Adaptive Identification of System with Bouc-Wen Hysteresis Modifications

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Abstract: The adaptive identification method developed to evaluate the parameters of the Bouc-Wen hysteresis (BWH). The adaptive approach based on the use of adaptive observer. We synthesize adaptive identification algorithms using the second Lyapunov method. Requirements for the input of the system which guarantee the identification of parameters considered. We propose BWH modifications (BWHM). Adaptive algorithms for estimating BWHM parameters developed. The boundedness of adaptive system processes shown in coordinate and parametric spaces. We prove the exponential dissipativity of processes in an adaptive system by using the Lyapunov vector function method. Estimating method proposed for signaling uncertainty in the system.

Keywords: hysteresis, Bouc-Wen model, adaptive observer, exponential dissipativity, Lyapunov vector function, uncertainty

MSC: 93B30, 34C55, 93C40

1. Introduction

Various models [1] apply to the description of the hysteresis. But the Bouc-Wen model (BWM) [2-3] has the widest application. Many BWM modifications (BWMM) [4-9] were proposed. Each model considered the features of the object [10-15]. The BWM successful application depends on the identification of its parameters [16-21]. Various algorithms are used for BWM identification [22-24]. An adaptive parameter identification method [14] was proposed for the BW hysteresis model. Identification of BW hysteresis parameters is based on time data consider in [15]. The algorithm is based on the least squares method and the sensitivity analysis of the output.

In [15, 17], adaptive algorithms propose for the BWM parameters estimation with the data forgetting [8]. Paper [18] presents an adaptive on-line identification methodology with a variable trace method to adjust the adaptation gain matrix. Most BWMM are based on the BWH approximation in some working areas of the object [19-24]. The approximation method choice depends on the requirements of the control and the workspace. Parametric identification procedures apply to obtain simplified BWH models.

Most proposed approaches use the derivative measurement by the output of the system. This possibility does not always exist in practical applications. There are studies [25] which estimates of BWM parameters do not coincide with the results obtained for other input data. Explain it with the fact that the BWM should be stable and ensure the adequacy
of a physical process [4].

The conditions to be satisfied by the Bouc-Wen model are considered in [26]. The major difficulties of the BWM parameters estimation are (i) the ensuring model stability (ii) the input choice. The stability imposes restrictions on the ranges of changes in model parameters. The choice of parameters belonging to the stability domain does not always give an adequate BWM [26].

So, the set of algorithms and procedures proposed for the Bouc-Wen model parameters identification. The models reflect the features of the system under study. As a rule, the change area BWM parameters set a priori. This is also true for some system parameters. It is often assumed that all system derivatives are measured. This assumption is not always true, which makes the algorithms unrealizable. Most identification procedures are valid only in some areas. Therefore, the design of identification algorithms is an urgent problem for BWH under uncertainty.

Below, we propose the adaptive identification based on adaptive observer. It is used for the problem solution stability and identification for system (1)-(4). Method are based on the approach proposed in [27-28] and does not require measuring derivatives of the system output. We believe that only the input and output of the system can be measured. BWH modifications are considered. They reduce the use of the model and remove the stability problem.

The paper has the following structure. Section 2 contains the problem statement. Section 3 considers the proposed modifications of the BWH (BWHM). These modifications guarantee the stable solving of the system with BWHM and its identification. Section 4 contains requirements on an input that guarantee the structural identifiability of the system with BWHM. The adaptive observer and the analysis of its properties are considered in sections 5, 6. We present modeling and discussion of results in sections 7, 8. Appendixes contain the stability proof of the adaptive system.

2. Problem statement

Consider the system $S_{BW}$

$$m\ddot{x} + c\dot{x} + F(x,z,t) = f(t),$$  \hspace{1cm} (1)

$$F(x,z,t) = a k x(t) + (1 - \alpha) kd z(t),$$  \hspace{1cm} (2)

$$\dot{z} = d^{-1} \left( a x - \beta |x| |\dot{x}|^{\alpha} \text{sign}(z) - \gamma |\dot{x}|^{\gamma} \right),$$  \hspace{1cm} (3)

$$y(t) = x(t),$$  \hspace{1cm} (4)

where $m > 0$ is mass, $c > 0$ is damping, $F(x, z, t)$ is the recovering force, $d > 0$, $n > 0$, $k > 0$, $\alpha \in (0, 1)$, $f(t)$ is exciting force, $\alpha, \beta, \gamma$ are some numbers. The system (1)-(4) are the basis for the classic BWH presentation. All further studies on BWH are based on the modification of this system. Equations (1)-(4) are used for the analysis of nonlinear mechanical systems. Adaptation of system $S_{BW}$ to real objects requires BWH modification.

The system (1)-(4) are widely used for the processes analysis in construction mechanics, control of complex mechanical systems, modeling the work of damping devices and the like. Equation (1) describes an object that is affected by the restoring force $F(x, z, t)$ and the exciting force $f(t)$. In applications, various approximations $F(x, z, t)$ are used, reflecting the specific of the system.

The set of the experimental data

$$I_o = \{ f(t), y(t), t \in J \},$$  \hspace{1cm} (5)

where $J \subset R$ is the given time interval. Denote the system parameters vector as $A = [m, c, a, k, \alpha, \beta, \gamma, n]^T$.

Problem: design the adaptive observer for vector $A$ estimation to
\[
\lim_{{t \to \infty}} |\hat{y}(t) - y(t)| \leq \pi_y
\]

where \( \hat{y} \in \mathbb{R} \) is the output of the adaptive observer, \( \pi_y \geq 0 \).

3. System \( S_{BW} \) modifications

Various modifications of BWH have been proposed (see, for example, \([9, 11, 26]\)). They consider features and properties of the system. System (1)-(3) are the basis for modifications. The analysis shows that the last term in (3) guarantees “fine-tuning” the BW hysteresis in the saturation or switching areas. If this is not critical for the system, then by selecting parameters of the S1-system, this term in the equation (3) can be compensated. In addition, some modifications are simplified and increase the system (1)-(3) stability. The main purpose of making structural changes is to simplify the system and improve its properties. We propose the following modifications of the Bouc-Wen model (3) \([28]\)

\[
\mathcal{M}_{\rho\mu\beta}\hat{z} = \pi |x|^\rho \text{sign}(\dot{x}) - \beta |z|^\mu \text{sign}(z),
\]

(7)

\[
\mathcal{M}_{\mu\beta}\hat{z} = \pi |\dot{x}|^\mu \text{sign}(\dot{x}) - \beta |z|^\mu \text{sign}(z),
\]

(8)

\[
\mathcal{M}_{\mu\mu\beta}\hat{z} = \pi |\dot{x}|^\mu \text{sign}(\dot{x}) - \beta |z|^\mu \text{sign}(z).
\]

(9)

The linear component on \( z \) in (7) increases the feasibility model, and stability of the system. As the system is nonlinear, the function \( |\dot{x}(t)|^\mu \) is introduced to ensure the required hysteresis state. It guarantees a change \( z \) in the specified boundaries. Parameters \( \rho > 0, \omega > 0 \) are some numbers.

We have not tried to due reproduce BWH (3) using modifications (7)-(9). A detailed analysis of the models (7)-(9) parameters effect of on hysteresis is given in \([28]\).

