



Asymptotic Behavior of a Stochastic E-rumor Model with Lévy Jump

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Abstract: In this paper, we investigate the effects of a Lévy jump on the dynamic of propagation of a rumor on a social network. The random environment is characterized by white noises and Lévy jump and we establish sufficient conditions for extinction and persistence in the mean of an e-rumor. At the end, we compare our study with our previous one^[7] to see the difference with only white noises.

Keywords: stochastic e-rumor models, Brownian motion, lévy jump, Itô-Lévy's formula, extinction, persistence in the mean

1. Introduction

Nowadays, social networks are the main mode of communication and these new tools have made the spread of information quick and easy. The propagation of fake news can be seen as the one of a virus and controlling them has been an increasingly complex issue in recent years. Contrary to an attack of a computer with an infected file that you need to open to be infected, the infection is direct from the moment you have contact with a spreader. Thus the spread of information has been seen earlier as the spread of an infectious disease in a population^[9, 10, 11, 25, 26, 27].

In our first approach, we study the problem of propagation of rumors in the context of optimal control theory. Using and modifying a model introduced in^[13], we control some parameters involved in the model^[3, 4] or some external actions^[5], such as the introduction of a counter-information or the isolation of the more active spreaders. This is done in order to minimize the spread of bad information on a social network. However the dynamic of a rumor in a network is not exactly the same as one of the epidemics in a country since an infected is not necessarily infectious, contrary to the epidemic case. Indeed, someone who knows the rumor has different choices: he can spread it, keep the information and spread it after or keep it and never spread it. This is due to the mediatic context^[15], the impact of education^[1], the forgetting mechanism^[34], the hesitating mechanism^[28], ... Based on this, we proposed a new deterministic e-rumor model for which we made a stability analysis in [6]. In the following, as it has been done in epidemic cases^[8, 12, 16, 17, 18, 32, 35], we add the random side in our model with the introduction of white noises. We use results from our last paper^[7] in which we studied the dynamic of a stochastic e-rumor model, pointed out the thresholds of extinction and persistence of the spreaders densities, and highlighted the benefit of the introduction of white noises with respect to the deterministic case. In fact, in the present paper, we begin with this last stochastic model, in which we add a Lévy jump as it has been done in the case of epidemics in [2, 20, 33, 36] for example. The reader could also refer to [14, 29, 30, 31] for the fractional-order systems. Indeed, the rumor model may suffer environmental perturbations which lead to a sudden evolution from the ignorants to the spreaders and cannot be modeled only by the stochastic model. In Section 2, we study the existence and uniqueness of a global solution of the obtained e-rumor model by using general results on stochastic differential equations^[21, 22, 23, 24]. Then we focus on its asymptotic behavior by pointing out the threshold of extinction and the one of persistence in Sections 3 and 4 respectively. Finally, a numerical example is carried out to illustrate the theoretical results and we conclude with some perspectives for future works.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, that is it is increasing and right continuous while \mathcal{F}_0 contains all \mathbb{P} -null sets. Also let $(B_i(\cdot)), i = 1, 2$, be scalar Brownian motions defined on the probability space. Moreover, let us fix the notations:

$$a \wedge b = \min(a, b), a \vee b = \max(a, b), \text{ for all } a, b \in \mathbb{R} \text{ and } \langle X(t) \rangle = \frac{1}{t} \int_0^t X(\xi) d\xi,$$

that we will use in the following.

Let N be the total number of persons on the social network and μ the rate for which an individual enters or leaves the social network. This population is divided into three groups: one of the ignorants, one of the spreaders, one of the stiflers. The stiflers are those who know the rumor but do not spread it for the moment. In the following, we denote by I the number of ignorants, S the number of spreaders and R the number of stiflers. We assume that an ignorant can become a spreader or a stifier with rates β_1 or β_2 respectively. A spreader can become a stifier with rate θ_1 and a stifier can become a spreader with rate θ_2 . Note that obviously a spreader or a stifier can not become an ignorant, however they can just leave the social network, as all the individuals of the social network with rate μ . We begin with the stochastic e-rumor model introduced in [7]:

$$\begin{cases} dI(t) = \left(\mu N - \beta_1 \frac{I(t)S(t)}{N} - \beta_2 \frac{I(t)R(t)}{N} - \mu I(t) \right) dt - \gamma_1 \frac{I(t)S(t)}{N} dB_1(t), \\ dS(t) = \left(\beta_1 \frac{I(t)S(t)}{N} - \theta_1 \frac{S(t)R(t)}{N} + \theta_2 \frac{S(t)R(t)}{N} - \mu S(t) \right) dt + \gamma_1 \frac{I(t)S(t)}{N} dB_1(t) + \gamma_2 \frac{R(t)S(t)}{N} dB_2(t), \\ dR(t) = \left(\beta_2 \frac{I(t)R(t)}{N} + \theta_1 \frac{S(t)R(t)}{N} - \theta_2 \frac{S(t)R(t)}{N} - \mu R(t) \right) dt - \gamma_2 \frac{R(t)S(t)}{N} dB_2(t), \end{cases}$$

where $\mu, \beta_1, \beta_2, \theta_1, \theta_2$ are strictly non negatives, B_1 and B_2 are independent standard Brownian motions and γ_1, γ_2 are constants (intensities of the white noises). By replacing $\theta = \theta_1 - \theta_2, i = \frac{I}{N}, s = \frac{S}{N}$ and $r = \frac{R}{N}$, we obtain the following stochastic e-rumor model as in [7]:

$$\begin{cases} di(t) = (\mu - \beta_1 i(t)s(t) - \beta_2 i(t)r(t) - \mu i(t)) dt - \gamma_1 i(t)s(t) dB_1(t), \\ ds(t) = (\beta_1 i(t)s(t) - \theta s(t)r(t) - \mu s(t)) dt + \gamma_1 i(t)s(t) dB_1(t) + \gamma_2 r(t)s(t) dB_2(t), \\ dr(t) = (\beta_2 i(t)r(t) + \theta s(t)r(t) - \mu r(t)) dt - \gamma_2 r(t)s(t) dB_2(t). \end{cases} \quad (1)$$

The novelty here is the consideration of a Lévy jump and (1) becomes:

$$\begin{cases} di(t) = (\mu - \beta_1 i(t)s(t) - \beta_2 i(t)r(t) - \mu i(t)) dt - \gamma_1 i(t)s(t) dB_1(t) - \int_{\mathbb{Z}} C(u) i(t^-) s(t^-) \tilde{N}(dt, du), \\ ds(t) = (\beta_1 i(t)s(t) - \theta s(t)r(t) - \mu s(t)) dt + \gamma_1 i(t)s(t) dB_1(t) + \gamma_2 r(t)s(t) dB_2(t) + \int_{\mathbb{Z}} C(u) i(t^-) s(t^-) \tilde{N}(dt, du), \\ dr(t) = (\beta_2 i(t)r(t) + \theta s(t)r(t) - \mu r(t)) dt - \gamma_2 r(t)s(t) dB_2(t), \end{cases} \quad (2)$$

where $i(t)$ and $s(t)$ are the left limits of $i(t)$ and $s(t)$ respectively; $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, with N a Poisson counting measure; λ is the characteristic measure of N on a measurable subset Z of $(0, +\infty)$ with $\lambda(Z) < +\infty$ and $C(u) : Z \times \Omega \rightarrow \mathbb{R}_+$ is bounded and continuous. This model can be written as:

$$dX(t) = F(t, X(t))dt + M(t, X(t))dB(t) + \int_{\mathbb{Z}} h(X(t^-), u) \tilde{N}(dt, du), \quad (3)$$

with

$$X(t) = \begin{pmatrix} i(t) \\ s(t) \\ r(t) \end{pmatrix}, F(t, X(t)) = \begin{pmatrix} \mu - \beta_1 i(t)s(t) - \beta_2 i(t)r(t) - \mu i(t) \\ \beta_1 i(t)s(t) - \theta s(t)r(t) - \mu s(t) \\ \beta_2 i(t)r(t) + \theta s(t)r(t) - \mu r(t) \end{pmatrix}, M(t, X(t)) = \begin{pmatrix} -\gamma_1 i(t)s(t) & 0 & 0 \\ \gamma_1 i(t)s(t) & \gamma_2 r(t)s(t) & 0 \\ 0 & -\gamma_2 r(t)s(t) & 0 \end{pmatrix}$$

and

$$h(X(t^-), u) = \begin{pmatrix} -C(u)i(t^-)s(t^-) & 0 & 0 \\ 0 & C(u)i(t^-)s(t^-) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that we will not put t every time for convenience of writing. Moreover, in order to state our results, we need to make the following assumptions on the jump-diffusion coefficients:

(H₁) For all $m > 0$, there is $L_m > 0$ such that:

$$\int_Z |h(x, u) - h(y, u)|^2 \lambda(du) \leq L_m |x - y|^2, \text{ with } |x| \vee |y| \leq m;$$

(H₂) There is a non-negative constant K such that:

$$|\ln(1 + C(u))| \leq K.$$

Let us notice that the assumption (H₁) will be useful in the proof of the existence result, whereas the assumption (H₂) will be used to investigate the properties of extinction and persistence after showing the existence and uniqueness of the solution.

