligned}
&\leq \|x\|_{\beta} \max\{1, T^{p^* - p_*}\} \left[ a_3 \int_0^t (t-s)^{\beta - p^* - 1} ds + a_2 a_4 a_5 \int_0^t (t-s)^{-p^*} ds \right] \\
&\leq \|x\|_{\beta} \max\{1, T^{p^* - p_*}\} \left[ \frac{a_3 T^{\beta - p^*}}{\beta - p^*} + \frac{a_2 a_4 a_5 T^{1-p^*}}{1-p^*} \right] \\
&= B_2 \|x\|_{\beta}, \quad -p^* \leq \omega \leq -p_*, \quad 0 \leq s < \theta < t \leq T, \tag{59}
\end{aligned}$$

$$|\omega_3(x(t))| \leq \frac{|x(t)|}{\Gamma(1-p(t))} t^{-p(0)} \leq a_1 a_5 \|x\|_{\beta} t^{-p(0)}, \tag{60}$$

$$\begin{aligned}
|g_1(x(t))| &\leq \int_0^t \frac{|x(s) - x(t)|}{|\Gamma(-\alpha_1)|} (t-s)^{-\alpha_1 - 1} ds \\
&\leq \frac{\|x\|_{\beta}}{|\Gamma(-\alpha_1)|} \int_0^t (t-s)^{\beta - \alpha_1 - 1} ds \\
&\leq \frac{T^{\beta - \alpha_1}}{(\beta - \alpha_1) |\Gamma(-\alpha_1)|} \|x\|_{\beta}, \tag{61}
\end{aligned}$$

and

$$|g_2(x(t))| \leq \frac{a_5 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} \|x\|_{\beta}. \tag{62}$$

From (38), (52) and (58)-(62), we have

$$\begin{aligned}
&\left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) \right| \\
&= \left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) - f(0, 0, 0) + f(0, 0, 0) \right| \\
&\leq L_1 |x(t)| + L_2 |g_1(x(t)) + g_2(x(t))| + L_3 \left| \sum_{i=1}^3 \omega_i(x(t)) \right| + |f(0, 0, 0)| \\
&\leq L_1 a_5 \|x\|_{\beta} + \frac{L_2 T^{\beta - \alpha_1} \|x\|_{\beta}}{(\beta - \alpha_1) |\Gamma(-\alpha_1)|} + \frac{L_2 a_5 t^{-\alpha_1} \|x\|_{\beta}}{\Gamma(1-\alpha_1)} + L_3 (B_1 + B_2) \|x\|_{\beta} \\
&\quad + L_3 a_1 a_5 t^{-p(0)} \|x\|_{\beta} + |f(0, 0, 0)| \\
&\leq (L + L t^{-\alpha_1} + L t^{-p(0)}) \|x\|_{\beta} + |f(0, 0, 0)|. \tag{63}
\end{aligned}$$

From (41), (57) and Proposition 2.1, we have

$$\begin{aligned}
Ax(t) &= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds \\
&\quad - I_{0+}^{\alpha} \left[ h(t) f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) \right] \\
&= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds \\
&\quad - I_{0+}^1 I_{0+}^{\alpha-1} \left[ h(t) f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) \right] \\
&= \frac{T^{1-\alpha} t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds \\
&\quad - \int_0^t I_{0+}^{\alpha-1} \left[ h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right] ds. \tag{64}
\end{aligned}$$

For  $x \in H_{\beta}[0, T]$ ,  $t_1, t_2 \in [0, T]$  with  $t_1 \neq t_2$  (without loss of generality, we assume that  $0 \leq t_1 < t_2 \leq T$ ), in view of (53), (54), (63) and (64) together with  $t_2^{\delta} - t_1^{\delta} \leq (t_2 - t_1)^{\delta}$  for  $t_2 > t_1 \geq 0$  and  $0 < \delta \leq 1$ , we have

$$\begin{aligned}
&|Ax(t_2) - Ax(t_1)| \\
&= \left| \frac{T^{1-\alpha} (t_2^{\alpha-1} - t_1^{\alpha-1})}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) ds \right. \\
&\quad \left. - \int_{t_1}^{t_2} I_{0+}^{\alpha-1} \left[ h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right] ds \right| \\
&\leq \frac{T^{1-\alpha} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |h(s)| \left| f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right| ds \\
&\quad + \int_{t_1}^{t_2} \left| I_{0+}^{\alpha-1} \left[ h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right] \right| ds \\
&\leq \frac{(t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s)| \left[ (L + Ls^{-\alpha_1} + Ls^{-p(0)}) \|x\|_{\beta} + |f(0, 0, 0)| \right] ds \\
&\quad + \int_{t_1}^{t_2} I_{0+}^{\alpha-1} \left[ |h(s)| \left( (L + Ls^{-\alpha_1} + Ls^{-p(0)}) \|x\|_{\beta} + |f(0, 0, 0)| \right) \right] ds
\end{aligned}$$

$$\begin{aligned}
&\leq (3LL_{1,h}\|x\|_\beta + |f(0,0,0)|L_{1,h})(t_2-t_1)^{\alpha-1} + (3LL_{2,h}\|x\|_\beta + |f(0,0,0)|L_{2,h})(t_2-t_1) \\
&\leq [(3LL_{1,h}T^{\alpha-1-\beta} + 3LL_{2,h}T^{1-\beta})\|x\|_\beta + (L_{1,h}T^{\alpha-1-\beta} + L_{2,h}T^{1-\beta})|f(0,0,0)|](t_2-t_1)^\beta. \tag{65}
\end{aligned}$$