4. About influence \( f(t) \) on BWH parameters identifiability

The input choice is an important stage in the nonlinear systems identification. These issues are discussed in \([28-29]\). The input \( f(t) \) of the system must be constantly excited and have the property of S-synchronizability. These conditions are the basis for the structural identifiability of the system (1)-(3). They guarantee the system parameters evaluation using adaptive algorithms.

5. Design of adaptive observer

5.1 System \( S_{BW} \)

Let \( d = 1, a = 1 \). Substitute \( F(x, z, t) \) in (1) and write it as

\[
\left( s^2 + \bar{a}_1 s + \bar{a}_2 \right) x + \bar{a}_3 z = bf',
\]

(10)

where
Reduce (10) to an identification form on \( \dot{x} \). Divide the left and right parts (10) into \( s + \mu \), where \( \mu > 0 \) does not coincide with roots of the polynomial \( s^2 + a_1 s + a_2 \). Then (10)

\[
\begin{align*}
\dot{x} &= a_1 x + a_2 p_x + a_3 p_z + bp_f, \\
\dot{p}_x &= -\mu p_x + x, \\
\dot{p}_f &= -\mu p_f + f, \\
\dot{p}_z &= -\mu p_z + z,
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{c - \mu m}{m}, \\
a_2 &= \frac{-ak - \mu(c - \mu m)}{m}, \\
a_3 &= \frac{(1-\alpha)k}{m}.
\end{align*}
\]

Variables \( p_i, i = x, f, z \) obtained from equations \( p_i = i(s + \mu) \). Equations (11), (12) contain only measurable variables except \( z \). It complicates the identification of the system \( S_{BW} \) parameters.

**Remark 1.** Simplifications \( d = 1 \) and \( a = 1 \) do not affect the parameters (11) identification. Consideration \( d, a \) increases the number of estimated parameters. The system (10)-(12) are used to guarantee the system (1)-(4) parameters identification on the set (5). It excludes the use of the non-measurable derivative \( \dot{x} \) in parametric identification.

Apply the model for parameters estimate of equation (11)

\[
\dot{x} = -k_x (\tilde{x} - x) + \tilde{a}_1 x + \tilde{a}_2 p_x + \tilde{a}_3 p_z + \tilde{b} p_f,
\]

where \( k_x > 0 \) is specified number, \( \tilde{a}_i(t), i = 1, 2, 3 \) and \( \tilde{b}(t) \) are adjusted parameters.

Designate \( e = \dot{x} - x \) and obtain the equation for the identification error from (11), (13)

\[
\dot{e} = -k_x e + \Delta a_1 x + \Delta a_2 p_x + \Delta a_3 p_z + \Delta b p_f,
\]

where \( \Delta a_1 = \tilde{a}_1(t) - a_1, \Delta a_2 = \tilde{a}_2(t) - a_2, \Delta a_3 = \tilde{a}_3(t) - a_3, \Delta b = \tilde{b}(t) - b. \)

The (14) is not solvable as the variable \( z \) is unknown in (12). Receive the current estimate for \( z \). Consider the model

\[
\dot{\tilde{x}} = -k_x (\tilde{x} - x) + \tilde{a}_1 x + \tilde{a}_2 p_x + \tilde{b} p_f.
\]

Determine the residual \( \varepsilon = x - \tilde{x} \) and use it for the variable \( z \) estimation. Apply the model

\[
\dot{\tilde{z}} = -k_z (\tilde{z} - \varepsilon) + \tilde{\beta} \| \tilde{z} \| \text{sign}(\tilde{z}) - \gamma |\tilde{z}|^{\beta},
\]

where \( \tilde{z} = (x(t + \tau) - x(t))/\tau; k_z > 0 \) is specified number; \( \tilde{\beta}, \gamma \) are the hysteresis (3) parameters estimations; \( \tau \) is the
integration step.

Introduce the residual $\varepsilon = \hat{x} - x$, and obtain the equation for $\varepsilon$

$$\dot{\varepsilon} = -k_{\varepsilon}\varepsilon + \Delta x + \Delta \beta \|\hat{\varepsilon}\|^p \operatorname{sign}(\hat{\varepsilon}) + \beta \eta_\beta + \Delta \gamma \hat{\varepsilon} \|\hat{\varepsilon}\|^p + \gamma \eta_\gamma,$$

$$\eta_\beta = |\hat{\varepsilon}||\varepsilon|^p \operatorname{sign}(\hat{\varepsilon}) - |\hat{\varepsilon}||\varepsilon|^p \operatorname{sign}(\hat{\varepsilon}),$$

$$\eta_\gamma = \dot{x} \|\varepsilon\|^p - \hat{\varepsilon} \|\hat{\varepsilon}\|^p,$$

where $\Delta x = \hat{x} - x$, $\Delta \beta = \beta - \hat{\beta}$, $\Delta \gamma = \gamma - \hat{\gamma}$.

Then the equation (13)

$$\dot{x} = -k_x(x - x) + \tilde{a}_1 x + \tilde{a}_2 p_x + \tilde{a}_3 p_z + \tilde{b} p_f,$$

where

$$\dot{p}_z = -\mu p_z + \dot{\varepsilon}.$$

Then (15)

$$\dot{\varepsilon} = -k_{\varepsilon} e + \Delta a_1 x + \Delta a_2 p_x + \Delta a_3 p_z + \Delta b p_f.$$

Synthesize algorithms for tuning parameters of adaptive models. Consider the Lyapunov function (LF) $V_{\varepsilon}(t) = 0.5 \varepsilon^2(t)$ and obtain for $\dot{V}_{\varepsilon}$

$$\dot{V}_{\varepsilon} = -k_{\varepsilon} \varepsilon^2 + e \left(\Delta a_1 x + \Delta a_2 p_x + \Delta a_3 p_z + \Delta b p_f\right).$$

Obtain adaptive algorithms from the condition $\dot{V}_{\varepsilon} \leq 0$

$$\Delta \hat{a}_1 = -\gamma_1 e_x,$$

$$\Delta \hat{a}_2 = -\gamma_2 p_x,$$

$$\Delta \hat{a}_3 = -\gamma_3 e p_z,$$

$$\Delta b = -\gamma_b e p_f.$$

where $\gamma_i > 0$, $i = 1, 2, 3$; $\gamma_b > 0$.

Synthesize algorithms for tuning model (16) parameters. Consider $V_{\varepsilon}(t) = 0.5 \varepsilon^2(t)$ and equation (17). Then $\dot{V}_{\varepsilon}$

$$\dot{V}_{\varepsilon} = \dot{\varepsilon}^2 + e \left(\Delta \dot{x} + \Delta \beta \|\hat{\varepsilon}\|^p \operatorname{sign}(\hat{\varepsilon}) + \beta \eta_\beta + \Delta \gamma \hat{\varepsilon} \|\hat{\varepsilon}\|^p + \gamma \eta_\gamma\right).$$
where $\varepsilon$ satisfies equation (17).

We receive from (25) 

$$
\Delta \hat{\beta} = -\chi_{\beta} \varepsilon \left| \tilde{x} \right|^{\mu} \text{sign}(\hat{z}),
$$

$$
\Delta \hat{\gamma} = -\chi_{\gamma} \varepsilon \left| \tilde{x} \right|^{n},
$$

(26)

where $\chi_{\beta} > 0, \chi_{\gamma} > 0$ are parameters that ensure the algorithms convergence.

