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Research Article

Freudenthal Suspension Theorem And James-Hopf Invariant of Spheres

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Abstract: This paper relates more precisely the image of the suspension map $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ with the kernel of James-Hopf invariant $h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1})$ for $k \leq 9$.

***Keywords*:** EHP-sequence, James construction, James-Hopf invariant, suspension homomorphism

MSC: 55Q25, 55P40, 55P45, 55Q40

1. Introduction

In the field of homotopy theory, the spherical Freudenthal suspension theorem is the fundamental result leading to the concept of stabilization of homotopy groups and ultimately to stable homotopy theory. As it is shown in [1, Chapter 4], it explains the behavior of simultaneously taking suspensions and increasing the index of the homotopy groups of the space in question. More precisely, the suspension homomorphism $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ was introduced by Freudenthal [2], who proved that Σ is an epimorphism if $k \leq -1$ and an isomorphism if $k < -1$. These bounds are sharp in general, i.e., the suspension map $\Sigma : \pi_{2n}(\mathbb{S}^n) \to \pi_{2n+1}(\mathbb{S}^{n+1})$ need not be surjective. The following theorem proved by Freudenthal in [2] says that the image of Σ is exactly the set of elements whose Hopf invariant $h_2 : \pi_{2n+1}(\mathbb{S}^{n+1}) \to \pi_{2n+1}(\mathbb{S}^{2n+1}) \approx \mathbb{Z}$ is zero.

Theorem. (Freudenthal) The image of the suspension homomorphism

$$\Sigma : \pi_{2n}(\mathbb{S}^n) \to \pi_{2n+1}(\mathbb{S}^{n+1})$$

is $\{\alpha \in \pi_{2n+1}(\mathbb{S}^{n+1});\, h_2(\alpha) = 0\}$.

Freudenthal also obtained some results on the kernel of Σ for $k = -1$. Namely, the kernel of the epimorphism $\Sigma : \pi_{2n-1}(\mathbb{S}^n) \to \pi_{2n}(\mathbb{S}^{n+1})$ is the cyclic group generated by the Whitehead product $[\iota_n, \iota_n]$ for the homotopy class ι_n of the identity map $\mathrm{id}_{\mathbb{S}^n}$. The latter results on the suspension homomorphism were completed by Whitehead, G.W. [3].

Putman [4] has written up a modern account of Pontryagin's approach [5] to calculating the homotopy groups

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$\pi_{n+1}(\mathbb{S}^n)$ and $\pi_{n+2}(\mathbb{S}^n)$ of the n-th sphere $\mathbb{S}^n$ using techniques from low-dimensional. In particular, [4, Section 9] contains a detailed account of Pontryagin's proof of the theorem of Freudenthal above. It is shown how to use the Hopf invariant to sharpen the Freudenthal suspension theorem.

It is a fundamental theorem of Adams [6] that a map $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ of Hopf invariant one exists only when $n = 2, 4, 8$. This has a number of very interesting consequences, for example:

- $\mathbb{R}^n$ is a division algebra only for $n = 1, 2, 4, 8$.
- $\mathbb{S}^n$ is an H-space only for $n = 0, 1, 3, 7$.
- $\mathbb{S}^n$ has n linearly independent tangent vector fields only for $n = 0, 1, 3, 7$.
- The only fiber bundles $\mathbb{S}^p \to \mathbb{S}^q \to \mathbb{S}^r$ occur when $(p, q, r) = (0, 1, 1), (1, 3, 2), (3, 7, 4)$, and $(7, 15, 8)$.
- If $\mathbb{S}^{n-1}$, with its usual differentiable structure, is parallelizable then $n = 2, 4$, or 8.

James [7] constructs a functorial homotopy decomposition $\Sigma\Omega\Sigma(X) \simeq \bigvee_{m=1}^{\infty}\Sigma(X^{\wedge m})$ for path-connected, pointed CW-complexes X. The paper [8] generalizes this to a p-local functorial decomposition of $\Sigma\Omega(Y)$ for a co-H-space Y and shows that the wedge summands of $\Sigma\Omega(Y)$ functorially decompose by using an action of an appropriate symmetric group. This is used to construct James-Hopf invariants, being generalizations of Hopf invariants, in a more general context. As a valuable example, an application to the theory of quasi-symmetric functions is presented. Furthermore, results stated in [9, Chapter XI] emphasize a significant role of James-Hopf invariants in homotopy theory.