The inequality (65) implies that  $Ax(t) \in H_\beta[0, T]$ . Then,  $A : H_\beta[0, T] \rightarrow H_\beta[0, T]$ .

Secondly, we prove that  $A : H_\beta[0, T] \rightarrow H_\beta[0, T]$  is a contractive operator. For  $x, y \in H_\beta[0, T]$ , by the similar way as the first step, we get

$$|\omega_1(x(t)) - \omega_1(y(t))| \leq B_1 \|x - y\|_\beta,$$

$$|\omega_2(x(t)) - \omega_2(y(t))| \leq B_2 \|x - y\|_\beta,$$

$$|\omega_3(x(t)) - \omega_3(y(t))| \leq a_1 a_5 t^{-p(0)} \|x - y\|_\beta,$$

$$|g_1(x(t)) - g_1(y(t))| \leq \frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|} \|x - y\|_\beta$$

and

$$|g_2(x(t)) - g_2(y(t))| \leq \frac{a_5 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} \|x - y\|_\beta.$$

From (38) and the obtained above inequalities, we obtain

$$\begin{aligned}
&\left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) - f\left(y(t), g_1(y(t)) + g_2(y(t)), \sum_{i=1}^3 \omega_i(y(t))\right) \right| \\
&\leq L_1 |x(t) - y(t)| + L_2 |g_1(x(t)) - g_1(y(t)) + g_2(x(t)) - g_2(y(t))| \\
&\quad + L_3 \left| \sum_{i=1}^3 \omega_i(x(t)) - \sum_{i=1}^3 \omega_i(y(t)) \right| \\
&\leq (L + Lt^{-\alpha_1} + Lt^{-p(0)}) \|x - y\|_\beta. \tag{66}
\end{aligned}$$

From (66), we have

$$\begin{aligned}
&|(Ax - Ay)(t_2) - (Ax - Ay)(t_1)| \\
&\leq \frac{(t_2 - t_1)^{\alpha-1} \|x - y\|_\beta}{\Gamma(\alpha)} \int_0^T |h(s)| (L + Ls^{-\alpha_1} + Ls^{-p(0)}) ds \\
&\quad + \|x - y\|_\beta \int_{t_1}^{t_2} I_0^{\alpha-1} [ |h(s)| (L + Ls^{-\alpha_1} + Ls^{-p(0)}) ] ds
\end{aligned}$$

$$\leq (3LL_{1,h}T^{\alpha-1-\beta} + 3LL_{2,h}T^{1-\beta})\|x-y\|_{\beta}(t_2-t_1)^{\beta}. \quad (67)$$

Obviously,  $Ax(0) = 0$  and  $Ay(0) = 0$ . Therefore, from (67), we have

$$\begin{aligned} \|Ax-Ay\|_{\beta} &= |(Ax-Ay)(0)| + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{|(Ax-Ay)(t_2) - (Ax-Ay)(t_1)|}{|t_1 - t_2|^{\beta}} \\ &\leq (3LL_{1,h}T^{\alpha-1-\beta} + 3LL_{2,h}T^{1-\beta})\|x-y\|_{\beta}. \end{aligned}$$

We obtain that  $A : H_{\beta}[0, T] \rightarrow H_{\beta}[0, T]$  is a contractive operator together with (56). Then  $A$  exists a unique fixed point  $x^* \in H_{\beta}[0, T]$  according to Banach fixed point theorem. Lemma 3.1 indicates that the obtained unique fixed pointed  $x \in H_{\beta}[0, T]$  is the solution of the boundary value problem (10).  $\square$

**Theorem 3.2** Suppose that  $(A_1)$ ,  $(A_2)$ ,  $(A_3)$ ,  $(A_4)$  and  $(A_6)$  hold. Then the boundary value problem (10) exists a solution  $x \in H_{\gamma}[0, T]$  with  $\alpha_1 < \gamma < \beta < \alpha - 1$ .