2. Existence of a global solution

In this section, we first establish the existence and uniqueness of a local solution of (3) and then prove that it is global.

Theorem 1 For all initial condition $X(0) = X_0 \in (0, 1)^3$, the stochastic differential equation (3) has a unique local solution.

Proof. Let us set $|A| = \sqrt{\sum_{1 \leq i, j \leq n} a_{i,j}^2}$, for any n -order matrix A . Following the same steps as in [7], using standard inequalities and the fact that any solution $X(\cdot) = (i(\cdot), s(\cdot), r(\cdot))$ is bounded, we can take all the components in $(0, 1)$ without loss of generality. We then obtain:

$$|F(t, X(t))|^2 \leq \mu^2 + 3(\beta_1^2 + \beta_2^2 + \mu^2)i^2(t) + (\beta_1^2 + 2(\theta^2 + \mu^2))s^2(t) + (2(\beta_2^2 + \theta^2) + \mu^2)r^2(t) \quad (4)$$

and

$$|M(t, X(t))|^2 \leq 2(\gamma_1^2 + \gamma_2^2). \quad (5)$$

Moreover, by taking $y = 0$ in (H₁), we obtain:

$$\int_Z |h(X, u)|^2 \lambda(du) \leq L_m |X|^2, \quad (6)$$

since $h(0, u) = 0$. By adding (4), (5) and (6), one has:

$$\begin{aligned} &|F(t, X(t))|^2 + |M(t, X(t))|^2 + \int_Z |h(X, u)|^2 \lambda(du) \leq \mu^2 + 3(\beta_1^2 + \beta_2^2 + \mu^2)i^2(t) \\ &+ (\beta_1^2 + 2(\theta^2 + \mu^2))s^2(t) + (2(\beta_2^2 + \theta^2) + \mu^2)r^2(t) + 2(\gamma_1^2 + \gamma_2^2) + L_m |X(t)|^2, \end{aligned}$$

that is:

$$|F(t, X(t))|^2 + |M(t, X(t))|^2 + \int_Z |h(X, u)|^2 \lambda(du) \leq C_1 (1 + |X(t)|)^2, \quad (7)$$

with

$$C_1 = (3(\beta_1^2 + \beta_2^2 + \mu^2)) \vee (\beta_1^2 + 2(\theta^2 + \mu^2)) \vee (2(\beta_2^2 + \theta^2) + \mu^2) \vee (\mu^2 + 2(\gamma_1^2 + \gamma_2^2)) \vee L_m.$$

Then, let us set $X(t) = (i_1(t), s_1(t), r_1(t))$ and $Y(t) = (i_2(t), s_2(t), r_2(t))$. With similar arguments, as in [7], we show that:

$$\begin{aligned} |F(t, X(t)) - F(t, Y(t))|^2 \leq & 3 \left[(3\mu^2 + 4(\beta_1^2 + \beta_2^2))(i_2(t) - i_1(t))^2 + (2(\mu^2 + \beta_1^2) + 3\theta^2)(s_2(t) - s_1(t))^2 \right. \\ & \left. + (\mu^2 + 3(\theta^2 + \beta_2^2))(r_2(t) - r_1(t))^2 \right] \end{aligned}$$

and

$$|M(t, X(t)) - M(t, Y(t))|^2 \leq 4 \left[\gamma_1^2 (i_2(t) - i_1(t))^2 + (\gamma_1^2 + \gamma_2^2)(s_2(t) - s_1(t))^2 + \gamma_2^2 (r_2(t) - r_1(t))^2 \right].$$

Moreover, by using (H_1) we obtain:

$$\begin{aligned} & |F(t, X(\cdot)) - F(t, Y(\cdot))|^2 + |M(t, X(\cdot)) - M(t, Y(\cdot))|^2 + \int_Z |h(X(\cdot), u) - h(Y(\cdot), u)|^2 \lambda(du) \\ & \leq 3 \left[(3\mu^2 + 4(\beta_1^2 + \beta_2^2))(i_2 - i_1)^2 + (2(\mu^2 + \beta_1^2) + 3\theta^2)(s_2 - s_1)^2 + (\mu^2 + 3(\theta^2 + \beta_2^2))(r_2 - r_1)^2 \right] \\ & \quad + 4 \left[\gamma_1^2 (i_2 - i_1)^2 + (\gamma_1^2 + \gamma_2^2)(s_2 - s_1)^2 + \gamma_2^2 (r_2 - r_1)^2 \right] + L_m |X(\cdot) - Y(\cdot)|^2, \end{aligned}$$

that is:

$$|F(t, X(t)) - F(t, Y(t))|^2 + |M(t, X(t)) - M(t, Y(t))|^2 + \int_Z |h(X(t), u) - h(Y(t), u)|^2 \lambda(du) \leq C_2 |X(t) - Y(t)|^2, \quad (8)$$

with

$$C_2 = \left[3 \left((3\mu^2 + 4(\beta_1^2 + \beta_2^2)) \vee (2(\mu^2 + \beta_1^2) + 3\theta^2) \vee (\mu^2 + 3(\theta^2 + \beta_2^2)) \right) \right] \vee \left[4(\gamma_1^2 + \gamma_2^2) \right] \vee L_m.$$

With inequalities (7) and (8) and by using a general result of [24], we can claim that there is one and only one local solution for (3).

Theorem 2 For any given initial condition $X(0) = (i_0, s_0, r_0)$ in $(0, 1)^3$, there is one and only one global solution of equation (3), $X(t)$ in $(0, 1)^3$ for all $t \geq 0$, with probability 1, that is:

$$\mathbb{P}(\{X(t) \in (0, 1)^3, \text{ for all } t \geq 0\}) = 1.$$

Proof. This result is inspired by [7] and [33]. The previous theorem gave the uniqueness of a local solution $(i(t), s(t), r(t))$ for $t \in [0, T_e)$ where T_e is the explosion time (see [21]). In order to show that this solution is global, we have to show that $T_e = +\infty$ a.s. Let $n_0 > 0$ be sufficiently large such that $i_0, s_0, r_0 \in \left(\frac{1}{n_0}, 1 - \frac{1}{n_0}\right)$. For each integer $n \geq n_0$, let us define the stopping time as:

$$\mathcal{T}_n = \inf \left\{ t \in [0, T_e) : (i(t) \wedge s(t)) \leq \frac{1}{n} \text{ or } (i(t) \vee s(t)) \geq 1 - \frac{1}{n} \right\},$$

where $\inf \emptyset = +\infty$. It is clear that \mathcal{T}_n is increasing as $n \rightarrow +\infty$. Let us set $\mathcal{T}_\infty = \lim_{n \rightarrow +\infty} \mathcal{T}_n$, then $\mathcal{T}_\infty \leq T_e$ a.s. If we can show that $\mathcal{T}_\infty = +\infty$ a.s. then $T_e = +\infty$ a.s. Let us assume that it is not the case. Then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $\mathbb{P}(\mathcal{T}_n \leq T) > \epsilon$. Consequently, there is an integer $n_1 \geq n_0$ such that:

$$\mathbb{P}(\mathcal{T}_n \leq T) \geq \epsilon, \text{ for all } n \geq n_1. \quad (9)$$

Let us define the set $D = \{(i, s) \in (0, 1)^2 / i + s < 1\}$ and introduce the function $V : D \rightarrow \mathbb{R}_+$ defined by:

$$V(i(t), s(t)) = \frac{1}{i(t)} + \frac{1}{s(t)} + \frac{1}{1-i(t)-s(t)}.$$

Moreover, let us transform the e-rumor model (2) into a system of two equations by replacing r by $1 - i - s$ to obtain the following:

$$\begin{cases} di(t) = (\mu - \beta_1 i(t)s(t) - \beta_2 i(t)(1-i(t)-s(t)) - \mu i(t))dt - \gamma_1 i(t)s(t)dB_1(t) \\ \quad - \int_Z C(u)i(t^-)s(t^-)\tilde{N}(dt, du), \\ ds(t) = (\beta_1 i(t)s(t) - \theta s(t)(1-i(t)-s(t)) - \mu s(t))dt + \gamma_1 i(t)s(t)dB_1(t) \\ \quad + \gamma_2 s(t)(1-i(t)-s(t))dB_2(t) + \int_Z C(u)s(t^-)\tilde{N}(dt, du). \end{cases} \quad (10)$$