Several algorithms are used to estimate the indicator $n$ in (11). Their effectiveness depends on several factors. A simple algorithm has the form 

$$
\hat{n} = \begin{cases} 
-\gamma_n \varepsilon \hat{\beta} \left| \tilde{x} \right|^{\beta - 1} \hat{z}, & \text{if } \frac{\varepsilon}{\varepsilon z} \in [\nu_0, \nu_1], \\
0, & \text{if } \frac{\varepsilon}{\varepsilon z} \notin [\nu_0, \nu_1].
\end{cases}
$$

(27)

where $\nu_0, \nu_1$ are set positive numbers, $\gamma_n > 0$.

So, equations (12), (17), (21), (22), (24), (26), (27) describe the adaptive identification system for the $S_{BW}$-system. Denote this system as $AS_{BW}$.

### 5.2 System (1), (2) with hysteresis $M_{\rho \omega \mu \nu \beta \theta n}, M_{\mu \beta n}, M_{\mu \mu \beta n}$

1. Model $M_{\rho \omega \mu \nu \beta \theta n}$. Equations (16)-(19) have the form in this case 

$$
\dot{\hat{z}} = -k_n \left( \hat{z} - e_n \right) - \rho \hat{z} \left| \tilde{x} \right|^{\rho} + \mu \left| \tilde{x} \right|^{\mu} \text{sign} \left( \tilde{x} \right) - \beta \left| \tilde{x} \right|^{\beta} \text{sign}(\hat{z}),
$$

(28)

$$
\dot{\hat{\varepsilon}} = -k_n \varepsilon - \Delta \rho \hat{z} \left| \tilde{x} \right|^{\rho} + \Delta \pi \left| \tilde{x} \right|^{\mu} \text{sign} \left( \tilde{x} \right) - \Delta \beta \left| \tilde{x} \right|^{\beta} \text{sign}(\hat{z}) + \rho \tilde{\pi}_\rho + \pi \tilde{\pi}_\pi + \beta \tilde{\pi}_\beta,
$$

(29)

$$
\tilde{\pi}_\rho = \left| \tilde{x} \right|^{\rho} z - \left| \tilde{x} \right|^{\rho} \hat{z},
$$

(30)

$$
\tilde{\pi}_\pi = \left| \tilde{x} \right|^{\mu} \text{sign}(\tilde{x}) - \left| \tilde{x} \right|^{\mu} \text{sign}(\hat{x}),
$$

(31)

$$
\tilde{\pi}_\beta = \left| \tilde{x} \right|^{\beta} \text{sign}(\tilde{x}) - \left| \tilde{x} \right|^{\beta} \text{sign}(\hat{z}).
$$

(32)

Consider $\dot{V}_e$ 

$$
\dot{V}_e = -k_n \varepsilon^2 + e \left( -\Delta \rho \hat{z} \left| \tilde{x} \right|^{\rho} + \Delta \pi \left| \tilde{x} \right|^{\mu} \text{sign} \left( \tilde{x} \right) - \Delta \beta \left| \tilde{x} \right|^{\beta} \text{sign}(\hat{z}) + \rho \tilde{\pi}_\rho + \pi \tilde{\pi}_\pi + \beta \tilde{\pi}_\beta \right)
$$

(33)

and obtain algorithms
\[ \Delta \beta = \chi \beta e^{|x_0|} \text{sign}(z). \]

\[ \Delta \hat{\pi} = -\chi \pi e^{|x_0|} \text{sign}(z). \]

\[ \Delta \hat{\rho} = -\chi \rho e^{|z_0|}, \]  \hspace{1cm} (34)

where \( \chi > 0, \chi > 0, \chi > 0 \) are parameters guaranteed convergence of algorithms, \( \Delta \rho = \hat{\rho}(t) - \rho, \Delta \pi = \hat{\pi}(t) - \pi. \)

The structure of algorithms for estimating \( n, \omega, \mu \) coincides with (27).

2. Model \( M_{\mu \beta n} \). Equations (16)-(19) have the form

\[ \dot{z} = -k_z (z - e_z) + \tilde{\pi} |x| e^{|x|} \text{sign}(\tilde{z}) - \tilde{\beta} |z| e^{|z|} \text{sign}(\tilde{z}), \]  \hspace{1cm} (16.2)

\[ \dot{e} = -k_e e - \Delta \beta |x| e^{|z|} \text{sign}(\tilde{z}) + \Delta \pi |x| e^{|z|} \text{sign}(\tilde{z}) + \beta \tilde{\eta}_\beta + \pi \tilde{\eta}_\pi, \]  \hspace{1cm} (17.2)

\[ \tilde{\eta}_\beta = |x| e^{|x|} \text{sign}(z) - |\tilde{x}| e^{|\tilde{x}|} \text{sign}(\tilde{z}), \]  \hspace{1cm} (18.2)

\[ \tilde{\eta}_\pi = |x| e^{|x|} \text{sign}(\tilde{x}) - |\tilde{x}| e^{|\tilde{x}|} \text{sign}(\tilde{z}). \]  \hspace{1cm} (19.2)

3. Model \( M_{\mu \pi \eta} \). Equations (16)-(19) have the form

\[ \dot{z} = -k_z (z - e_z) + \tilde{\pi} |x| e^{|x|} \text{sign}(\tilde{z}) - \tilde{\beta} |x| e^{|z|} \text{sign}(\tilde{z}), \]  \hspace{1cm} (16.3)

\[ \dot{e} = -k_e e - \Delta \beta |x| e^{|z|} \text{sign}(\tilde{z}) + \Delta \pi |x| e^{|z|} \text{sign}(\tilde{z}) + \beta \tilde{\eta}_\beta + \pi \tilde{\eta}_\pi, \]  \hspace{1cm} (17.3)

\[ \tilde{\eta}_\beta = |x| e^{|x|} \text{sign}(z) - |\tilde{x}| e^{|\tilde{x}|} \text{sign}(\tilde{z}), \]  \hspace{1cm} (18.3)

\[ \tilde{\eta}_\pi = |x| e^{|x|} \text{sign}(\tilde{x}) - |\tilde{x}| e^{|\tilde{x}|} \text{sign}(\tilde{z}). \]  \hspace{1cm} (19.3)

Algorithms structurally coincide with (34) for (16.2) and (16.3).

6. Properties \( AS_{BW} \)

Evaluate properties of the \( AS_{BW} \) system. Consider the subsystem \( AS_\chi \) described by equations (22), (24). Let

\[ \Delta K(t) = [\Delta a(t), \Delta z(t), \Delta a_0(t), \Delta b(t)]^T, \]
\[ V_K(t) = 0.5 \Delta K^T(t) \Gamma^{-1} \Delta K(t), \]  
(35)

\[ V(t) = V_e(t) + V_K(t), \]  
(36)

where \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4). \)

**Assumption 1.** The input of the system (1)-(3) is constantly excited and bounded, i.e. the condition

\[ \mathcal{P} \mathcal{E} f : f^2(t) \geq \eta \]  
(37)

is valid for \( \exists \eta > 0 \) and \( \forall t \geq t_0 \) on some interval \([0, T]\).