**Proof.** According to Lemma 3.1, it is sufficient to consider the existence of fixed point of operator  $A$  defined by

$$Ax(t) = \int_0^T G(t, s)h(s)f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right)ds, \quad 0 \leq t \leq T.$$

In the next analysis, we let

$$\begin{aligned} \bar{B}_2 &= \max\{1, T^{p^*-p_*}\} \left[ \max\left\{\frac{a_3 T^{\beta-p^*}}{\beta-p^*}, \frac{a_3 T^{\gamma-p^*}}{\gamma-p^*}\right\} + \frac{a_2 a_4 a_5 T^{1-p^*}}{1-p^*} \right]. \\ \bar{M} &= \max \left\{ M_1 a_5^{\mu_1}, M_2 \left( \frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1) |\Gamma(-\alpha_1)|} \right)^{\mu_2}, M_2 \left( \frac{T^{\gamma-\alpha_1}}{(\gamma-\alpha_1) |\Gamma(-\alpha_1)|} \right)^{\mu_2}, \right. \\ &\quad \left. \frac{M_2 a_5^{\mu_2}}{[\Gamma(1-\alpha_1)]^{\mu_2}}, M_3 (B_1 + \bar{B}_2)^{\mu_3}, M_3 a_1^{\mu_3} a_5^{\mu_3} \right\}, \end{aligned} \quad (68)$$

$$M_{1,h} = \max \left\{ \frac{\int_0^T |h(s)| ds}{\Gamma(\alpha)}, \frac{\int_0^T s^{-\alpha_1 \mu_2} |h(s)| ds}{\Gamma(\alpha)}, \frac{\int_0^T s^{-p(0)\mu_3} |h(s)| ds}{\Gamma(\alpha)} \right\}, \quad (69)$$

$$M_{2,h} = \max \left\{ \sup_{0 \leq t \leq T} I_0^{\alpha-1} |h(t)|, \sup_{0 \leq t \leq T} I_0^{\alpha-1} |t^{-\alpha_1 \mu_2} h(t)|, \sup_{0 \leq t \leq T} I_0^{\alpha-1} |t^{-p(0)\mu_3} h(t)| \right\}, \quad (70)$$

$$M = \max \left\{ \bar{M}(T^{\alpha-1-\beta} M_{1,h} + T^{1-\beta} M_{2,h}), \bar{M}(T^{\alpha-1-\gamma} M_{1,h} + T^{1-\gamma} M_{2,h}) \right\}, \quad (71)$$

where  $a_1$ ,  $a_5$  and  $B_1$  are the constants given by (48)-(50).

Firstly, we show that  $A : H_{\beta}[0, T] \rightarrow H_{\beta}[0, T]$  with  $\alpha_1 < \gamma < \beta < \alpha - 1$ . In fact, for  $x \in H_{\beta}[0, T]$ , we obtain the same estimation of  $\omega_1(x(t))$ ,  $\omega_2(x(t))$ ,  $\omega_3(x(t))$ ,  $g_1(x(t))$  and  $g_2(x(t))$  as the Theorem 3.1.

From (58)-(62), (39) and (68) together with the inequality  $(a + b)^\delta \leq a^\delta + b^\delta$  for  $0 < \delta < 1, a, b \geq 0$ , we have

$$\begin{aligned}
& \left| f\left(x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t))\right) \right| \\
& \leq M_1 |x(t)|^{\mu_1} + M_2 |g_1(x(t)) + g_2(x(t))|^{\mu_2} + M_3 \left| \sum_{i=1}^3 \omega_i(x(t)) \right|^{\mu_3} \\
& \leq M_1 a_5^{\mu_1} \|x\|_{\beta}^{\mu_1} + M_2 \left( |g_1(x(t))|^{\mu_2} + |g_2(x(t))|^{\mu_2} \right) \\
& \quad + M_3 \left( |\omega_1(x(t))|^{\mu_3} + |\omega_2(x(t))|^{\mu_3} + |\omega_3(x(t))|^{\mu_3} \right) \\
& \leq M_1 a_5^{\mu_1} \|x\|_{\beta}^{\mu_1} + M_2 \left( \frac{T^{\beta-\alpha_1}}{(\beta-\alpha_1)|\Gamma(-\alpha_1)|} \right)^{\mu_2} \|x\|_{\beta}^{\mu_2} + \frac{M_2 a_5^{\mu_2} t^{-\alpha_1 \mu_2}}{[\Gamma(1-\alpha_1)]^{\mu_2}} \|x\|_{\beta}^{\mu_2} \\
& \quad + M_3 (B_1 + \bar{B}_2)^{\mu_3} \|x\|_{\beta}^{\mu_3} + M_3 a_1^{\mu_3} a_5^{\mu_3} t^{-p(0)\mu_3} \|x\|_{\beta}^{\mu_3} \\
& \leq \bar{M} (\|x\|_{\beta}^{\mu_1} + \|x\|_{\beta}^{\mu_2} + \|x\|_{\beta}^{\mu_3} + t^{-\alpha_1 \mu_2} \|x\|_{\beta}^{\mu_2} + t^{-p(0)\mu_3} \|x\|_{\beta}^{\mu_3}). \tag{72}
\end{aligned}$$