The use of Itô-Lévy's formula^[24] for V and the two previous equations leads to:

$$\begin{aligned} dV(i(t), s(t)) &= BV(i(t), s(t))dt + \gamma_1 \left(\frac{s(t)}{i(t)} - \frac{i(t)}{s(t)} \right) dB_1(t) + \gamma_2 \left(\frac{s(t)}{1-i(t)-s(t)} - \frac{1-i(t)-s(t)}{s(t)} \right) dB_2(t) \\ &\quad + \int_Z \left(\frac{1}{i(t^-) - C(u)i(t^-)s(t^-)} + \frac{1}{s(t^-) + C(u)i(t^-)s(t^-)} - \frac{1}{i(t^-)} - \frac{1}{s(t^-)} \right) \tilde{N}(dt, du), \end{aligned} \quad (11)$$

with

$$BV(i, s) = LV(i, s) + \int_Z \left[\frac{1}{i(t^-) - C(u)i(t^-)s(t^-)} + \frac{1}{s(t^-) + C(u)i(t^-)s(t^-)} - \frac{1}{i(t^-)} - \frac{1}{s(t^-)} - \frac{C(u)s(t^-)}{i(t^-)} + \frac{C(u)i(t^-)}{s(t^-)} \right] \lambda(du),$$

and

$$\begin{aligned} LV(i, s) &= \left(\frac{1}{(1-i-s)^2} - \frac{1}{i^2} \right) (\mu - \beta_1 is - \beta_2 i(1-i-s) - \mu i) \\ &\quad + \left(\frac{1}{(1-i-s)^2} - \frac{1}{s^2} \right) (\beta_1 is - \theta s(1-i-s) - \mu s) + \gamma_1^2 i^2 s^2 \left(\frac{1}{i^3} + \frac{1}{(1-i-s)^3} \right) \\ &\quad + \left(\frac{1}{s^3} + \frac{1}{(1-i-s)^3} \right) (\gamma_1^2 i^2 s^2 + \gamma_2^2 (1-i-s)^2 s^2) - \frac{2\gamma_1^2 i^2 s^2}{(1-i(t)-s(t))^3}. \end{aligned}$$

Consequently,

$$\begin{aligned} BV(i, s) &= LV(i, s) + \int_Z \left[\frac{1}{i(t^-)} \times \frac{C^2(u)s^2(t^-)}{1-C(u)s(t^-)} + \frac{1}{s(t^-)} \times \frac{C^2(u)i^2(t^-)}{1+C(u)i(t^-)} \right] \lambda(du) \\ &\leq LV(i, s) + \int_Z \left[\frac{1}{i(t^-)} \times \frac{C^2(u)}{1-C(u)} + \frac{1}{s(t^-)} \times \frac{C^2(u)i^2(t^-)}{C(u)i(t^-)} \right] \lambda(du) \\ &\leq LV(i, s) + \int_Z \left[\frac{1}{i(t^-)} \times \frac{C^2(u)}{1-C(u)} + \frac{1}{s(t^-)} \times C(u) + \frac{1}{1-i(t^-)-s(t^-)} \right] \lambda(du) \end{aligned}$$

$$\leq LV(i, s) + \int_Z \left(\frac{C^2(u)}{1-C(u)} \vee C(u) \vee 1 \right) \lambda(du) V(i(t^-), s(t^-))$$

but

$$\begin{aligned} LV(i, s) &= -\frac{\mu}{i^2} + \frac{\beta_1 s}{i} + \frac{\beta_2(1-i-s)}{i} + \frac{\mu}{i} + \frac{\mu}{(1-i-s)^2} - \frac{\beta_1 i s}{(1-i-s)^2} - \frac{\beta_2 i}{1-i-s} \\ &\quad - \frac{\mu i}{(1-i-s)^2} - \frac{\beta_1 i}{s} + \frac{\theta(1-i-s)}{s} + \frac{\mu}{s} + \frac{\beta_1 i s}{(1-i-s)^2} - \frac{\theta s}{1-i-s} \\ &\quad - \frac{\mu s}{(1-i-s)^2} + \frac{\gamma_1^2 s^2}{i} + \frac{\gamma_1^2 i^2}{s} + \frac{\gamma_2^2(1-i-s)^2}{s} + \frac{\gamma_2^2 s^2}{1-i-s}, \end{aligned}$$

so:

$$\begin{aligned} LV(i, s) &\leq \frac{\beta_1 s}{i} + \frac{\beta_2(1-i-s)}{i} + \frac{\mu}{i} + \frac{\gamma_1^2 s^2}{i} + \frac{\theta(1-i-s)}{s} + \frac{\mu}{s} \\ &\quad + \frac{\gamma_1^2 i^2}{s} + \frac{\gamma_2^2(1-i-s)^2}{s} + \frac{\mu(1-i-s)}{(1-i-s)^2} + \frac{\gamma_2^2 s^2}{1-i-s} \\ &\leq \frac{1}{i}(\beta_1 + \beta_2 + \mu + \gamma_1^2) + \frac{1}{s}(\theta + \mu + \gamma_1^2 + \gamma_2^2) + \frac{1}{1-i-s}(\mu + \gamma_2^2), \end{aligned}$$

since $(i, s) \in D$. Then we obtain:

$$BV(i(t), s(t)) \leq K_1 V(i(t), s(t)) + K_2 V(i(t^-), s(t^-)),$$

with

$$K_1 = (\beta_1 + \beta_2 + \mu + \gamma_1^2) \vee (\theta + \mu + \gamma_1^2 + \gamma_2^2)$$

and

$$K_2 = \int_Z \left(\frac{C^2(u)}{1-C(u)} \vee C(u) \vee 1 \right) \lambda(du).$$

By replacing in (11), one has:

$$\begin{aligned} dV(i(t), s(t)) &\leq (K_1 V(i(t), s(t)) + K_2 V(i(t^-), s(t^-))) dt \\ &\quad + \gamma_1 \left(\frac{s(t)}{i(t)} - \frac{i(t)}{s(t)} \right) dB_1(t) + \gamma_2 \left(\frac{s(t)}{1-i(t)-s(t)} - \frac{1-i(t)-s(t)}{s(t)} \right) dB_2(t) \\ &\quad + \int_Z \left(\frac{1}{i(t^-) - C(u)i(t^-)s(t^-)} + \frac{1}{s(t^-) + C(u)i(t^-)s(t^-)} - \frac{1}{i(t^-)} - \frac{1}{s(t^-)} \right) \tilde{N}(dt, du). \end{aligned} \tag{12}$$

By integrating (12) between 0 and $\mathcal{T}_n \wedge t$, we obtain:

$$\begin{aligned} \int_0^{\mathcal{T}_n \wedge t} dV(i(\xi), s(\xi)) &\leq \int_0^{\mathcal{T}_n \wedge t} (K_1 V(i(\xi), s(\xi)) + K_2 V(i(\xi^-), s(\xi^-))) d\xi \\ &+ \int_0^{\mathcal{T}_n \wedge t} \gamma_1 \left(\frac{s(\xi)}{i(\xi)} - \frac{i(\xi)}{s(\xi)} \right) dB_1(\xi) + \int_0^{\mathcal{T}_n \wedge t} \gamma_2 \left(\frac{s(\xi)}{1-i(\xi)-s(\xi)} - \frac{1-i(\xi)-s(\xi)}{s(\xi)} \right) dB_2(\xi) \\ &+ \int_0^{\mathcal{T}_n \wedge t} \int_Z \left(\frac{1}{i(\xi^-) - C(u)i(\xi^-)s(\xi^-)} + \frac{1}{s(\xi^-) + C(u)i(\xi^-)s(\xi^-)} - \frac{1}{i(\xi^-)} - \frac{1}{s(\xi^-)} \right) \tilde{N}(d\xi, du). \end{aligned}$$