**Theorem 1.** Let (i) functions \( V_e(t) = 0.5 e^2(t) \), \( V_K(t) \) are positive definite and satisfy conditions \( \inf V_e(e) \rightarrow \infty \), \( \inf V_K(\Delta K) \rightarrow \infty \); (ii) assumption 1 for \( f(t) \) satisfied. Then (i) all trajectories of the system \( AS_X \) bounded, (ii) belong area

\[ G_f = \{(e, \Delta K) : V(t) \leq V(t_0)\}, \]

(a3) the estimation

\[ \int_{t_0}^{t} 2k_e V_e(\tau) d\tau \leq V(t_0) - V(t) \]  
(38)

is fair.

We give the proof of Theorem 1 in Appendix A.

Theorem 1 shows the restriction of adaptive system \( AS_X \) trajectories. Ensuring of asymptotic stability in the system demands to impose additional conditions. Consider these conditions. Let \( B(t) \) is stable, and the input \( f(t) \) is restricted. Therefore, present the property \( \mathcal{P} \mathcal{E} \) for the matrix \( B \) as

\[ \mathcal{P} \mathcal{E}_v : P(t) P^T(t) \geq v I_4 \]  
(39)

fairly for \( \exists v > 0 \) and \( \forall t \geq t_0 \) on some interval \( T > 0 \), where \( I_4 \in \mathbb{R}^4 \) is the unity matrix.

If the vector \( P(t) \) has property \( \mathcal{P} \mathcal{E}_v \), then we will write \( P(t) \in \mathcal{P} \mathcal{E}_v \).

The system \( S_{BP} \) is stable, and the input \( f(t) \) is restricted. Therefore, present the property \( \mathcal{P} \mathcal{E}_v \) for the matrix \( B p(t) = P(t) P^T(t) \) as

\[ \mathcal{P} \mathcal{E}_{v,\sigma} : v I_4 \leq B p(t) \leq \sigma I_4 \quad \forall t \geq t_0, \]  
(40)

where \( \sigma > 0 \) is some number.

Let the estimate for \( V_K(t) \) be fair

\[ 0.5 \beta_1^{-1}(\Gamma) \| \Delta K(t) \|^2 \leq V_K(t) \leq 0.5 \beta_2^{-1}(\Gamma) \| \Delta K(t) \|^2, \]  
(41)

where \( \beta_1(\Gamma), \beta_2(\Gamma) \) are minimal and maximum eigenvalues of the matrix \( \Gamma \).
Apply (40), (41) and get estimations for $V \cdot e$, $V \cdot K$

\[
\dot{V}_e \leq -k_e V_e + \frac{\mathbb{P} \beta_1 (\Gamma)}{k_e} V_K,
\]

\[
\dot{V}_K \leq -\frac{3}{4} \mathcal{G} \nu \beta_1 (\Gamma) V_K + \frac{8}{3} \mathcal{G} V_e,
\]

where $\mathcal{G} > 0$ is some number. We describe the method of obtaining estimates (42), (43) in [30].

Theorem 2. Let conditions be satisfied (i) positive definite Lyapunov functions $V_e(t) = 0.5 \varepsilon^2(t)$ and $V_K(t) = 0.5 \Delta K^T(t)\Gamma^{-1}\Delta K(t)$ allow the indefinitely small highest limit at $|e(t)| \to 0$, $||\Delta K(t)|| \to 0$; (ii) $P(t) \in \mathcal{PE}_{\nu, \mathcal{G}}$; (iii) equality $e\Delta K^T P = \mathcal{G} (\Delta K^T B \Delta K + \varepsilon^2)$ is fair in the area $O(O)$ with $0 < \mathcal{G}$, where $O = \{0, 0^{3m} \} \subset R \times R^{3m} \times J_{0, \mathcal{G}}$, $O$ is some neighborhood of the point $O$; (iv) the function $V_K(t)$ satisfies (41); (v) $\dot{V}_e, \dot{V}_K$ satisfy the system of inequalities

\[
\begin{bmatrix}
\dot{V}_e \\
\dot{V}_K
\end{bmatrix} \leq
\begin{bmatrix}
-k_e & \frac{\mathbb{P} \beta_1 (\Gamma)}{k_e} \\
\frac{8}{3} \mathcal{G} + 3 \nu \beta_1 (\Gamma) & -4
\end{bmatrix}
\begin{bmatrix}
V_e \\
V_K
\end{bmatrix},
\]

(vi) the upper solution for $V_{e,K}(t) = [V_e(t) V_K(t)]^T$ satisfies to the comparison equation $\dot{S} = A_\mathcal{G} S$ if

\[
V_{\rho}(t) \leq s_{\rho}(t) \quad \forall (t \geq t_0) \& \{ V_{\rho}(t_0) \leq s_{\rho}(t_0) \},
\]

where $\rho = e, K$, $S = [s_e s_K]^T$, $A_\mathcal{G} \in R^{2 \times 2}$ is M-matrix [31]. Then the system $A S_X$ is exponentially stable with the estimation

\[
V_{e,K}(t) \leq e^{A_\mathcal{G}(t-t_0)} S(t_0), \quad V_{e,K} = [V_e V_K]^T,
\]

if

\[
k_e > 0, \quad k_e \geq \frac{4}{3} \frac{2^7 \mathbb{P} \beta_1 (\Gamma)}{\nu \beta_1 (\Gamma)}.
\]

Theorem 2 shows if $P(t) \in \mathcal{PE}_{\nu, \mathcal{G}}$, then the adaptive system $A S_X$ gives accurate estimates of system (11) parameters. The system parameters satisfy condition (47). We suppose that the variable $P_t$ bounded.

The boundedness of the variable $\hat{x}_e$ follows from the system stability.

Consider subsystem $A S_Y$ described by equations (17), (25) and (26). Introduce Lyapunov functions

\[
V_{\phi \mathcal{G}}(t) = V_{\phi}(t) + V_{\beta \mathcal{G}}(t),
\]

\[
V_{\beta \mathcal{G}}(t) = 0.5 \chi_{\beta}^{-1} (\Delta \beta(t))^2 + 0.5 \chi_{\gamma}^{-1} (\Delta \gamma(t))^2.
\]
Theorem 3. Let (1) functions $V_\varepsilon(t) = 0.5\varepsilon^2(t)$, $V_{\beta_\varepsilon}(t)$ are positive definite and satisfy condition
\[
\inf_{|\xi| \to \infty} V_\varepsilon(\varepsilon) \to \infty, \quad \inf_{\Delta \beta, \Delta \gamma} V_{\beta_\varepsilon}(\Delta \beta, \Delta \gamma) \to \infty;
\]
(2) the function $V_{\phi_\varepsilon}(t)$ has the form (88); (3) the function
\[
\tilde{g}_1(t) = \sup_{\varepsilon \in \Omega} \frac{|\varepsilon|^{p+1}}{V_\varepsilon(t, \varepsilon)}, \quad g_1 = \sup_{\varepsilon \in \Omega} \tilde{g}_1(t),
\]
exists, where $\Omega$ is the definition range of the subsystem $AS_\gamma$; (4) $|\Delta \dot{x}| \leq \delta_\Delta$, $\delta_\Delta \geq 0$, $|\dot{x}| \leq \nu > 0$; (5) the assumption 1 holds for the system (1)-(3). Then (i) all trajectories of the system $AS_\varepsilon$ bounded, (ii) trajectories belong in the area
\[
G_\varepsilon = \{(\varepsilon, \Delta \beta, \Delta \gamma) : V_{\phi_\varepsilon}(t) \leq V_{\phi_\varepsilon}(t_0)\},
\]
(iii) the estimation
\[
\int_{t_0}^{t} (k_z - \nu(\beta + \gamma)g_1)V_\varepsilon(\tau) d\tau + \frac{1}{2(k_z - \nu(\beta + \gamma)g_1)(t-t_0)}(\delta_\Delta)^2 \leq V_{\phi_\varepsilon}(t_0) - V_{\phi_\varepsilon}(t),
\]
is fair if
\[
k_z > \nu(\beta + \gamma)g_1.
\]
We give the proof of Theorem 1 in Appendix B.