For  $x \in H_{\beta}[0, T]$ ,  $t_1, t_2 \in [0, T]$  with  $t_1 \neq t_2$  (without loss of generality, we assume that  $0 \leq t_1 < t_2 \leq T$ ), in a similar fashion to (65) and together with (68)-(72), we have

$$\begin{aligned}
& |Ax(t_2) - Ax(t_1)| \\
& \leq \frac{T^{1-\alpha} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |h(s)| \left| f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right| ds \\
& \quad + \int_{t_1}^{t_2} \left| I_{0+}^{\alpha-1} \left[ h(s) f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right] \right| ds \\
& \leq \frac{\bar{M} (t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s)| (\|x\|_{\beta}^{\mu_1} + \|x\|_{\beta}^{\mu_2} + \|x\|_{\beta}^{\mu_3} + s^{-\alpha_1 \mu_2} \|x\|_{\beta}^{\mu_2} + s^{-p(0)\mu_3} \|x\|_{\beta}^{\mu_3}) ds \\
& \quad + \bar{M} \int_{t_1}^{t_2} I_{0+}^{\alpha-1} \left[ |h(s)| (\|x\|_{\beta}^{\mu_1} + \|x\|_{\beta}^{\mu_2} + \|x\|_{\beta}^{\mu_3} + s^{-\alpha_1 \mu_2} \|x\|_{\beta}^{\mu_2} + s^{-p(0)\mu_3} \|x\|_{\beta}^{\mu_3}) \right] ds \\
& \leq \bar{M} M_{1,h} (\|x\|_{\beta}^{\mu_1} + 2\|x\|_{\beta}^{\mu_2} + 2\|x\|_{\beta}^{\mu_3}) T^{\alpha-1-\beta} (t_2 - t_1)^{\beta} \\
& \quad + \bar{M} M_{2,h} (\|x\|_{\beta}^{\mu_1} + 2\|x\|_{\beta}^{\mu_2} + 2\|x\|_{\beta}^{\mu_3}) T^{1-\beta} (t_2 - t_1)^{\beta} \\
& \leq M (\|x\|_{\beta}^{\mu_1} + 2\|x\|_{\beta}^{\mu_2} + 2\|x\|_{\beta}^{\mu_3}) (t_2 - t_1)^{\beta}. \tag{73}
\end{aligned}$$

The inequality (73) implies that  $Ax(t) \in H_\beta[0, T]$ . Then,  $A : H_\beta[0, T] \rightarrow H_\beta[0, T]$ . Let the ball  $\Omega_\rho$  in  $H_\beta[0, T]$  be  $\Omega_\rho = \{x \in H_\beta; \|x\|_\beta \leq \rho\}$ , where

$$\rho \geq \max \left\{ (3M)^{\frac{1}{1-\mu_1}}, (6M)^{\frac{1}{1-\mu_2}}, (6M)^{\frac{1}{1-\mu_3}}, (3Ma_5^{\mu_1})^{\frac{1}{1-\mu_1}}, (6Ma_5^{\mu_2})^{\frac{1}{1-\mu_2}}, (6Ma_5^{\mu_3})^{\frac{1}{1-\mu_3}} \right\}. \quad (74)$$

Secondly, we prove  $A : \Omega_\rho \rightarrow \Omega_\rho$  in the space  $H_\beta[0, T]$ . For  $x \in \Omega_\rho$ , by (73) together with  $Ax(0) = 0$ , we have

$$\begin{aligned} \|Ax\|_\beta &= |Ax(0)| + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{|Ax(t_1) - Ax(t_2)|}{|t_1 - t_2|^\beta} \\ &\leq M(\|x\|_\beta^{\mu_1} + 2\|x\|_\beta^{\mu_2} + 2\|x\|_\beta^{\mu_3}) \\ &\leq M\rho^{\mu_1} + 2M\rho^{\mu_2} + 2M\rho^{\mu_3} \\ &\leq \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3} \leq \rho. \end{aligned} \quad (75)$$

Hence  $A(\Omega_\rho)$  is a bounded subset in  $H_\beta[0, T]$ . Then we obtain that  $A(\Omega_\rho)$  is relatively compact in  $H_\gamma[0, T]$  by Lemma 2.3 for  $\alpha_1 < \gamma < \beta$ . According to Lemma 2.2, the ball  $\Omega_\rho$  is contained in the space  $H_\gamma[0, T]$  for  $\alpha_1 < \gamma < \beta$ .