The function V is continuous on D and $(i(\cdot), s(\cdot))$ are continuous on $[0, \mathcal{T}_n \wedge t]$ so V is continuous on $[0, \mathcal{T}_n \wedge t]$, which gives $V(i(\xi^-), s(\xi^-)) = V(i(\xi), s(\xi))$ a.s., for all ξ in $[0, \mathcal{T}_n \wedge t]$. Consequently, one has:

$$\begin{aligned} \int_0^{\mathcal{T}_n \wedge t} dV(i(\xi), s(\xi)) &\leq \int_0^{\mathcal{T}_n \wedge t} (K_1 + K_2) V(i(\xi), s(\xi)) d\xi + \int_0^{\mathcal{T}_n \wedge t} \gamma_1 \left(\frac{s(\xi)}{i(\xi)} - \frac{i(\xi)}{s(\xi)} \right) dB_1(\xi) \\ &+ \int_0^{\mathcal{T}_n \wedge t} \gamma_2 \left(\frac{s(\xi)}{1-i(\xi)-s(\xi)} - \frac{1-i(\xi)-s(\xi)}{s(\xi)} \right) dB_2(\xi) \\ &+ \int_0^{\mathcal{T}_n \wedge t} \int_Z \left(\frac{1}{i(\xi^-) - C(u)i(\xi^-)s(\xi^-)} + \frac{1}{s(\xi^-) + C(u)i(\xi^-)s(\xi^-)} - \frac{1}{i(\xi^-)} - \frac{1}{s(\xi^-)} \right) \tilde{N}(d\xi, du), \end{aligned}$$

and the expectation of each terms leads to:

$$\begin{aligned} E[V(i(\mathcal{T}_n \wedge t), s(\mathcal{T}_n \wedge t))] &\leq V(i_0, s_0) + (K_1 + K_2) E \left[\int_0^{\mathcal{T}_n \wedge t} V(i(\xi), s(\xi)) d\xi \right] \\ &\leq V(i_0, s_0) + (K_1 + K_2) \int_0^t E[V(i(\mathcal{T}_n \wedge \xi), s(\mathcal{T}_n \wedge \xi))] d\xi. \end{aligned}$$

By using Gronwall's lemma, we obtain:

$$E[V(i(\mathcal{T}_n \wedge T), s(\mathcal{T}_n \wedge T))] \leq V(i_0, s_0) e^{(K_1 + K_2)T}. \quad (13)$$

Let us set $\Omega_n = \{\mathcal{T}_n \leq T\}$ for $n \geq n_1$, then $\mathbb{P}(\Omega_n) \geq \epsilon$ from (9). For all $\omega \in \Omega_n$, there is at least one of the terms $i(\mathcal{T}_n, \omega)$ or $s(\mathcal{T}_n, \omega)$ which is equal to $1 - \frac{1}{n}$ or $\frac{1}{n}$. If $i(\mathcal{T}_n, \omega) = 1 - \frac{1}{n}$ or $\frac{1}{n}$ then

$$V(i(\mathcal{T}_n), s(\mathcal{T}_n)) \geq n,$$

and if $s(\mathcal{T}_n, \omega)$ is equal to $1 - \frac{1}{n}$ or $\frac{1}{n}$ then:

$$V(i(\mathcal{T}_n), s(\mathcal{T}_n)) \geq n.$$

In each cases, we have $V(i(\mathcal{T}_n), s(\mathcal{T}_n)) \geq n$ and from (9) and (13), one obtains:

$$V(i_0, s_0) e^{(K_1 + K_2)T} \geq E[\mathbb{1}_{\Omega_n(\omega)} V(i(\mathcal{T}_n, \omega), s(\mathcal{T}_n, \omega))] \geq \epsilon n,$$

where $\mathbb{1}_{\Omega_n}$ is the indicator function of Ω_n . Letting $n \rightarrow \infty$ leads to the contradiction that $\infty > V(i_0, s_0) e^{(K_1 + K_2)T} = \infty$. So we must have $\mathcal{T}_\infty = \infty$ a.s., which completes the proof.

3. Extinction

In order to state the extinction result, we first need three lemmas.

Lemma 1 Let $(i(\cdot), s(\cdot), r(\cdot))$ be the solution of (2) with the initial condition $X(0) = (i_0, s_0, r_0) \in (0, 1)^3$. Then:

$$\lim_{t \rightarrow +\infty} \frac{i(t) + s(t) + r(t)}{t} = 0 \text{ a.s.},$$

that is:

$$\lim_{t \rightarrow +\infty} \frac{i(t)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{s(t)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{r(t)}{t} = 0 \text{ a.s.}$$

Proof. The proof of this lemma is the same as the one of [7] since the sum of the stochastic differential equations with and without the Lévy jump is the same. Moreover, these two ones are inspired by [35] for a stochastic SIS epidemic model. We choose to develop it here to help the reader's understanding. Let us set $u(\cdot) = i(\cdot) + s(\cdot) + r(\cdot)$ and define $w(u) = (1 + u)^p$, where p is a non negative constant to be determined later. By using Itô's formula, one has:

$$dw(u) = \frac{\partial w}{\partial u} du + \frac{1}{2} \frac{\partial^2 w}{\partial u^2} (du)^2 = p(1+u)^{p-1} du + \frac{1}{2} p(p-1)(1+u)^{p-2} (du)^2.$$

Moreover,

$$du = di + ds + dr \text{ and } (du)^2 = (di)^2 + (ds)^2 + (dr)^2 + 2(dids + didr + dsdr).$$

From the equations of system (2) and the standard formulas $dt^2 = 0$, $dB_i dB_j = \delta_i^j dt$ (δ_i^j is the Kronecker symbol) and $dB_i dt = dt dB_i = 0$, for $i, j = 1, 2$, one has $(di)^2 = \gamma_1^2 i^2 s^2 dt$, $(ds)^2 = (\gamma_1^2 i^2 + \gamma_2^2 r^2) s^2 dt$, $(dr)^2 = \gamma_2^2 s^2 r^2 dt$, $dids = -\gamma_1^2 i^2 s^2 dt$, $didr = 0$ and $dsdr = -\gamma_2^2 s^2 r^2 dt$, then:

$$(du)^2 = \gamma_1^2 i^2 s^2 dt + \gamma_1^2 i^2 s^2 dt + \gamma_2^2 r^2 s^2 dt + \gamma_2^2 s^2 r^2 dt + 2(-\gamma_1^2 i^2 s^2 dt - \gamma_2^2 s^2 r^2 dt) = 0.$$

Consequently,

$$dw(u) = p(1+u)^{p-1} [\mu - \mu(i+s+r)] dt = p(1+u)^{p-1} (\mu - \mu u) dt = p(1+u)^{p-2} (\mu - \mu u^2) dt,$$

that is:

$$dw(u(t)) = Lw(u(t)) dt, \tag{14}$$

with $Lw(u) = p(1+u)^{p-2} (\mu - \mu u^2)$. For $0 < k < p\mu$, by replacing $w(u(t))$ by $e^{kt} w(u(t))$ in (14), one has:

$$d[e^{kt} w(u(t))] = L[e^{kt} w(u(t))] dt.$$

Then, by taking the integral between 0 and t and the expectation, one obtains:

$$E[e^{kt} w(u(t))] = w(u(0)) + \int_0^t L[e^{k\zeta} w(u(\zeta))] d\zeta, \tag{15}$$

with

$$L[e^{kt} w(u)] = ke^{kt} w(u) + e^{kt} Lw(u) = ke^{kt} (1+u)^p + pe^{kt} (1+u)^{p-2} (\mu - \mu u^2)$$

$$= pe^{kt} (1+u)^{p-2} \left[\frac{k}{p} (1+u)^2 + \mu(1-u^2) \right] < pe^{kt} H,$$

and $H = \sup_{u \in \mathbb{R}_+} \left\{ (1+u)^{p-2} \left(\frac{k}{p} + \mu + \frac{2k}{p} u - \left(\mu - \frac{k}{p} \right) u^2 \right) \right\} + 1$. By using (15), one has:

$$E \left[e^{kt} (1+u)^p \right] \leq (1+u(0))^p + \frac{pHe^{kt}}{k}$$

so:

$$E \left[(1+u(t))^p \right] \leq \frac{(1+u(0))^p}{e^{kt}} + \frac{pH}{k}.$$

Then,

$$\lim_{t \rightarrow +\infty} \sup E \left[(1+u(t))^p \right] \leq \frac{pH}{k} \text{ a.s.}$$

As $u(\cdot)$ is a continuous function, there is a constant $M > 0$ such that:

$$E \left[(1+u(t))^p \right] \leq M, \text{ for all } t \geq 0. \tag{16}$$

Note that (14) yields, for $\delta > 0$ small enough and $k = 1, 2, 3, \dots$ so:

$$E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \right] = E \left[(1+u(k\delta))^p \right] + I_1,$$

with

$$I_1 = E \left[\sup_{k\delta \leq t \leq (k+1)\delta} \int_{k\delta}^t p(1+u(\xi))^{p-2} (\mu - \mu u^2(\xi)) d\xi \right].$$