So, the boundedness of trajectories in the adaptive system $AS_{BW}$ was proved. The trajectories limitation of the subsystem $AS_\varepsilon$ is a more complex problem in the parametric and output spaces. The estimation (51) shows that the quality of $AS_\varepsilon$-system processes depends on the output derivative of the $S_{BW}$-system. The guarantee of the $AS_\varepsilon$-system stability is the fulfillment of the condition (52). This conclusion explains problems in implementing various procedures for BWM identifying. The following result gives more exact estimations for $AS_\varepsilon$-system.

Theorem 4. Let (i) positive definite Lyapunov functions
\[
V_\varepsilon(t) = 0.5\varepsilon^2(t), \quad V_{\beta_\varepsilon}(t) = 0.5\chi_\beta^{-1}(\Delta \beta)^2 + 0.5\chi_\gamma^{-1}(\Delta \gamma)^2
\]
allow the indefinitely small highest limit at $|\varepsilon(t)| \to 0, ||[\Delta \beta(t), \Delta \gamma(t)]|| \to 0$; (ii) $P(t) \in \mathcal{PE}_{\varepsilon,\beta}$; (iii) $c_1 > 0, c_2 > 0$ exist such that
\[
\varepsilon \Delta \gamma \tilde{x} \tilde{z}^p = c_2 \left( (\Delta \gamma)^2 \left( \tilde{x} \tilde{z}^p \right)^2 + \varepsilon^2 \right),
\]
\[
\varepsilon \Delta \beta \tilde{x} \tilde{z}^p \text{sign}(\tilde{z}) = c_1 \left( (\Delta \beta)^2 \left( \tilde{x} \tilde{z}^p \right)^2 + \varepsilon^2 \right)
\]
in area $O_\varepsilon(O)$, where $O = \{0, 0^2\} \subset R \times R^2 \times J_{\beta,\varepsilon}$, $O_\varepsilon$ is some neighborhood of a point $O$; (iv) the inequality $(\varepsilon - \varepsilon_c)^2 \geq \varepsilon_2$. 
$c_z$ holds for almost all $t$, where $c_z \geq 0$; (v) $\pi_z \geq 0$ and $\omega > 0$ exist such that $(\tilde{X})^2 \geq \pi_z$ and $|\varepsilon - c_z| \leq \omega|\varepsilon|$; (vi) the function

$$g_2(t) = \sup_{\varepsilon \in \Omega} \left| \varepsilon \right|^{2(n+1)}(t), \quad g_2 = \sup_{\varepsilon \in \Omega} \tilde{g}_2(t),$$

exists, where $\Omega$ the definition range of the subsystem; (vii) $\dot{V}_e$, $\dot{V}_{\beta\gamma}$ satisfy the system of inequalities

$$
\begin{bmatrix}
\dot{V}_e \\
\dot{V}_{\beta\gamma}
\end{bmatrix} \leq
\begin{bmatrix}
-(k_x - 2\tilde{u}g_1 - \omega v g_2) \\
-c
\end{bmatrix}
\begin{bmatrix}
V_e \\
V_{\beta\gamma}
\end{bmatrix}
+ \frac{1}{2}\frac{d_s}{B_e}
\left(\delta_{\Lambda}\right)^2.
\end{bmatrix}
$$

(viii) the upper solution for $V_{e,\beta\gamma} = [V_e(t)V_{\beta\gamma}(t)]^T$ satisfies to the equation

$$\dot{\tilde{S}} = A_e \tilde{S} + B_e \left(\delta_{\Lambda}\right)^2,$$

if

$$V_{\rho}(t) \leq \tilde{s}_{\rho}(t) \quad \forall(t \geq t_0) \& \left(V_{\rho}(t_0) \leq \tilde{s}_{\rho}(t_0)\right),$$

where $\tilde{S} = [\tilde{s}_e \tilde{s}_{\beta\gamma}]^T$, $\tilde{s} = e, (\beta, \gamma)$, $A_e \in R^{2\times2}$ is M-matrix. Then the system $AS_Z$ is exponentially dissipative with the estimate

$$V_{e,\beta\gamma}(t) \leq e^{A_e(t-t_0)} \tilde{S}(t_0) + \left(\delta_{\Lambda}\right)^2 \int_{t_0}^{t} e^{A_e(t-\tau)} B_e d\tau,$$

if

$$k_x > 2\tilde{u}g_1 - \omega v g_2, \quad (k_x - 2\tilde{u}g_1 - \omega v g_2) d_s > 2c \chi_{\omega \varepsilon \varepsilon}, \quad d_s > 0,$$

where

$$\overline{\chi} = \min\left(\chi_e, \chi_{\gamma}\right), \quad \overline{c} = \min\left(c_1, c_2\right), \quad \chi = \max\left(\chi_e, \chi_{\gamma}\right), \quad d_s = \chi \pi, \varepsilon c_z.$$

So, the system $AS_Z$ is exponentially dissipative. The dissipativity area depends on the informational set $I_o$ of the $S_{BW}$-system.

Get results that show the possibility of using adaptive observers to parameters identification of the $S_{BW}$-system. Properties of system (1) with BWHM supervene from the presented theorems.
7. Simulation results

Consider the engine control system (1)-(3) with parameters: \( n = 1.5, c = 2, m = 1, \beta = 0.5, \gamma = 0.2, \alpha = 0.7, k = 0.6. \)
Let \( d = a = 1. \) Exciting force \( f(t) = 2 - 2\sin(0.15\pi t). \) The system is modeled with initial conditions \( x(0) = 1, x(0) = 0, z(0) = 1. \) Form the set \( I_0. \) The system phase portrait and output of the hysteresis shown in Figure 1.

Estimate the structural identifiability of the system (1)-(3). Construct the structure \( S_{ey} \) (Figure 2) using the method [32]. A variable \( \hat{e} \in R \) is \( \hat{e} = \dot{x} - \dot{x}_h, \dot{x}_h \) is an estimation of the steady state (process) in the \( S_{BH} \)-system for \( \forall t \geq 9.85s, \) and \( \hat{e} \) is the nonlinearity estimation in the corresponding space.

![Figure 1. System phase portrait and hysteresis change](image1)

![Figure 2. Structure \( S_{ey} \) for assessing possibility of solving identification problem](image2)
As follows from Figures 1, 2, definition areas \( z \) and \( \tilde{e} \) coincide. Analysis \( S_{ey} \) shows that the system \( S_{BWW} \) is structurally identifiable, and input \( f(t) \) is \( S \)-stabilizing.