Thirdly, we prove the operator  $A : \Omega_\rho \rightarrow \Omega_\rho$  according to the norm  $\|\cdot\|_\gamma$  in the space  $H_\gamma[0, T]$ . For  $x \in \Omega_\rho \subset H_\gamma[0, T]$ , in a similar fashion to Theorem 3.1, we have

$$|\omega_1(x(t))| \leq B_1 \|x\|_\gamma, \quad |\omega_2(x(t))| \leq \bar{B}_2 \|x\|_\gamma, \quad |\omega_3(x(t))| \leq a_1 a_5 \|x\|_\gamma t^{-p(0)} \quad (76)$$

and

$$|g_1(x(t))| \leq \frac{T^{\gamma-\alpha_1}}{(\gamma-\alpha_1)|\Gamma(-\alpha_1)|} \|x\|_\gamma, \quad |g_2(x(t))| \leq \frac{a_5 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} \|x\|_\gamma. \quad (77)$$

From (76), (77) and (68), in similar fashion to (72), we have

$$\begin{aligned} &\left| f \left( x(t), g_1(x(t)) + g_2(x(t)), \sum_{i=1}^3 \omega_i(x(t)) \right) \right| \\ &\leq \bar{M} (\|x\|_\gamma^{\mu_1} + \|x\|_\gamma^{\mu_2} + \|x\|_\gamma^{\mu_3} + t^{-\alpha_1 \mu_2} \|x\|_\gamma^{\mu_2} + t^{-p(0) \mu_3} \|x\|_\gamma^{\mu_3}). \end{aligned} \quad (78)$$

From (78) and (71), in similar fashion to (73), we have

$$|Ax(t_2) - Ax(t_1)| \leq M(\|x\|_\gamma^{\mu_1} + 2\|x\|_\gamma^{\mu_2} + 2\|x\|_\gamma^{\mu_3})(t_2 - t_1)^\gamma. \quad (79)$$

From (79) and (74) together with  $Ax(0) = 0$ , in similar fashion to (75), we have

$$\begin{aligned}
\|Ax\|_{\gamma} &= |Ax(0)| + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{|Ax(t_1) - Ax(t_2)|}{|t_1 - t_2|^{\gamma}} \\
&\leq M(\|x\|_{\gamma}^{\mu_1} + 2\|x\|_{\gamma}^{\mu_2} + 2\|x\|_{\gamma}^{\mu_3}) \\
&\leq M(a_5^{\mu_1} \|x\|_{\beta}^{\mu_1} + 2a_5^{\mu_2} \|x\|_{\beta}^{\mu_2} + 2a_5^{\mu_3} \|x\|_{\beta}^{\mu_3}) \\
&\leq M(a_5^{\mu_1} \rho^{\mu_1} + 2a_5^{\mu_2} \rho^{\mu_2} + 2a_5^{\mu_3} \rho^{\mu_3}) \\
&\leq \frac{\rho}{3} + \frac{\rho}{3} + \frac{\rho}{3} \leq \rho,
\end{aligned} \tag{80}$$

which indicates that the operator  $A : \Omega_{\rho} \rightarrow \Omega_{\rho}$  according to the norm  $\|\cdot\|_{\gamma}$ . Hence, as a result, we obtain that the operator  $A$  maps  $\Omega_{\rho}$  into itself for the norm  $\|\cdot\|_{\gamma}$  and  $A(\Omega_{\rho})$  is relatively compact in  $H_{\gamma}[0, T]$ .

In the end, we prove that the operator  $A : \Omega_{\rho} \rightarrow \Omega_{\rho}$  is continuous according to the norm  $\|\cdot\|_{\gamma}$ . Consider any  $x_n \in \Omega_{\rho}$  such that  $x_n \rightarrow x$  in  $\Omega_{\rho}$ . By virtue of the continuity of function  $f(x, y, z)$ , then  $\forall \varepsilon > 0$ ,  $\exists N > 0$ , for  $n > N$ , we have

$$\left| f\left(x_n(s), g_1(x_n(s)) + g_2(x_n(s)), \sum_{i=1}^3 \omega_i(x_n(s))\right) - f\left(x(s), g_1(x(s)) + g_2(x(s)), \sum_{i=1}^3 \omega_i(x(s))\right) \right| < \varepsilon. \tag{81}$$

From (81), in a similar fashion to (73), for  $n > N$ , we have

$$\begin{aligned}
&|(Ax_n - Ax)(t_2) - (Ax_n - Ax)(t_1)| \\
&\leq \frac{\varepsilon(t_2 - t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s)| ds + \varepsilon \int_{t_1}^{t_2} I_{0+}^{\alpha-1} |h(s)| ds \\
&\leq \varepsilon(M_{1,h} T^{\alpha-1-\gamma} + M_{2,h} T^{1-\gamma})(t_2 - t_1)^{\gamma}, \quad 0 \leq t_1 < t_2 \leq T.
\end{aligned} \tag{82}$$

Since  $(Ax_n - Ax)(0) = 0$ , by (82), for  $n > N$ , we have

$$\begin{aligned}
\|Ax_n(t) - Ax(t)\|_{\gamma} &= |(Ax_n - Ax)(0)| + \sup_{\substack{t_1, t_2 \in [0, T] \\ t_1 \neq t_2}} \frac{|(Ax_n - Ax)(t_2) - (Ax_n - Ax)(t_1)|}{|t_2 - t_1|^{\gamma}} \\
&\leq \varepsilon(M_{1,h} T^{\alpha-1-\gamma} + M_{2,h} T^{1-\gamma}),
\end{aligned} \tag{83}$$

which indicates that  $A$  is continuous at the any point  $x \in \Omega_{\rho}$  with the norm  $\|\cdot\|_{\gamma}$ .