Moreover, there is $c_1 > 0$ such that:

$$I_1 \leq c_1 E \left[\sup_{k\delta \leq t \leq (k+1)\delta} \int_{k\delta}^t (1+u(\xi))^p d\xi \right] \leq c_1 E \left[\int_{k\delta}^{(k+1)\delta} (1+u(\xi))^p d\xi \right] \leq c_1 \delta E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \right].$$

Then,

$$E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \right] \leq E \left[(1+u(k\delta))^p \right] + c_1 \delta E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \right].$$

Consequently, for $\delta > 0$ such that $c_1 \delta \leq 1$, from (16), one has:

$$E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \right] \leq 2E \left[(1+u(k\delta))^p \right] \leq 2M.$$

Let $\epsilon_u > 0$ be arbitrary. By Chebyshev's inequality^[21], one has:

$$\mathbb{P}\left(\left\{\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p > (k\delta)^{1+\epsilon_u}\right\}\right) \leq \frac{E\left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p\right]}{(k\delta)^{1+\epsilon_u}} \leq \frac{2M}{(k\delta)^{1+\epsilon_u}}, \quad k = 1, 2, 3, \dots$$

Moreover, by using Borel-Cantelli's lemma^[21], we obtain for almost all $\omega \in \Omega$:

$$\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^p \leq (k\delta)^{1+\epsilon_u}, \quad k = 1, 2, 3, \dots \quad (17)$$

As this last inequality is true for all but finitely many k , there is a $k_0(\omega)$, for almost all $\omega \in \Omega$, such that the inequality (17) is true for $k \geq k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \geq k_0$ and $k\delta \leq t \leq (k+1)\delta$, then:

$$\frac{\log(1+u(t))^p}{\log t} \leq \frac{(1+\epsilon_u)\log(k\delta)}{\log(k\delta)} = 1 + \epsilon_u,$$

so:

$$\limsup_{t \rightarrow +\infty} \frac{\log(1+u(t))^p}{\log t} \leq 1 + \epsilon_u, \text{ a.s.}$$

Letting $\epsilon_u \rightarrow 0$, one obtains:

$$\limsup_{t \rightarrow +\infty} \frac{\log(1+u(t))^p}{\log t} \leq 1, \text{ a.s.}$$

For $1 < p < 1 + 2\mu$, one has $\mu > \frac{p-1}{2}$ and

$$\limsup_{t \rightarrow +\infty} \frac{\log(u(t))}{\log t} \leq \limsup_{t \rightarrow +\infty} \frac{\log(1+u(t))}{\log t} \leq \frac{1}{p}, \text{ a.s.,}$$

that is, for $0 < \sigma < 1 - \frac{1}{p}$, there are a constant $T = T(\omega)$ and a set Ω_σ such that $\mathbb{P}(\Omega_\sigma) \geq 1 - \sigma$ and for $t \geq T$, $\omega \in \Omega_\sigma$:

$$\log u(t) \leq \left(\frac{1}{p} + \sigma\right) \log t$$

and so:

$$\limsup_{t \rightarrow +\infty} \frac{u(t)}{t} \leq \limsup_{t \rightarrow +\infty} \frac{t^{\frac{1}{p} + \sigma}}{t} = 0.$$

With the nonnegativity of i , s and r a.s., we have u non-negative a.s. also, which completes the proof.

Lemma 2 Let $(i(\cdot), s(\cdot), r(\cdot))$ be the solution of (2) with the initial condition $X(0) = (i_0, s_0, r_0) \in (0, 1)^3$. Then:

$$\lim_{t \rightarrow +\infty} \frac{\int_0^t i(\xi) dB_1(\xi)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\int_0^t i(\xi) dB_2(\xi)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\int_0^t s(\xi) dB_2(\xi)}{t} = 0 \text{ a.s.}$$

and

$$\lim_{t \rightarrow +\infty} \frac{\int_0^t i(\xi) s(\xi) dB_2(\xi)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\int_0^t s^2(\xi) dB_2(\xi)}{t} = 0 \text{ a.s.}$$

Proof. Once again, the proof of this lemma is the same as the one of [7] but we choose once again to develop it here to help the reading. It is inspired by [35] and we also use some results obtained in the proof of Lemma 1.

$$\text{Let } X(t) = \int_0^t i(\xi)dB_1(\xi), Y(t) = \int_0^t i(\xi)dB_2(\xi), Z(t) = \int_0^t s(\xi)dB_2(\xi), H(t) = \int_0^t i(\xi)s(\xi)dB_2(\xi)$$

and $R(t) = \int_0^t s^2(\xi)dB_2(\xi)$. From Burkholder-Davis-Gundy's and Hölder's inequalities, we obtain, for $2 < p < 1 + 2\mu$,

$$E \left[\sup_{0 \leq \xi \leq t} |H(\xi)|^p \right] \leq C_p E \left[\int_0^t i^2(\xi)s^2(\xi)d\xi \right]^{\frac{p}{2}} \leq C_p t^{\frac{p}{2}} E \left[\sup_{0 \leq \xi \leq t} i^p(\xi) \sup_{0 \leq \xi \leq t} s^p(\xi) \right] \leq 2M^2 C_p t^{\frac{p}{2}}.$$

Let ϵ_H be an arbitrary nonnegative constant. Then,

$$\mathbb{P} \left(\left\{ \omega : \sup_{k\delta \leq t \leq (k+1)\delta} |H(t)|^p > (k\delta)^{1+\epsilon_H + \frac{p}{2}} \right\} \right) \leq \frac{E \left[|H((k+1)\delta)|^p \right]}{(k\delta)^{1+\epsilon_H + \frac{p}{2}}} \leq \frac{2M^2 C_p [(k+1)\delta]^{\frac{p}{2}}}{(k\delta)^{1+\epsilon_H + \frac{p}{2}}} \leq \frac{2^{1+\frac{p}{2}} C_p M^2}{(k\delta)^{1+\epsilon_H}},$$

according to Doob's martingale inequality^[21]. Using Borel-Cantelli's lemma, one has for almost all $\omega \in \Omega$,

$$\sup_{k\delta \leq t \leq (k+1)\delta} |H(t)|^p \leq (k\delta)^{1+\epsilon_H + \frac{p}{2}},$$

holds for all but finitely many k . Hence, there exists a non-negative $k_{H_0}(\omega)$, for almost all $\omega \in \Omega$, for which (3) holds whenever $k \geq k_{H_0}$. Then, for almost all $\omega \in \Omega$, if $k \geq k_{H_0}$ and $k\delta \leq t \leq (k+1)\delta$, then:

$$\frac{\ln |H(t)|^p}{\ln t} \leq \frac{(1 + \epsilon_H + \frac{p}{2}) \ln(k\delta)}{\ln(k\delta)} = 1 + \epsilon_H + \frac{p}{2},$$

so

$$\limsup_{t \rightarrow +\infty} \frac{\ln |H(t)|}{\ln t} \leq \frac{1 + \epsilon_H + \frac{p}{2}}{p}.$$

Letting $\epsilon_H \rightarrow 0$ yields:

$$\lim_{t \rightarrow +\infty} \frac{\ln |H(t)|}{\ln t} \leq \frac{1 + \frac{p}{2}}{p} = \frac{1}{2} + \frac{1}{p}.$$

Then, for an arbitrary small $0 < \eta < \frac{1}{2} - \frac{1}{p}$, there are a constant $\bar{T} = \bar{T}(\omega) > 0$ and a set Ω_η such that $\mathbb{P}(\Omega_\eta) \geq 1 - \eta$ and for $t \geq \bar{T}$, $\omega \in \Omega_\eta$,

$$\ln |H(t)| \leq \left(\frac{1}{2} + \frac{1}{p} + \eta \right) \ln t \text{ and } \limsup_{t \rightarrow +\infty} \frac{H(t)}{t} \leq \limsup_{t \rightarrow +\infty} \frac{t^{\frac{1}{2} + \frac{1}{p} + \eta}}{t} = 0,$$

since $\eta + \frac{1}{2} + \frac{1}{p} < 1$. But $\liminf_{t \rightarrow +\infty} \frac{|H(t)|}{t} \geq 0$ so $\lim_{t \rightarrow +\infty} \frac{|H(t)|}{t} = 0$ a.s. Therefore

$$\lim_{t \rightarrow +\infty} \frac{H(t)}{t} = 0.$$

The proof is exactly the same to obtain that:

$$\lim_{t \rightarrow +\infty} \frac{X(t)}{t} = 0, \lim_{t \rightarrow +\infty} \frac{Y(t)}{t} = 0, \lim_{t \rightarrow +\infty} \frac{Z(t)}{t} = 0 \text{ and } \lim_{t \rightarrow +\infty} \frac{R(t)}{t} = 0.$$

Lemma 3 ^[21] Let $M = \{M(t)\}_{t \geq 0}$ be a real martingale locally continuous such that $M(0) = 0$ and $\langle M(t) \rangle$ be its quadratic variation.

$$\text{If } \lim_{t \rightarrow +\infty} \langle M(t), M(t) \rangle = \infty \text{ a.s. then } \lim_{t \rightarrow +\infty} \frac{M(t)}{\langle M(t), M(t) \rangle} = 0 \text{ a.s.}$$

and

$$\text{if } \limsup_{t \rightarrow +\infty} \frac{\langle M(t), M(t) \rangle}{t} < \infty \text{ a.s. then } \lim_{t \rightarrow +\infty} \frac{M(t)}{t} = 0 \text{ a.s.}$$

Let us remark that this lemma is a general result. This result will be used in the particular case of a Brownian motion B since $\limsup_{t \rightarrow +\infty} \frac{\langle B(t), B(t) \rangle}{t} = 1 < +\infty$ a.s. implies that $\lim_{t \rightarrow +\infty} \frac{B(t)}{t} = 0$ a.s.