Consider the system parameters identification. Determine the parameter \( \mu \) of the system (13) using the transient process analysis for \( \tilde{e} \) and \( t < 9.85 \text{s} \). Calculate Lyapunov exponents (LE) \([33]\). The estimation for the maximum LE is \(-0.9\). Therefore, we set \( \mu = 0.8 \). Initial conditions in (12) are equal to zero.

Adaptive system work results are presented in Figures 3-5. Parameters \( k_x, k_z \) equal to 2.5 and 0.75. The tuning process of \( AS_x \)-systems parameters (the model (12)) is shown in Figure 3. Figure 4 showed the model (16) parameters tuning.

![Figure 3. Tuning of model (13) parameters](image3)

![Figure 4. Tuning of model (16) parameters: 1 is tuning \( \hat{\beta} \), 2 is tuning \( \hat{\gamma} \)](image4)
Show the modification of identification errors $e, \varepsilon$ in Figure 5. We see that the accuracy of obtained estimations depends on the numbers of tuned parameters, and the level $\hat{x}$ and properties $f(t)$. Obtained results confirm statements of theorems 3 and 4. The $AS_x$-system work results influence the tuning processes in the $AS_z$-system. Gain coefficients in (25), (26) and (27) are $\chi_\beta = 0.0000002, \chi_z = 0.0000002, \gamma_4 = 0.00005, \gamma_1 = 0.0002, \gamma_2 = 0.0001, \gamma_3 = 0.00002$. The parameter $n$ is 1.5 in (16).

**Remark 2.** Modeling results of the system $AS_{BW}$ with the algorithm (27) showed that the algorithm is sensitive to various perturbations, increases the adaptation time and requires further study.

The hysteresis output estimation is shown in Figure 6. Comparison of determination coefficients $r_{xz} = 0.864$ for the
reference BWH in (Figure 1) and the resulting BWH (Figure 6) $r_{z_{i}} = 0.764$ confirms the effectiveness of the proposed approach.

Figure 7 presents comparing results estimates $\hat{z}$ and $\varepsilon_z$, obtaining in subsystems $AS_X$ and $AS_Z$ on the interval $[25; 70]$s. We analyze the dependence $\hat{\varepsilon}_z(\hat{z})$ and show the approach effectiveness as the coefficient of determination is $r_{z_{i}\varepsilon_z} = 0.91$. In Figure 7, we represent the secant $\hat{\varepsilon}_z(\hat{z})$. Results confirm the adequacy of the obtained estimate $\hat{z}$.

![Figure 7. Comparison of estimates $\hat{z}$ and uncertainty $\varepsilon_z$.](image)

Figures 8-12 represent the work of the adaptive system with (8), $\pi = 1$. Tuning of models (20) and (16.2) parameters shows in Figures 8, 9.

![Figure 8. Tuning of model (13) parameters](image)
Figure 9. Tuning of model (16) parameters

Figure 10. Outputs modification of systems $AS_X, AS_Z$

Figure 10 shows the change in errors $e, \varepsilon$. The accuracy of obtained parameter estimates is shown in Figure 11, where $\mathcal{N}_{\beta,\mu,n}(t) = ||(\Delta \beta(t))^2 + (\Delta \mu(t))^2 + (\Delta n(t))^2||$, $||.||$ is the Euclidean norm.

Figure 12 demonstrates the adaptive system work with $M_{\text{适}}$ in $(e, N_{\beta,\mu,n})$ and $(e, \hat{\beta})$ spaces. We see that the tuning process is nonlinear. It depends on the main circuit $AS_X$ work of the adaptive system and the uncertainty estimation.

So, simulation results confirm the exponential dissipativity of the designed system. The obtained results are applicable to the analysis of robotic and macaronis systems.
Figure 11. Changing $N_{\beta,\mu,n}(t)$ for adaptive system with $M_{\beta,\mu,n}$

Figure 12. Adaptive system tuning with $M_{\beta,\mu,n}$ in $(c, N_{\beta,\mu,n})$ and $(c, \hat{\beta})$ spaces

8. Conclusion

We propose the adaptive identification method of system parameters with the Bouc-Wen hysteresis. We relate the fundamental problem of the BWH identification to ensuring the stability of the assessment system. The proposed identification method is based on the use of adaptive observers. Algorithms for the adaptive observer are designed and the trajectories limitation in the adaptive system is shown. An approach is proposed to estimate the uncertainty about the hysteresis state. This estimation is used to adjust the parameters of the hysteresis model. We consider BWH modifications and propose adaptive algorithms for estimating their parameters. The Lyapunov vector function method are used to evaluate the identification system quality in coordinate and parametric spaces. We prove processes
exponential dissipativity of in an adaptive system. It shows that the exponential dissipation domain of the system determines by the level of derivative output. We study the influence of input on BWH parameters identification.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References


Appendix A

A.1 Proof of Theorem 1

Consider the Lyapunov function $V(t)$ (36). Then $\dot{V}(t)$

$$\dot{V} = -k_x e^2 + \dot{V}_K - \dot{V}_K \leq -2k_s V_e.$$  \hspace{1cm} (A.1)

Apply the condition (i) theorem 1. As $\dot{V}(t) < 0$, the $\mathcal{AS}_X$-system is stable. Integrate $\dot{V}(t)$ on the time and obtain

$$V(t_0) - 2k_s \int_{t_0}^{t} V_e(\tau)d\tau \geq V(t).$$ \hspace{1cm} (A.2)

Get from (A.2) to all trajectories of the system $\mathcal{AS}_X$ belong to the area $G_t = \{(e, \Delta K) : V(t) \leq V(t_0)\}$. We get an estimate for the $\mathcal{AS}_X$-system

$$\int_{t_0}^{t} 2k_s V_e(\tau)d\tau \leq V(t_0) - V(t).$$ \hspace{1cm} (A.3)

Appendix B

B.1 Proof of Theorem 3

Determine $\dot{V}_{\epsilon, \beta, \gamma}$

$$\dot{V}_{\epsilon, \beta, \gamma} = -k_x e^2 + \epsilon \left( \beta \eta_{\beta} + \gamma \eta_{\gamma} + \Delta \dot{x} \right) \dot{V}_{\beta, \gamma} - \dot{V}_{\beta, \gamma}$$

$$= -k_x e^2 + \epsilon \left( \beta \eta_{\beta} + \gamma \eta_{\gamma} + \Delta \dot{x} \right).$$ \hspace{1cm} (B.1)

Since, $\dot{\bar{x}}$ is function $x$, $\dot{\bar{x}} = \sigma \dot{x}$, where $\sigma \approx 1$. We have showed that $\epsilon_x$ is the estimation $z$. Therefore, present $\eta_{\beta}$ as

$$\eta_{\beta} = |\dot{\bar{x}}| \|e\|^{\alpha} \text{sign}(z) - |\dot{\bar{x}}| \|e\|^{\alpha} \text{sign}(\dot{z}) \equiv |\dot{\bar{x}}| \|e\|^{\alpha},$$ \hspace{1cm} (B.2)

for $\forall t > t_c$. Similarly

$$\eta_{\beta} = |\dot{\bar{x}}| \|e\|^{\alpha} \text{sign}(z) - |\dot{\bar{x}}| \|e\|^{\alpha} \text{sign}(\dot{z}) \equiv |\dot{\bar{x}}| \|e\|^{\alpha},$$ \hspace{1cm} (B.3)

Considering the assumption 1 and the boundedness of trajectories $\mathcal{AS}_X$-system, we obtain $|\dot{x}| \leq \upsilon$ for $\forall t \geq t_0$ where $\upsilon > 0$. Then
\begin{align*}
\dot{V}_{eff} &\leq -k_z e^2 + \beta|\varepsilon||\eta_\beta| + \gamma |\varepsilon||\eta_\gamma| + |\varepsilon||\Delta t| \\
&\leq -k_z e^2 + \beta|\varepsilon||n^{p+1} + \gamma |\varepsilon||n^{p+1} + |\varepsilon||\Delta t| \\
&\leq -k_z e^2 + \nu(\beta + \gamma)|\varepsilon||n^{p+1} + |\Delta |e| \\
\end{align*}
(B.4)

where $|\Delta t| \leq \delta_\Delta$, $\delta_\Delta \geq 0$.