Hence,  $A : \Omega_{\rho} \rightarrow \Omega_{\rho}$  is completely continuous in  $H_{\gamma}[0, T]$ . Then we deduce that the boundary value problem (10) exists a solution  $x \in H_{\gamma}[0, T]$  ( $\alpha_1 < \gamma < \beta < \alpha - 1$ ) according to Schauder fixed point theorem.  $\square$

**Example 3.1** We consider the following boundary value problem

$$\begin{cases} D_{0+}^{1.9}x(t) + t^{-0.1} \left( \frac{1}{400(1+x^2)} + \frac{(D_{0+}^{0.7}x)^2}{600(1+(D_{0+}^{0.7}x)^2)} + \frac{(D_{0+}^{0.6-0.3t}x)^2}{600(1+(D_{0+}^{0.6-0.3t}x)^2)} \right) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \quad (84)$$

then we claim that the boundary value problem (84) exists a unique nontrivial solution  $x \in H_{0.8}[0, 1]$ .

**Proof.** Let  $\alpha = 1.9$ ,  $\alpha_1 = 0.7$ ,  $p(t) = 0.6 - 0.3t$ ,  $\beta = 0.8$ ,  $h(t) = t^{-0.1}$ ,  $f(x, y, z) = \frac{1}{400(1+x^2)} + \frac{y^2}{600(1+y^2)} + \frac{z^2}{600(1+z^2)}$ . Obviously,  $p^* = 0.6$ ,  $p_* = 0.3$  and  $p^* < \alpha_1 < \beta < \alpha - 1$ . Since  $h(t) \in L(0, 1)$ ,

$$\frac{\int_0^1 |h(s)| ds}{\Gamma(\alpha)} = \frac{1}{0.9\Gamma(1.9)}, \quad \frac{\int_0^1 s^{-\alpha_1} |h(s)| ds}{\Gamma(\alpha)} = \frac{1}{0.2\Gamma(1.9)}, \quad \frac{\int_0^1 s^{-p(0)} |h(s)| ds}{\Gamma(\alpha)} = \frac{1}{0.3\Gamma(1.9)}$$

and

$$\sup_{0 \leq t \leq 1} I_{0+}^{\alpha-1} |h(t)| = \frac{\Gamma(0.9)}{\Gamma(1.8)} \approx 1.1474, \quad \sup_{0 \leq t \leq 1} I_{0+}^{\alpha-1} |t^{-0.7} h(t)| = \frac{\Gamma(0.2)}{\Gamma(1.1)} \approx 4.8256,$$

$$\sup_{0 \leq t \leq 1} I_{0+}^{\alpha-1} |t^{-0.6} h(t)| = \frac{\Gamma(0.3)}{\Gamma(1.2)} \approx 3.2582,$$

then  $p(t)$ ,  $\alpha$ ,  $\alpha_1$ ,  $h(t)$  and  $f$  satisfy the assumptions  $(A_1)$ ,  $(A_2)$ ,  $(A_3)$  and  $(A_4)$ .

Since

$$\begin{aligned} & \left| \frac{1}{400(1+x_1^2)} - \frac{1}{400(1+x_2^2)} \right| \\ &= \frac{|x_1 + x_2| |x_1 - x_2|}{400(1+x_1^2)(1+x_2^2)} \\ &\leq \left( \frac{|x_1|}{400(1+x_1^2)} + \frac{|x_2|}{400(1+x_2^2)} \right) |x_1 - x_2| \\ &\leq \frac{|x_1 - x_2|}{200}, \end{aligned}$$

by the same token, we have

$$\left| \frac{y_1^2}{600(1+y_1^2)} - \frac{y_2^2}{600(1+y_2^2)} \right| \leq \frac{|y_1 - y_2|}{300}.$$