Let us introduce the following notations:

$$\sigma_2 = \frac{1}{2} \gamma_2^2 + \int_Z [C(u) - \ln(1 + C(u))] \lambda(du) \text{ and } \sigma_2^1 = \frac{1}{2} \gamma_2^2$$

and

$$k(t) = \int_0^t \int_Z \ln(1 + C(u) i(\xi^-)) \tilde{N}(d\xi, du).$$

Let us now set:

$$R_{01}^J = \frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2^1)}.$$

Theorem 3 Let $(i(\cdot), s(\cdot), r(\cdot))$ be the solution of (2) with the initial condition $X(0) = (i_0, s_0, r_0) \in (0, 1)^3$. If $R_{01}^J < 1$ then:

$$\limsup_{t \rightarrow +\infty} \frac{\ln s(t)}{t} \leq (\mu + \theta + \sigma_2^1)(R_{01}^J - 1) < 0 \text{ a.s.},$$

which means that $s(\cdot)$ tends exponentially to zero a.s., that is the rumor will not spread with probability one.

Proof. We transform (2) in a system of two equations as:

$$\begin{cases} di(t) = \left[\mu - (\beta_1 - \beta_2) i(t) s(t) + \beta_2 i^2(t) - (\beta_2 + \mu) i(t) \right] dt - \gamma_1 i(t) s(t) dB_1(t) \\ \quad - \int_Z C(u) i(t^-) s(t^-) \tilde{N}(dt, du), \\ ds(t) = \left[(\beta_1 + \theta) i(t) s(t) + \theta s^2(t) - (\theta + \mu) s(t) \right] dt + \gamma_1 i(t) s(t) dB_1(t) + \gamma_2 s(t) dB_2(t) \\ \quad - \gamma_2 i(t) s(t) dB_2(t) - \gamma_2 s^2(t) dB_2(t) + \int_Z C(u) i(t^-) s(t^-) \tilde{N}(dt, du), \end{cases} \quad (18)$$

with $r(\cdot) = 1 - i(\cdot) - s(\cdot)$. We integrate the two equations of (18) between 0 and t and add the obtained equalities in order to write:

$$i(t) - i_0 + s(t) - s_0 = \mu t + (\beta_2 + \theta) \int_0^t i(\xi) s(\xi) d\xi + \beta_2 \int_0^t i^2(\xi) d\xi + \theta \int_0^t s^2(\xi) d\xi - (\beta_2 + \mu) \int_0^t i(\xi) d\xi \\ - (\theta + \mu) \int_0^t s(\xi) d\xi + \gamma_2 \int_0^t s(\xi) dB_2(\xi) - \gamma_2 \int_0^t i(\xi) s(\xi) dB_2(\xi) - \gamma_2 \int_0^t s^2(\xi) dB_2(\xi),$$

which gives:

$$\int_0^t i(\xi) d\xi = \frac{\mu t}{\beta_2 + \mu} + \frac{\beta_2 + \theta}{\beta_2 + \mu} \int_0^t i(\xi) s(\xi) d\xi + \frac{\beta_2}{\beta_2 + \mu} \int_0^t i^2(\xi) d\xi + \frac{\theta}{\beta_2 + \mu} \int_0^t s^2(\xi) d\xi - \frac{\theta + \mu}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + \phi(t), \quad (19)$$

with

$$\phi(t) = -\frac{1}{\beta_2 + \mu} \left[i(t) - i_0 + s(t) - s_0 - \gamma_2 \int_0^t s(\xi) dB_2(\xi) + \gamma_2 \int_0^t i(\xi) s(\xi) dB_2(\xi) + \gamma_2 \int_0^t s^2(\xi) dB_2(\xi) \right].$$

The application of Itô-Lévy's formula to the function $\ln s(t)$ leads to:

$$d(\ln s(t)) = \left((\beta_1 + \theta + \gamma_2^2) i(t) + (\theta + \gamma_2^2) s(t) - (\theta + \mu) - \frac{1}{2} (\gamma_1^2 + \gamma_2^2) i^2(t) - \frac{1}{2} \gamma_2^2 s^2(t) \right. \\ \left. - \gamma_2^2 i(t) s(t) - \frac{1}{2} \gamma_2^2 - \int_Z [C(u) i(t^-) - \ln(1 + C(u) i(t^-))] \lambda(du) \right) dt \\ + \gamma_1 i(t) dB_1(t) + \gamma_2 dB_2(t) - \gamma_2 i(t) dB_2(t) - \gamma_2 s(t) dB_2(t) + \int_Z \ln(1 + C(u) i(t^-)) \tilde{N}(dt, du).$$

The integration between 0 and t leads to:

$$\ln s(t) = (\beta_1 + \theta + \gamma_2^2) \int_0^t i(\xi) d\xi + (\theta + \gamma_2^2) \int_0^t s(\xi) d\xi - (\theta + \mu)t - \frac{1}{2} (\gamma_1^2 + \gamma_2^2) \int_0^t i^2(\xi) d\xi \\ - \frac{1}{2} \gamma_2^2 \int_0^t s^2(\xi) d\xi - \gamma_2^2 \int_0^t i(\xi) s(\xi) d\xi - \frac{1}{2} \gamma_2^2 t - \int_0^t \int_Z [C(u) i(t^-) - \ln(1 + C(u) i(t^-))] \lambda(du) d\xi \\ + \gamma_1 \int_0^t i(\xi) dB_1(\xi) + \gamma_2 B_2(t) - \gamma_2 \int_0^t i(\xi) dB_2(\xi) - \gamma_2 \int_0^t s(\xi) dB_2(\xi) + \int_0^t \int_Z \ln(1 + C(u) i(\xi^-)) \tilde{N}(d\xi, du) + \ln s_0.$$

The use of the expression of $\int_0^t i(\xi) d\xi$ of (19) gives:

$$\ln s(t) = \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - (\theta + \mu)t - \frac{1}{2} \gamma_2^2 t - \int_0^t \int_Z [C(u) i(t^-) - \ln(1 + C(u) i(t^-))] \lambda(du) d\xi \\ + \frac{(\beta_2 + \theta)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t i(\xi) s(\xi) d\xi + \frac{\beta_2(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t i^2(\xi) d\xi + \frac{\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s^2(\xi) d\xi \\ + (\theta + \gamma_2^2) \int_0^t s(\xi) d\xi - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi - \frac{1}{2} \gamma_2^2 \int_0^t s^2(\xi) d\xi - \frac{1}{2} (\gamma_1^2 + \gamma_2^2) \int_0^t i^2(\xi) d\xi \\ - \gamma_2^2 \int_0^t i(\xi) s(\xi) d\xi + \gamma_2 B_2(t) + \psi(t) + k(t) + (\beta_1 + \theta + \gamma_2^2) \phi(t), \quad (20)$$

with

$$\psi(t) = \gamma_1 \int_0^t i(\xi) dB_1(\xi) - \gamma_2 \int_0^t i(\xi) dB_2(\xi) - \gamma_2 \int_0^t s(\xi) dB_2(\xi) + \ln s_0.$$

Then we use the fact that i and s are between 0 and 1 a.s. and the non negativity of $C(u)$ and of the function $x \mapsto x - \ln(1+x)$ on \mathbb{R}_+ to have:

$$\begin{aligned} \ln s(t) \leq & \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - (\theta + \mu + \sigma_2^1) t + \frac{2(\beta_2 + \theta)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi \\ & + (\theta + \gamma_2^2) \int_0^t s(\xi) d\xi - \frac{1}{2} \gamma_2^2 \int_0^t s^2(\xi) d\xi + \gamma_2 B_2(t) + \psi(t) + k(t) + (\beta_1 + \theta + \gamma_2^2) \phi(t). \end{aligned}$$