Let

$$\bar{g}_1(t) = \sup_{e \in \Omega} \frac{|e|^{p+1}}{V_e(t, e)}, \quad g_1 = \max_t \bar{g}_1(t).$$
(B.5)

Then $|e|^{p+1} \leq g_1 e^2(t)$ and transform (B.4) to the form

$$\dot{V}_{eff} \leq -k_z e^2 + \nu(\beta + \gamma) g_1 e^2 + \delta_\Delta |e|$$
(B.6)

where $k_z - \nu(\beta + \gamma) g_1 > 0$.

Apply the inequality

$$-aq^2 + bq \leq \frac{a}{2} q^2 + \frac{b^2}{2a}.$$ (B.7)

Then (B.6)

$$\dot{V}_{eff} \leq -(k_z - \nu(\beta + \gamma) g_1) e^2 + \delta_\Delta |e|$$

$$\leq -\frac{k_z - \nu(\beta + \gamma) g_1}{2} e^2 + \frac{1}{2(k_z - \nu(\beta + \gamma) g_1)} (\delta_\Delta)^2$$
$$\leq -(k_z - \nu(\beta + \gamma) g_1) V_e + \frac{1}{2(k_z - \nu(\beta + \gamma) g_1)} (\delta_\Delta)^2.$$ (B.8)

Integrate (B.8) and obtain the estimation

$$\int_{t_0}^t (k_z - \nu(\beta + \gamma) g_1) V_e(t) + \frac{1}{2(k_z - \nu(\beta + \gamma) g_1)} (t - t_0) (\delta_\Delta)^2 \leq V_{eff}(t) - V_{eff}(t_0).$$ (B.9)

The left part (B.9) is nonnegative and $V_e(t)$ satisfies conditions of theorem 3. Therefore, all trajectories $AS_e$-system is limited. □
B.2 Proof of Theorem 4

Consider $\dot{V}_\epsilon$

$$\dot{V}_\epsilon = -k_2 \epsilon^2 + \epsilon \left( \beta \eta_\beta + \gamma \eta_\gamma + \Delta \dot{x} \right) + \epsilon \left( \Delta \beta \|x\|^n \text{sign}(\dot{z}) + \Delta \gamma \dot{x} \|x\|^n \right). \quad (C.1)$$

Evaluate the second and third summands on the right side (C.1).

Lemma C1. The estimation

$$|\eta_\beta| \leq \nu |\epsilon|^p$$

is fair for $\eta_\beta = \|\dot{x}\|^n \text{sign}(z) - \|\dot{z}\|^n \text{sign}(\dot{z})$.

Lemma C1 proof.

As $|\dot{x}| \leq \nu$, then

$$|\eta_\beta| \leq \nu \|\dot{z}\|^n \text{sign}(z) - \sigma \|\dot{z}\|^n \text{sign}(\dot{z})$$

$$\leq \nu (|z - \dot{z}|^n + |\dot{z}|^n - |\dot{z}|^n) \leq \nu |z - \dot{z}| = \nu |\epsilon|^p. \quad \square$$

Similarly, the estimation has the form $|\eta_\gamma| \leq \nu |\epsilon|^p$ for $\eta_\gamma = \|\dot{x}\|^n - \|\dot{z}\|^n$. It is based on the proof of the lemma C1. Then

$$|\epsilon (\beta \eta_\beta + \gamma \eta_\gamma + \Delta \dot{x})| \leq \nu (\beta + \gamma) |\epsilon|^p + \delta \nu |\epsilon|, \quad (C.2)$$

where $\bar{\delta} = \nu (\beta + \gamma)$.

Consider the last item in the right member (C.1).

Lemma C2. The estimation

$$\left| \epsilon \left( \Delta \beta \|\dot{x}\|^n \text{sign}(\dot{z}) + \Delta \gamma \dot{x} \|x\|^n \right) \right| \leq \omega \nu |\epsilon|^{p+1} \left( |\Delta \beta| + |\Delta \gamma| \right), \quad (C.3)$$

is fair for $\epsilon (\Delta \beta \|\dot{x}\|^n \text{sign}(\dot{z}) + \Delta \gamma \dot{x} \|x\|^n)$, where $\omega > 0$ is such that $|\epsilon - \epsilon_z| \leq \omega |\epsilon|$.

Lemma C2 proof. Transform (C.3) to the form

$$\left( \epsilon \delta \beta \|\dot{x}\|^n \text{sign}(\dot{z}) + \epsilon \Delta \gamma \dot{x} \|x\|^n \right) |_{\epsilon = \epsilon_z} = \epsilon \left( \delta \beta \text{sign}(\dot{z}) + \epsilon \Delta \gamma \text{sign}(\dot{z}) \right). \quad (C.4)$$

Let $\omega > 0$ exist such that $|\epsilon - \epsilon_z| \leq \omega |\epsilon|$. Then (C.4)
\[ |e|^{\alpha} |\varepsilon - \varepsilon_0| |\Delta \beta \text{sign}(\varepsilon - \varepsilon_0) + \Delta \gamma \text{sign}(\tilde{x})| \]

\[ \leq |\omega| |e|^{\alpha+1} |\Delta \beta \text{sign}(\varepsilon - \varepsilon_0) + \Delta \gamma \text{sign}(\tilde{x})| \]

\[ \leq |\omega| |e|^{\alpha+1} (|\Delta \beta| + |\Delta \gamma|). \]  

(C.5)

(C.3) is result (C.5). □

Consider (C.5) and transform \( |\omega| |e|^{\alpha+1} (|\Delta \beta| + |\Delta \gamma|) \) into the form

\[ |\omega| |e|^{\alpha+1} (|\Delta \beta| + |\Delta \gamma|) \leq 0.5\varepsilon^{2(\alpha+1)} + 0.5\lambda (\Delta \beta^2 + \Delta \gamma^2), \]

where \( \lambda > 1 \).