It seems to be no other approach for this Freudenthal suspension theorem and James-Hopf invariants of spheres being a generalization of Hopf invariant. The main objective of this paper is to generalize the Freudenthal theorem above and relate more precisely the image of $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ with the kernel of $h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1})$ of James-Hopf invariant for $k \le 9$.

In Section 1, we set stages for developments to come. This introductory section is devoted to a general discussion and establishes notations used in the rest of the paper.

Section 2 reviews some basic results on maps of co-H-spaces needed in the next section.

Section 3 is devoted to the main result on the image of Σ and the kernel of h_2.

Theorem 3.4 (1) If $k < 0$ then the suspension homomorphism $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is surjective,

(2) If $k = 0, 2, 3$ then $\mathrm{Im}(\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1}))$ for $n \ge 1$,

(3) If $k = 1$ then $\mathrm{Im}(\Sigma : \pi_{2n+1}(\mathbb{S}^n) \to \pi_{2n+2}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+2}(\mathbb{S}^{n+1}) \to \pi_{2n+2}(\mathbb{S}^{2n+1}))$ for $n \ne 2$.

The image $\mathrm{Im}(\Sigma : \pi_5(\mathbb{S}^2) \to \pi_6(\mathbb{S}^3)) = \mathbb{Z}_2\{2\nu'\} \subsetneqq \mathrm{Ker}(h_2 : \pi_6(\mathbb{S}^3) \to \pi_6(\mathbb{S}^5)) = \mathbb{Z}_6\{2\nu'^{+}\}$.

Furthermore, the image of $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ and the kernel of $h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1})$ for $4 \le k \le 9$ are investigated as well.

To make techniques used in this paper more plausible, we refer to [10] and [11].

2. Prerequisites

Throughout this paper, all spaces are assumed to be connected, based and of the homotopy type of CW-complexes, and all maps are based maps unless stated otherwise. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the (based) loop (resp. suspension) space of a space X and $[Y, X]$ for the set of (based) homotopy classes of maps $Y \to X$. We do not distinguish notationally between a continuous map and its homotopy class and we use freely notations from the books [12] and [13].

Let X be a pointed connected topological space. For any prime p, we write $\pi_k(X; p)$ for the p-primary component of the k-th homotopy group $\pi_k(X)$ for $k \ge 1$. For a nilpotent space X, we write $X_{(p)}$ for its p-localization.

Given a space X, James [7] has shown that the extension $J(X) \xrightarrow{\simeq} \Omega\Sigma(X)$ of the counit map $\eta_X : X \to \Omega\Sigma(X)$ to the infinite reduced product (the James construction [7] of X) $J(X) = \mathrm{colim}_{n\ge1} J_n(X)$ is a homotopy equivalence and $\Sigma(J(X)) \simeq \bigvee_{m=1}^{\infty}\Sigma(X^{\wedge m})$, where $X^{\wedge m}$ denotes the m-fold smash power of X. This means that $J(\mathbb{S}^n) \simeq \Omega(\mathbb{S}^{n+1})$ for the n-th sphere $\mathbb{S}^n$ and the splitting map above leads to the projection maps $\Sigma\Omega(\mathbb{S}^{n+1}) \to \mathbb{S}^{kn+1}$ for $k \ge 0$ which are adjoint to the maps

$$H_k : \Omega(\mathbb{S}^{n+1}) \to \Omega(\mathbb{S}^{kn+1})$$

known as James-Hopf maps. Those induce James-Hopf invariants

$$h_k : [Y,\ \Omega(\mathbb{S}^{n+1})] \to [Y,\ \Omega(\mathbb{S}^{kn+1})]$$

or equivalently

$$h_k : [\Sigma(Y),\ \mathbb{S}^{n+1})] \to [\Sigma(Y),\ \mathbb{S}^{kn+1}]$$

for any space Y.

Now, recall the fibration

$$\mathbb{S}^n_{(2)} \xrightarrow{E} \Omega(\mathbb{S}^{n+1}_{(2)}) \xrightarrow{H} \Omega(\mathbb{S}^{2n+1}_{(2)})$$

found by James [14] and the fibrations

$$\hat{\mathbb{S}}^{2n}_{(p)} \to \Omega(\mathbb{S}^{2n+1}_{(p)}) \to \Omega(\mathbb{S}^{2np+1}_{(p)}),$$

$$\mathbb{S}^{2n-1}_{(p)} \to \Omega(\hat{\mathbb{S}}^{2n}_{(p)}) \to \Omega(\mathbb{S}^{2np-1}_{(p)})$$

found by Toda [13] for $p > 2$, where $\hat{\mathbb{S}}^{2n}$ is the $(2np - 1)$-skeleton of the loop space $\Omega(\mathbb{S}^{2n+1})$. Here, $E = \Sigma$ stands for Einfängung (suspension), $H = H_2$ for James-Hopf map, and P for Whitehead product. Thus, the fibrations above and the Serre result [15] lead to:

Theorem 1.1 (1) The fibre of James-Hopf map $H_2 : \Omega(\mathbb{S}^{2n}) \to \Omega(\mathbb{S}^{4n-1})$ is $\mathbb{S}^{2n-1}$ and there is an odd primary equivalence (due to Serre)

$$\Omega(\mathbb{S}^{2n}) \simeq \mathbb{S}^{2n-1} \times \Omega(\mathbb{S}^{4n-1}),$$

(2) the p-local fibre of $H_p : \Omega(\mathbb{S}^{2n+1}) \to \Omega(\mathbb{S}^{2pn+1})$ is $J_{p-1}(\mathbb{S}^{2n})$ for any prime p (due to James for $p = 2$ and Toda for $p > 2$).

The EHP-sequences are the long exact sequences of homotopy groups associated with fibrations considered above. Hence, we get the long exact sequence

$$\cdots \to \pi_{n+k}(\mathbb{S}^n_{(p)}) \xrightarrow{\Sigma} \pi_{n+k+1}(\mathbb{S}^{n+1}_{(p)}) \xrightarrow{H} \pi_{n+k+1}(\mathbb{S}^{2n+1}_{(p)}) \xrightarrow{P} \pi_{n+k-1}(\mathbb{S}^n_{(p)}) \to \cdots.$$

Thus, we may state:

Proposition 1.2 (1) If $p = 2$ and $n \geq 1$ or $p > 2$ and n odd then there is a fibration

$$\mathbb{S}^n_{(p)} \to \Omega(\mathbb{S}^{n+1}_{(p)}) \to \Omega(\mathbb{S}^{2n+1}_{(p)})$$

which yields the long exact sequence

$$\cdots \to \pi_{n+k}(\mathbb{S}^n_{(p)}) \xrightarrow{\Sigma} \pi_{n+k+1}(\mathbb{S}^{n+1}_{(p)}) \xrightarrow{H} \pi_{n+k+1}(\mathbb{S}^{2n+1}_{(p)}) \xrightarrow{P} \pi_{n+k-1}(\mathbb{S}^n_{(p)}) \to \cdots,$$

(2) if $p > 2$ and $n = 2m - 1$ then

$$\pi_{2m+k}(\mathbb{S}^{2m}_{(p)}) \approx \pi_{2m+k-1}(\mathbb{S}^{2m-1}_{(p)}) \oplus \pi_{2m+k}(\mathbb{S}^{4m-1}_{(p)}).$$

The spherical Freudenthal suspension theorem [2] is the fundamental result leading to the concept of stabilization of homotopy groups and ultimately to stable homotopy theory. It explains the behavior of simultaneously taking suspensions and increasing the index of the homotopy groups of the space in question. For its general formulation we need:

Lemma 1.3 ([16, Proposition 3.2]) Let X be an n-connected topological space. Then, the adjunction unit of the adjunction $X \to \Omega\Sigma(X)$ is $(2n + 1)$-connected.

The Freudenthal suspension theorem is about homotopy groups of n-spheres but, applying Lemma 1.3, we may state:

Proposition 1.4 The suspension homomorphism on homotopy groups of spheres $\pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is an epimorphism for $k \leq -1$ and an isomorphism for $k < -1$.

More generally, for X an n-connected CW-complex, then the suspension homomorphism on homotopy groups $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an epimorphism for $k \leq 2n+1$ and an isomorphism for $k < 2n + 1$.

Next, since the sphere $\mathbb{S}^k \approx \Sigma(\mathbb{S}^{k-1})$ and $\mathbb{S}^{k-1}$ is a co-H-space for $k > 1$, results stated in [17, Chapter III, (1.8) and (1.9) Corollaries] yield the following proved by Hilton [18].

Proposition 1.5 Let $\alpha \in \pi_k(\mathbb{S}^m)$ and $\beta, \gamma \in \pi_m(\mathbb{S}^n)$ with $k, m, n > 1$. Then

$$(\beta + \gamma) \circ \alpha = \beta\alpha + \gamma\alpha + [\beta, \gamma] \circ h_2(\alpha).$$

In particular, by induction on $t \in \mathbb{Z}$, it follows:

$$(t\iota_m) \circ \alpha = t\alpha + \left(\frac{t(t-1)}{2}\right)[\iota_m, \iota_m]h_2(\alpha).$$

3. Maps of co-*H*-spaces

To move to the main objective of the paper and generalize Freudenthal theorem announced in Introduction, some prerequisites are required.