But

$$-\frac{1}{2} \gamma_2^2 s^2 + (\theta + \gamma_2^2) s = -\frac{1}{2} \left(\gamma_2^2 s^2 - 2\gamma_2 s \frac{1}{\gamma_2} (\theta + \gamma_2^2) \right) = -\frac{1}{2} \left[\left(\gamma_2 s - \frac{\theta + \gamma_2^2}{\gamma_2} \right)^2 - \frac{(\theta + \gamma_2^2)^2}{\gamma_2^2} \right] \leq \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2}$$

then,

$$\begin{aligned} \ln s(t) \leq & \frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - (\theta + \mu + \sigma_2^1) t + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2} t \\ & - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + \gamma_2 B_2(t) + \psi(t) + k(t) + (\beta_1 + \theta + \gamma_2^2) \phi(t). \end{aligned}$$

Finally, by dividing by t , one obtains:

$$\begin{aligned} \frac{\ln s(t)}{t} \leq & (\theta + \mu + \sigma_2^1) \left[\frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2^1)} - 1 \right] \\ & - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \langle s(t) \rangle + \gamma_2 \frac{B_2(t)}{t} + (\beta_1 + \theta + \gamma_2^2) \frac{\phi(t)}{t} + \frac{k(t)}{t} + \frac{\psi(t)}{t}. \end{aligned} \quad (21)$$

By replacing R_{01}^J by its value, one has:

$$\frac{\ln s(t)}{t} \leq (\theta + \mu + \sigma_2^1)(R_{01}^J - 1) + \gamma_2 \frac{B_2(t)}{t} + (\beta_1 + \theta + \gamma_2^2) \frac{\phi(t)}{t} + \frac{k(t)}{t} + \frac{\psi(t)}{t}.$$

But

$$\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{k(t)}{t} = 0,$$

following Lemmas 1, 2 and 3. Moreover, there is a non negative constant c such that:

$$\langle k, k \rangle_t = \int_0^t \int_z^t \left[\ln(1 + C(u)i(\xi^-)) \right]^2 \lambda(du) d\xi < \int_0^t c d\xi = ct,$$

using Proposition 2.4 of [19] and assumption (H_2) , thus:

$$\lim_{t \rightarrow +\infty} \frac{\langle k, k \rangle_t}{t} < c,$$

and the application of Lemma 3 gives:

$$\lim_{t \rightarrow +\infty} \frac{k(t)}{t} = 0.$$

As $R_{01}^J < 1$, one has:

$$\limsup_{t \rightarrow \infty} \frac{\ln s(t)}{t} \leq (\mu + \theta + \sigma_2^1)(R_{01}^J - 1) < 0 \text{ a.s.}$$

Consequently $\lim_{t \rightarrow \infty} s(t) = 0$ a.s.

4. Persistence

We recall two lemmas that the reader could find in [16] and which will be useful in the proof of the persistence result.

Lemma 4 Let $f \in \mathcal{C}([0, \infty) \times \Omega, (0, \infty))$. If there are two non-negative constants λ_0 and λ such that:

$$\ln f(t) \leq \lambda t - \lambda_0 \int_0^t f(\xi) d\xi + F(t) \text{ a.s.}$$

for all $t \geq 0$, with $F \in \mathcal{C}([0, \infty) \times \Omega, \mathbb{R})$ and $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ a.s., then:

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t f(\xi) d\xi \leq \frac{\lambda}{\lambda_0} \text{ a.s.}$$

Lemma 5 Let $f \in \mathcal{C}([0, \infty) \times \Omega, (0, \infty))$. If there are two non-negative constants λ_0 and λ such that:

$$\ln f(t) \geq \lambda t - \lambda_0 \int_0^t f(\xi) d\xi + F(t) \text{ a.s.}$$

for all $t \geq 0$, with $F \in \mathcal{C}([0, \infty) \times \Omega, \mathbb{R})$ and $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ a.s., then:

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\xi) d\xi \geq \frac{\lambda}{\lambda_0} \text{ a.s.}$$

Let us set now:

$$R_{02}^J = \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)}.$$

We remark that:

$$\begin{aligned} R_{02}^J &= \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)} \\ &\leq \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} + \frac{2(\beta_2 + \theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2)} \end{aligned}$$

$$\leq \frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2)}$$

$$\leq \frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2^1)} = R_{01}^J$$

since $\sigma_2^1 \leq \sigma_2$ due to the non negativity of the function $x \mapsto x - \ln(1 + x)$ on IR_+ .

Theorem 4 Let $(i(\cdot), s(\cdot), r(\cdot))$ be the solution of (2) with the initial condition $X(0) = (i_0, s_0, r_0) \in (0, 1)^3$. If $R_{02}^J > 1$ then:

$$\frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2)(R_{02}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \leq \liminf_{t \rightarrow \infty} \langle s(t) \rangle \leq \limsup_{t \rightarrow \infty} \langle s(t) \rangle \leq \frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)(R_{01}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.,}$$

which is the persistence in the mean of the phenomenon of rumor spreading.

Proof. The proof has two steps. The first one concerns the limit sup and the second one the limit inf. We start from inequality (21) of the proof of the extinction result that we multiply by t to obtain:

$$\ln s(t) \leq (\theta + \mu + \sigma_2^1) \left[\frac{(\mu + 2(\beta_2 + \theta))(\beta_1 + \theta + \gamma_2^2)}{(\theta + \mu + \sigma_2^1)(\beta_2 + \mu)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2^1)} - 1 \right] t$$

$$- \frac{(\beta_1 + \theta + \gamma_2^2)(\theta + \mu)}{(\beta_2 + \mu)} \int_0^t s(\xi) d\xi + \gamma_2 B_2(t) + (\beta_1 + \theta + \gamma_2^2)\phi(t) + k(t) + \psi(t),$$

since $\langle s(t) \rangle = \frac{1}{t} \int_s^t s(\xi) d\xi$. Then we use the value of R_{01}^J to obtain:

$$\ln s(t) \leq (\mu + \theta + \sigma_2^1)(R_{01}^J - 1)t - \frac{(\beta_1 + \theta + \gamma_2^2)(\theta + \mu)}{(\beta_2 + \mu)} \int_0^t s(\xi) d\xi + \gamma_2 B_2(t) + (\beta_1 + \theta + \gamma_2^2)\phi(t) + k(t) + \psi(t).$$

Let us take:

$$F(t) = \gamma_2 B_2(t) + (\beta_1 + \theta + \gamma_2^2)\phi(t) + k(t) + \psi(t), \quad \lambda_1 = (\mu + \theta + \sigma_2^1)(R_{01}^J - 1) \text{ and } \lambda_0 = \frac{(\beta_1 + \theta + \gamma_2^2)(\theta + \mu)}{(\beta_2 + \mu)}.$$

One has $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$, following Lemmas 1, 2 and 3, and λ_1 and λ_0 are strictly non-negative since $R_{02}^J \leq R_{01}^J$ and $R_{02}^J > 1$.

The application of Lemma 4 gives:

$$\limsup_{t \rightarrow \infty} \langle s(t) \rangle \leq \frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)(R_{01}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.,}$$

which proves the first part of the result. Moreover, by using equation (20) and both the fact that i and s are between 0 and 1 a.s., which implies that $-i^2 \geq -1$, $-s^2 \geq -1$ and $-is \geq -1$ a.s. and the increase of the function $x \mapsto x \ln(1 + x)$ on IR_+ , one obtains:

$$\ln s(t) \geq \left[\frac{\mu(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_2^2 + 2\gamma_2^2) - (\theta + \mu + \sigma_2) \right] t - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + F(t),$$

that is:

$$\ln s(t) \geq (\theta + \mu + \sigma_2) \left[\frac{\mu(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)} - 1 \right] t - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + F(t).$$

By replacing R_{02}^J by its value, one has:

$$\ln s(t) \geq (\theta + \mu + \sigma_2)(R_{02}^J - 1)t - \lambda_0 \int_0^t s(\xi) d\xi + F(t).$$

Finally, by setting $\lambda_2 = (\theta + \mu + \sigma_2)(R_{02}^J - 1)$, one has $\lambda_2 > 0$ since $R_{02}^J > 1$ and the application of Lemma 5 gives:

$$\liminf_{t \rightarrow +\infty} \langle s(t) \rangle \geq \frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2)(R_{02}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.,}$$

which achieves the proof.