As

\[ (\Delta \beta)^2 + (\Delta \gamma)^2 = 2 \cdot 0.5 \left[ \chi_\beta^{-1} \chi_\beta (\Delta \beta)^2 + \chi_\gamma^{-1} \chi_\gamma (\Delta \gamma)^2 \right] \leq 2 \chi V_{\beta,\gamma}, \]

where \( \chi = \max(\chi_\beta, \chi_\gamma) \), then (C.6) receives the form

\[ |\omega| |e|^{\alpha+1} (|\Delta \beta| + |\Delta \gamma|) \leq 0.5\varepsilon^{2(\alpha+1)} + \lambda \chi \omega \nu V_{\beta,\gamma}. \]  

(C.7)

As

\[ -k_\varepsilon e^2 + \delta_\Delta |e| \leq -\frac{1}{2} k_\varepsilon e^2 + \frac{1}{2k_\varepsilon} (\delta_\Delta)^2, \]  

(C.8)

then (C.1), considering lemmas C1 and C2, and inequalities (C.7) and (C.8), we present as

\[ \dot{V}_e \leq -k_\varepsilon e^2 + \tilde{\nu} |e|^{\alpha+1} + \delta_\Delta |e| + 0.5\varepsilon^{2(\alpha+1)} + \lambda \chi \omega \nu V_{\beta,\gamma} \]

\[ \leq -\frac{1}{2} k_\varepsilon e^2 + \tilde{\nu} |e|^{\alpha+1} + 0.5\varepsilon^{2(\alpha+1)} + \lambda \chi \omega \nu V_{\beta,\gamma} + \frac{1}{2k_\varepsilon} (\delta_\Delta)^2. \]  

(C.9)

Consider functions \( \tilde{g}(t) \) (B.5) and

\[ g_2(t) = \sup_{\varepsilon \in \Omega} \frac{|e|^{2(\alpha+1)}(t)}{V_e(t, \varepsilon)} \cdot g_2 = \max_t \tilde{g}_2(t). \]

Then (C.9)

\[ \dot{V}_e \leq -\frac{1}{2} (k_\varepsilon - 2\tilde{\nu} - \omega \nu g_2) V_e + \lambda \chi \omega \nu V_{\beta,\gamma} + \frac{1}{2k_\varepsilon} (\delta_\Delta)^2. \]  

(C.10)
Obtain the estimation for the derivative $V_{\beta,\gamma}(t) = 0.5\chi^{-1}_\beta (\Delta \beta)^2 + 0.5\chi^{-1}_\gamma (\Delta \gamma)^2$:

$$V_{\beta,\gamma} = -\varepsilon\Delta\beta \beta^\prime \Pi^n \text{sign}(\tilde{z}) - \varepsilon \Delta\gamma \tilde{x}^n.$$  \hfill (C.11)

Let $c_1 > 0$, $c_2 > 0$ exist such that

$$\varepsilon \Delta\gamma \tilde{x}^n = c_2 \left( (\Delta \gamma)^2 \left( \tilde{x}^n \right)^2 + \varepsilon^2 \right).$$  \hfill (C.12)

Then (C.11)

$$\dot{V}_{\beta,\gamma} = -c\varepsilon^2 - \left( \tilde{x}^n \right)^2 \left( c_1 (\Delta \beta)^2 + c_2 (\Delta \gamma)^2 \right).$$  \hfill (C.13)

where $c = c_1 + c_2$.

Let $\bar{c} = \min(c_1, c_2)$. Then

$$c_1 (\Delta \beta)^2 + c_2 (\Delta \gamma)^2 \geq \bar{c} \left( (\Delta \beta)^2 + (\Delta \gamma)^2 \right) \geq 2\bar{c} \bar{\chi} V_{\beta,\gamma}.$$  \hfill (C.14)

where $\bar{\chi} = \min(\chi^\prime_\beta, \chi^\prime_\gamma)$.

Apply (C.14) and (C.15) write as

$$\dot{V}_{\beta,\gamma} \leq -c\varepsilon^2 - 2\bar{\chi} \left( \tilde{x}^n \right)^2 V_{\beta,\gamma}.$$  \hfill (C.16)

$\tilde{x}$ bounded $(\tilde{\tilde{x}}) \geq \pi_x \geq 0$, therefore,

$$\dot{V}_{\beta,\gamma} \leq -c\varepsilon^2 - 2\bar{\chi} \pi_x \tilde{x}^n V_{\beta,\gamma} \leq -c\varepsilon^2 - 2\chi \pi_x \varepsilon^2 V_{\beta,\gamma},$$

$$\dot{V}_{\beta,\gamma} \leq -c\varepsilon^2 - 2\chi \pi_x \varepsilon \varepsilon^2 V_{\beta,\gamma}.$$  \hfill (C.17)

The variable $\varepsilon$ boundedness follows from theorem 3, and the boundedness $\varepsilon^2$ is the boundedness result $\varepsilon$. Therefore, $(\varepsilon - \varepsilon_2)^2 \geq \varepsilon^2$, where $\varepsilon_2 \geq 0$. So

$$\dot{V}_{\beta,\gamma} \leq -c\varepsilon^2 - 2\chi \pi_x \varepsilon \varepsilon^2 V_{\beta,\gamma}.$$  \hfill (C.18)

Let $d_\varepsilon = \chi \pi_x \varepsilon$. Then (C.14) write as

$$\dot{V}_{\beta,\gamma} \leq -d_\varepsilon V_{\beta,\gamma} + 2\sqrt{d_\varepsilon \varepsilon} \sqrt{V_{\beta,\gamma}}.$$  \hfill (C.19)
Use the inequality (B.7) and obtain the estimation for $\dot{V}_{\beta,\gamma}$

$$\dot{V}_{\beta,\gamma} \leq -\frac{d_s}{2} V_{\beta,\gamma} + c V_v.$$  \hspace{1cm} (C.20)

So, the following system of inequalities is fair for the $AS_Z$-system

$$\begin{bmatrix} \dot{V}_v \\ \dot{V}_{\beta,\gamma} \end{bmatrix} \leq \begin{bmatrix} -(k_z - 2\tilde{\omega}g_1 - \omega v g_2) & \lambda \chi \omega v \\ c & -\frac{d_s}{2} \end{bmatrix} \begin{bmatrix} V_v \\ V_{\beta,\gamma} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left( \frac{1}{2k_z} \right) (\delta_\Lambda)^2.$$  \hspace{1cm} (C.21)

Let $V_{\beta}(t) \leq \tilde{s}_{\beta}(t), \ \forall (t \geq t_0) \ \& \ (V_{\beta}(t_0) \leq \tilde{s}_{\beta}(t_0)), \ \tilde{\beta} = \varphi(\beta, \gamma), \ \text{and} \ \tilde{S} = [\tilde{s}_v \tilde{s}_{\beta,\gamma}]^T$. The comparison system for (C.17) has the form

$$\dot{\tilde{S}} = A_e \tilde{S} + B_e (\delta_\Lambda)^2.$$  \hspace{1cm} (C.22)

We have the estimation for the system $AS_Z$ from (C.22)

$$V_{e,\beta,\gamma}(t) \leq e^{A_e(t-t_0)} \tilde{S}(t_0) + (\delta_\Lambda)^2 \int_{t_0}^{t} e^{A_e(t-t')} B_e dt'.$$  \hspace{1cm} (C.23)

where $V_{e,\beta,\gamma} = [V_v V_{\beta,\gamma}]^T$ if

$$k_z > 2\tilde{\omega}g_1 - \omega v g_2, \ (k_z - 2\tilde{\omega}g_1 - \omega v g_2) d_s > 2c \lambda \chi \omega v, \ d_s > 0.$$ \hspace{1cm} \Box