Given co-H-spaces (X, ν) and (X', ν'), we say that $\alpha : X \to X'$ is a co-H-map (or primitive) with respect to ν and ν' if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ \nu \downarrow & & \downarrow \nu' \\ X \vee X & \xrightarrow[\alpha\vee\alpha]{} & X' \vee X' \end{array}$$

commutes up to homotopy.

Remark 2.1 Let $\alpha : X \to X'$ be a map of co-H-spaces. Then, the following are equivalent:

(1) $\alpha : X \to X'$ is a co-H-map,

(2) $(i_1 + i_2)\alpha = i_1\alpha + i_2\alpha$ in $[X, X' \vee X']$, where $i_1, i_2 : X' \hookrightarrow X' \vee X'$ are first and second inclusion maps, respectively.

Certainly, the suspension $\Sigma(\beta) : \Sigma(X) \to \Sigma(X')$ of a map $\beta : X \to X'$ is a co-H-map with respect to the suspension structures on $\Sigma(X)$ and $\Sigma(X')$. It is easily seen that $h_k(\Sigma(\beta)) = 0$ for $k \geq 2$.

We say that a map $\alpha : X \to X'$ of co-H-spaces is a weak co-H-map if the induced map $\alpha^* : [X', Y] \to [X, Y]$ is a homomorphism for any space Y. Certainly, any co-H-map $\alpha : X \to X'$ of co-H-spaces is a weak co-H-map.

In particular, a map of spheres $\alpha : \mathbb{S}^k \to \mathbb{S}^m$ is a weak co-H-map if the induced map $\alpha^* : \pi_m(\mathbb{S}^n) \to \pi_k(\mathbb{S}^n)$ is a homomorphism for $n \geq 0$. For maps of spheres we have:

Proposition 2.2 Let $\alpha : \mathbb{S}^k \to \mathbb{S}^m$. Then:

(1) $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$ provided $[\beta, \gamma] = 0$ for $\beta, \gamma \in \pi_n(\mathbb{S}^n)$,

(2) if $h_2(\alpha) = 0$ then $\alpha : \mathbb{S}^k \to \mathbb{S}^m$ is a weak co-H-map,

(3) $(t[\beta, \gamma])\alpha = t([\beta, \gamma]\alpha)$ for $t \in \mathbb{Z}$,

(4) if $h_2(\alpha) = 0$ and m is even then $\alpha : \mathbb{S}^m \to \mathbb{S}^n$ is a suspension map,

(5) if the order $|\alpha| = t$ and m are odd then $\alpha : \mathbb{S}^k \to \mathbb{S}^m$ is a weak co-H-map.

Proof. (1) and (2): Those follow from Proposition 1.5.

(3): We proceed by induction on t.

First, by $(2[\beta, \gamma])\alpha = ([\beta, \gamma] + [\beta, \gamma])\alpha$ and $[[\beta, \gamma], [\beta, \gamma]] = 0$, the property (1) yields $(2[\beta, \gamma])\alpha = 2([\beta, \gamma]\alpha)$.

Next, let $(t[\beta, \gamma])\alpha = t([\beta, \gamma]\alpha)$ for some integer t. Then, again (1) yields $((t + 1)[\beta, \gamma])\alpha = ([t\beta, \gamma] + [\beta, \gamma])\alpha = (t[\beta, \gamma])\alpha + [\beta, \gamma])\alpha$. Hence, $((t + 1)[\beta, \gamma])\alpha = (t + 1)([\beta, \gamma]\alpha)$.

(4): It follows from *EHP*-sequence properties.

(5): If $\beta, \gamma \in \pi_m(\mathbb{S}^n)$ then, in view of Proposition 1.5,

$$(\beta + \gamma) \circ \alpha = \beta\alpha + \gamma\alpha + [\beta,\ \gamma] \circ h_2(\alpha).$$

But for $|\alpha| = t$, in view of (3), we have $(t[\beta, \gamma]) \circ h_2(\alpha) = [\beta, \gamma]) \circ (th_2(\alpha)) = 0$. Since $|\alpha| = t$ and m are odd, we deduce that $t[\beta, \gamma] = [\beta, \gamma]$. Consequently, $[\beta, \gamma] \circ h_2(\alpha) = 0$ implies $(\beta + \gamma) \circ \alpha = \beta\alpha + \gamma\alpha$ and the proof is complete.