Then, we have

$$|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)| \leq \frac{|x_1 - x_2|}{200} + \frac{|y_1 - y_2|}{300} + \frac{|z_1 - z_2|}{300}, \quad x, y, z \in \mathbb{R},$$

which implies that  $f$  satisfies the assumption  $(A_5)$  of Theorem 3.1 with  $L_1 = \frac{1}{200}$ ,  $L_2 = \frac{1}{300}$  and  $L_3 = \frac{1}{300}$ . By calculation, we have

$$a_1 = \max_{0 \leq t \leq 1} \left| \frac{1}{\Gamma(0.4 + 0.3t)} \right| \approx 0.7704, \quad a_2 = 0.3, \quad a_3 = \max_{0 \leq t \leq 1} \left| \frac{1}{\Gamma(-0.6 + 0.3t)} \right| \approx 0.2821,$$

$$a_4 = \max_{-0.6 \leq t \leq -0.3} \left| \left( \frac{1}{\Gamma(t)} \right)' \right| \approx 0.4884, \quad a_5 = 1, \quad K = \frac{5}{e}, \quad B_1 \approx 2.7034, \quad B_2 \approx 1.7768,$$

$$L \approx 0.0277, \quad L_{1,h} = \frac{1}{0.2\Gamma(1.9)} \approx 5.1988, \quad L_{2,h} \approx 4.8256,$$

then

$$3L(L_{1,h} + L_{2,h}) \approx 0.833 < 1,$$

which implies that the condition (56) is satisfied. According to Theorem 3.1, the boundary value problem (84) exists a unique nontrivial solution in  $H_{0.8}[0, 1]$ .  $\square$

**Example 3.2** We consider the following boundary value problem

$$\begin{cases} D_{0+}^{1.8}x(t) + t^{-0.2} \left( \frac{(\sin x)^{\frac{4}{3}}}{x} + \frac{(D_{0+}^{0.5}x)^{\frac{1}{3}}}{1+x^2} + \frac{(D_{0+}^{0.4-0.2t}x)^{\frac{2}{3}}}{1+(D_{0+}^{0.5}x)^2 + (D_{0+}^{0.4-0.2t}x)^2} \right), & 0 < t < 1, \\ x(0) = x(1) = 0. \end{cases} \quad (85)$$

Then we claim that the boundary value problem (85) exists a nontrivial solution  $x \in H_{0.6}[0, 1]$ .

**Proof.** Let  $\alpha = 1.8$ ,  $p(t) = 0.4 - 0.2t$ ,  $\alpha_1 = 0.5$ ,  $\beta = 0.7$ ,  $\gamma = 0.6$ ,  $f(x, y, z) = \frac{(\sin x)^{\frac{4}{3}}}{x} + \frac{y^{\frac{1}{3}}}{1+x^2} + \frac{z^{\frac{2}{3}}}{1+y^2+z^2}$ .

Obviously,  $p^* = 0.4$ ,  $p_* = 0.2$  and  $p^* < \alpha_1 < \gamma < \beta < \alpha - 1$ . Then,  $p(t)$ ,  $\alpha$ ,  $\alpha_1$ ,  $h(t)$  and  $f$  satisfy the assumptions  $(A_1)$ ,  $(A_2)$ ,  $(A_3)$  and  $(A_4)$ .

Since

$$\left| \frac{(\sin x)^{\frac{4}{3}}}{x} \right| \leq \frac{|x|^{\frac{4}{3}}}{|x|} = |x|^{\frac{1}{3}},$$

then

$$|f(x, y, z)| \leq |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{2}{3}}, \quad x, y, z \in \mathbb{R},$$

which implies that  $f$  satisfies the assumption  $(A_6)$  of Theorem 3.2 with  $M_1 = M_2 = M_3 = 1$ ,  $\mu_1 = \frac{1}{3}$ ,  $\mu_2 = \frac{1}{3}$  and  $\mu_3 = \frac{2}{3}$ .

According to Theorem 3.2, we obtain that the boundary value problem (85) exists a nontrivial solution  $x \in H_{0,6}[0, 1]$ .  $\square$

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## Conflict of interest

The authors declare that they have no competing interests.