Let us set:

$$R_{03}^J = \frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)}.$$

We remark that if $\beta_2 \geq \theta$ (it will be the case in the following theorem) then $\mu + 2(\beta_2 + \theta) \geq \mu + \beta_2 + 3\theta$, which leads to:

$$\frac{(\mu + 2(\beta_2 + \theta)(\beta_1 + \theta + \gamma_2^2))}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} \geq \frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} \geq \frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)},$$

since $\sigma_2^1 \leq \sigma_2$ due to the non negativity of the function $x \mapsto x - \ln(1 + x)$ on IR_+ , so:

$$\frac{(\mu + 2(\beta_2 + \theta)(\beta_1 + \theta + \gamma_2^2))}{(\beta_2 + \mu)(\theta + \mu + \sigma_2^1)} + \frac{(\theta + \gamma_2^2)^2}{2\gamma_2^2(\theta + \mu + \sigma_2^1)} \geq \frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)},$$

that is $R_{03}^J \leq R_{01}^J$.

Theorem 5 Let $(i(\cdot), s(\cdot), r(\cdot))$ be the solution of (2) with the initial condition $X(0) = (i_0, s_0, r_0) \in (0, 1)^3$. If

$$\beta_2 \geq \theta, \theta < \frac{\beta + \mu}{2}, \gamma_1^2 + \frac{\mu - \beta_2}{\beta_2 + \mu} \gamma_2^2 \geq \frac{2\beta_2(\beta_1 + \theta)}{\beta_2 + \mu}, \gamma_2^2 \geq \frac{2\theta(\beta_1 + \theta)}{\beta_2 + \mu - 2\theta} \text{ and } R_{03}^J > 1$$

then:

$$\frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2)(R_{03}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \leq \liminf_{t \rightarrow \infty} \langle s(t) \rangle \leq \limsup_{t \rightarrow \infty} \langle s(t) \rangle \leq \frac{(\beta_2 + \mu)(\mu + \theta + \sigma_2^1)(R_{01}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.,}$$

which is the persistence in the mean of the phenomenon of rumor spreading.

Proof. The right-hand side of the inequality is exactly the same as the one of Theorem 4 and is valid since the fact that $R_{03}^J > 1$ implies that $R_{01}^J > 1$. So it remains to prove that:

$$\liminf_{t \rightarrow \infty} \langle s(t) \rangle \geq \frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2)(R_{03}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.}$$

By using equation (20), the increase of the function $x \mapsto x - \ln(1 + x)$ on IR_+ , and the fact that $\beta_2 \geq \theta$, we obtain:

$$\begin{aligned} \ln s(t) \geq & \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - (\theta + \mu + \sigma_2)t + \left(\frac{2\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \gamma_2^2 \right) \int_0^t i(\xi)s(\xi)d\xi \\ & + \left(\frac{\beta_2(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}(\gamma_1^2 + \gamma_2^2) \right) \int_0^t i^2(\xi)d\xi + \left(\frac{\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}\gamma_2^2 \right) \int_0^t s^2(\xi)d\xi \\ & - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi)d\xi + \gamma_2 B_2(t) + k(t) + \psi(t) + (\beta_1 + \theta + \gamma_2^2)\phi(t). \end{aligned}$$

With the assumption $\gamma_1^2 + \frac{\mu - \beta_2}{\beta_2 + \mu} \gamma_2^2 \geq \frac{2\beta_2(\beta_1 + \theta)}{\beta_2 + \mu}$, we have:

$$\frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2 \geq \frac{\beta_2(\beta_1 + \theta)}{\beta_2 + \mu} + \frac{1}{2}\gamma_2^2 - \frac{\mu - \beta_2}{2(\beta_2 + \mu)}\gamma_2^2,$$

that is:

$$\frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2 \geq \frac{\beta_2(\beta_1 + \theta)}{\beta_2 + \mu} + \left(\frac{1}{2} - \frac{\mu - \beta_2}{2(\beta_2 + \mu)} \right) \gamma_2^2$$

and

$$\frac{\beta_2(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}\gamma_1^2 - \frac{1}{2}\gamma_2^2 \leq 0.$$

Moreover, the assumption $\gamma_2^2 \geq \frac{2\theta(\beta_1 + \theta)}{\beta_2 + \mu - 2\theta}$ implies that:

$$\frac{\beta_2 + \mu - 2\theta}{2(\beta_2 + \mu)} \gamma_2^2 \geq \frac{\theta(\beta_1 + \theta)}{\beta_2 + \mu}, \text{ that is } \frac{1}{2}\gamma_2^2 \geq \frac{\theta(\beta_1 + \theta)}{\beta_2 + \mu} + \frac{\theta}{\beta_2 + \mu} \gamma_2^2,$$

so:

$$\frac{\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}\gamma_2^2 \leq 0 \text{ and } \frac{2\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \gamma_2^2 \leq 0.$$

The fact that i and s are in $(0, 1)$ a.s. implies that $i^2 \leq 1$, $s^2 \leq 1$ and $is \leq 1$. Consequently,

$$\begin{aligned} \ln s(t) \geq & \frac{\mu(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} t - (\theta + \mu + \sigma_2)t + \left(\frac{2\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \gamma_2^2 \right) \int_0^t d\xi \\ & + \left(\frac{\beta_2(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}(\gamma_1^2 + \gamma_2^2) \right) \int_0^t d\xi + \left(\frac{\theta(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{1}{2}\gamma_2^2 \right) \int_0^t d\xi \\ & - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi)d\xi + F(t), \end{aligned}$$

where $F(t)$ is exactly the same as the one of the proof of Theorem 4. Thus:

$$\ln s(t) \geq \left[\frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} - \frac{\gamma_1^2 + 4\gamma_2^2}{2} - (\theta + \mu + \sigma_2) \right] t - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + F(t),$$

which gives:

$$\ln s(t) \geq (\theta + \mu + \sigma_2) \left[\frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\beta_2 + \mu)(\theta + \mu + \sigma_2)} - \frac{\gamma_1^2 + 4\gamma_2^2}{2(\theta + \mu + \sigma_2)} - 1 \right] t - \frac{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)}{\beta_2 + \mu} \int_0^t s(\xi) d\xi + F(t).$$

By replacing R_{03}^J by its value, we obtain:

$$\ln s(t) \geq (\theta + \mu + \sigma_2)(R_{03}^J - 1)t - \lambda_0 \int_0^t s(\xi) d\xi + F(t),$$

with the same λ_0 than the one of the proof of Theorem 4. Then we use Lemma 5 with $\lambda_3 = (\theta + \mu + \sigma_2)(R_{03}^J - 1)$ which is strictly non negative since $R_{03}^J > 1$ and obtain:

$$\liminf_{t \rightarrow \infty} \langle s(t) \rangle \geq \frac{(\beta_2 + \mu)(\theta + \mu + \sigma_2)(R_{03}^J - 1)}{(\theta + \mu)(\beta_1 + \theta + \gamma_2^2)} \text{ a.s.,}$$

which completes the proof.

5. Numerical example and concluding remarks

In the case where we take the following values:

β_1	β_2	θ_1	θ_2	$\theta = \theta_1 - \theta_2$	μ	γ_1	γ_2	σ_2
$\frac{1}{24}$	$\frac{1}{30}$	$\frac{21}{720}$	$\frac{7}{720}$	$\frac{14}{720}$	$\frac{1}{60}$	0.1	0.48	0.1153

we obtain the following thresholds:

R_0^d	R_{03}^s	R_{03}^J
0,84615	1,40143	1,09508

where R_0^d and R_{03}^s are the thresholds of persistence of the e-rumor in the deterministic case and in the stochastic one with only white noises respectively, defined in [7] as:

$$R_0^d = \frac{\mu(\theta + \beta_1)}{\beta_2(\theta + \mu)}$$

and

$$R_{03}^s = \frac{(\mu + \beta_2 + 3\theta)(\beta_1 + \theta + \gamma_2^2)}{(\theta + \mu)(\beta_2 + \mu)} - \frac{\gamma_1^2 + 5\gamma_2^2}{2(\theta + \mu)}.$$

With these values, we note that we obtain extinction of the rumor in the deterministic case and persistence in the two stochastic ones. However, the interval of persistence is smaller in presence of the Lévy jump since $R_{03}^J < R_{03}^s$ in this case. This numerical example points out the relevance of a stochastic study, as well as the presence of Lévy jump, but other values of parameters will give of course other interpretations.

In this paper, we discussed a stochastic e-rumor model with the Lévy jump. We showed that the system has a unique global solution. Sufficient conditions for extinction and persistence in the mean have been established. With a numerical example, we noted that the area of persistence of the phenomenon of e-rumor is bigger in the stochastic case without the Lévy jump. In future work, we plan to make an optimal control approach adapted to these stochastic e-rumor models in order to minimize the effects of fake news.

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