## References

- [1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.
- [2] Samko SG, Ross B. Integration and differentiation to a variable fractional order. *Integral Transforms and Special Functions*. 1993; 1(4): 277-300.
- [3] Lorenzo CF, Hartley TT. Variable order and distributed order fractional operators. *Nonlinear Dynamics*. 2002; 29(1): 57-98.
- [4] Coimbra CFM. Mechanics with variable-order differential operators. *Annalen der Physik*. 2003; 12(11-12): 692-703.
- [5] Patnaik S, Hollkamp JP, Semperlotti F. Applications of variable-order fractional operators: a review. *Proceedings of The Royal Society A Mathematical Physical and Engineering Sciences*. 2020; 476(2234): 20190498.
- [6] Jiang JF, Chen HT, Cao DQ, Guirao JLG. The global sliding mode tracking control for a class of variable order fractional differential systems. *Chaos, Solitons & Fractals*. 2022; 154: 111674.
- [7] Ali U, Ahmad H, Baili J, Botmart T, Aldahlan MA. Exact analytical wave solutions for space-time variable-order fractional modified equal width equation. *Results in Physics*. 2022; 33: 105216.
- [8] Jahanshahi H, Zambrano-Serrano E, Bekiros S, Wei ZC, Volos C, Castillo O, et al. On the dynamical investigation and synchronization of variable-order fractional neural networks: the Hopfield-like neural network model. *The European Physical Journal Special Topics*. 2022; 231: 1757-1769.
- [9] DarAssi MH, Safi MA, Khan MA, Beigi A, Aly AA, Alshahrani MY. A mathematical model for SARS-CoV-2 in variable-order fractional derivative. *The European Physical Journal Special Topics*. 2022; 231: 1905-1914.
- [10] Hosseiniinia M, Heydari MH, Avazzadeh Z. Orthonormal shifted discrete Legendre polynomials for the variable-order fractional extended Fisher-Kolmogorov equation. *Chaos, Solitons & Fractals*. 2022; 155: 111729.
- [11] Valério D, Da Costa JS. Variable-order fractional derivatives and their numerical approximations. *Signal Processing*. 2011; 91(3): 470-483.
- [12] Zhang SQ. The uniqueness result of solutions to initial value problems of differential equations of variable-order. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*. 2018; 112(2): 407-423.
- [13] Pandey P, Singh J. An efficient computational approach for nonlinear variable order fuzzy fractional partial differential equations. *Computational and Applied Mathematics*. 2022; 41: 38.
- [14] Hassani H, Machado JAT, Avazzadeh Z. An effective numerical method for solving nonlinear variable-order fractional functional boundary value problems through optimization technique. *Nonlinear Dynamics*. 2019; 97(4): 2041-2054.
- [15] Lin FR, Wang QY, Jin XQ. Crank-Nicolson-weighted-shifted-Grünwald-difference schemes for space Riesz variable-order fractional diffusion equations. *Numerical Algorithms*. 2021; 87(2): 601-631.
- [16] Zheng X, Wang H. An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes. *SIAM Journal on Numerical Analysis*. 2020; 58(1): 330-352.
- [17] Alrabaiah H, Ahmad I, Amin R, Shah K. A numerical method for fractional variable order pantograph differential

equations based on Haar wavelet. *Engineering with Computers*. 2022; 38(3): 2655-2668.

[18] Zhang SQ, Hu L. The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. *AIMS Mathematics*. 2020; 5(4): 2923-2943.

[19] Zheng XC, Wang H. Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems. *Applicable Analysis*. 2022; 101(6): 1848-1870.

[20] Ersoy MT, Furkan H. On the existence of the solutions of a Fredholm integral equation with a modified argument in Hölder spaces. *Symmetry*. 2018; 10(10): 522.

[21] Caballero J, Darwish MA, Sadarangani K. Solvability of a quadratic integral equation of Fredholm type in Hölder spaces. *Electronic Journal of Differential Equations*. 2014; 2014(31): 1-10.

[22] Ersoy MT, Furkan H, Sarıçek B. On the solutions of some nonlinear fredholm integral equations in topological Hölder spaces. *TWMS Journal of Applied and Engineering Mathematics*. 2020; 10(3): 657-668.

[23] Dal İ, Temizer ÖF. Solvability of a quadratic integral equation of Fredholm type via a modified argument. *Journal of Mathematics and Applications*. 2020; 43: 47-66.

[24] Okrasinska-P lociniczak H, Płociniczak Ł, Rocha J, Sadarangani K. Solvability in Hölder spaces of an integral equation which models dynamics of the capillary rise. *Journal of Mathematical Analysis and Applications*. 2020; 490(1): 124237.

[25] Özdemir İ. On the solvability of a class of nonlinear integral equations in Hölder spaces. *Numerical Functional Analysis and Optimization*. 2022; 43(4): 365-393.

[26] Cabrera I, Harjani J, Sadarangani K. Existence and uniqueness of solutions for a boundary value problem of fractional type with nonlocal integral boundary conditions in Hölder spaces. *Mediterranean Journal of Mathematics*. 2018; 15(3): 98.

[27] Caballero J, López B, Sadarangani K. Positive solutions in the space of Lipschitz functions to a class of fractional differential equations with infinite-point boundary value conditions. *Journal of Applied Analysis and Computation*. 2021; 11(2): 1039-1050.

[28] Banaś J, Nalepa R. On the space of functions with growths tempered by a modulus of continuity and its applications. *Journal of Function Spaces and Applications*. 2013; 2013: 820437.

[29] Zheng XC, Wang H. Wellposedness and smoothing properties of history-state-based variable-order time-fractional diffusion equations. *Zeitschrift für angewandte Mathematik und Physik*. 2020; 71(1): 1-25.

[30] Ross B, Samko S. Fractional integration operator of variable order in the Hölder spaces  $H^{(x)}$ . *International Journal of Mathematics and Mathematical Sciences*. 1995; 18(4): 777-788.

[31] Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives*. Yverdon-les-Bains: Gordon and breach science publishers; 1993.

[32] Zhang SQ, Su XW. Unique existence of solution to initial value problem for fractional differential equation involving with fractional derivative of variable order. *Chaos, Solitons & Fractals*. 2021; 148: 111040.