□

Example 2.3 Given an odd prime p, by [13, Lemma 13.5] there are $\alpha_k(p) \in \pi_{2k(p-1)+2}(\mathbb{S}^3; p)$ with $p\alpha_k(p) = 0$ for $k \geq 1$. Furthermore, in view of [13, (13.9)], we have $h_2(\alpha_k(p)) = t\alpha_{k-1}(p)$ for $1 < k < p$ and $t \not\equiv 0 \pmod p$. Then, by Proposition 2.2(5), the map $\alpha_k(p) : \mathbb{S}^{2k(p-1)+2} \to \mathbb{S}^3$ is a weak co-H-map, but it is not a co-H-map provided $k > 1$.

Co-H-maps $\alpha : \Sigma(Y) \to \Sigma(Y)$ need not be suspensions. But, in view of [17, (2.7) Proposition], we have:

Proposition 2.4 Let X be finite dimensional *CW*-complex. Then a map $\alpha : \Sigma(X) \to \Sigma(X')$ is a co-H-map with respect to the suspension structures on $\Sigma(X)$ and $\Sigma(X')$ if and only if James-Hopf invariants $h_k(\alpha)$ are trivial for $k \geq 2$.

Furthermore, by [19, Theorem 2], we have:

Proposition 2.5 Let p be an odd prime and X a space. Then $\alpha : \Sigma^2(X) \to \mathbb{S}^{2n+1}_{(p)}$ is a co-H-map with respect to the suspension structures on $\Sigma^2(X)$ and $\mathbb{S}^{2n+1}_{(p)}$ only if $h_p(\alpha)$ is trivial.

We point out that Proposition 2.5 might be easily extended to maps $\alpha : \Sigma(X) \to \mathbb{S}^{2n+1}_{(p)}$ provided X is a co-H-space.

The question as to whether co-H-spaces are suspensions leads naturally to the related question for maps: if co-H-map $\Sigma(X) \to \Sigma(X')$ with respect to the suspension structures, is a suspension? Examples have been given in [20] to show that the answer is in general negative.

Example 2.6 Let p be an odd prime and $\alpha_1(3) \in \pi_{2p}(\mathbb{S}^3; p)$ be an element of order p. Then, $\alpha_1(3)$ is not a suspension since $\pi_{2p-1}(\mathbb{S}^2) \approx \pi_{2p-1}(\mathbb{S}^3)$ contains no element of order p ([13, (13.6)']). Furthermore, the group $\pi_{2p}(\mathbb{S}^{k+1})$ with $k > 1$ contains no element of order p. Therefore, $h_k(\alpha_1(3)) = 0$ with $k \geq 2$ and, by Proposition 2.4, the map $\alpha_1(3) : \mathbb{S}^{2p} \to \mathbb{S}^3$ is a co-H-map.

Recall that by Section 1 the fibration

$$\mathbb{S}^n_{(p)} \to \Omega(\mathbb{S}^{n+1}_{(p)}) \to \mathbb{S}^{2n+1}_{(p)}$$

for $p = 2$ and $n \geq 1$ or $p > 2$ and n odd yields the *EHP*-sequence

$$\cdots \to \pi_{n+k}(\mathbb{S}^n_{(p)}) \xrightarrow{\Sigma} \pi_{n+k+1}(\mathbb{S}^{n+1}_{(p)}) \xrightarrow{H} \pi_{n+k+1}(\mathbb{S}^{2n+1}_{(p)}) \xrightarrow{P} \pi_{n+k-1}(\mathbb{S}^n_{(p)}) \to \cdots.$$

Here, $H = h_2$ for Hopf invariant, and P for Whitehead product. If $n = 2m - 1$ is odd then the fibration is valid for all primes p and it splits at odd primes, so for $p > 2$ we have

$$\pi_{2m+k}(\mathbb{S}^{2m}_{(p)}) \approx \pi_{2m+k-1}(\mathbb{S}^{2m-1}_{(p)}) \oplus \pi_{2m+k}(\mathbb{S}^{4m-1}_{(p)}).$$

4. The main result

It is well-known that the spherical Freudenthal suspension theorem says that the suspension map $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is surjective for $k \geq -1$ and an isomorphism for $k < -1$. These bounds are sharp in general, i.e., the suspension map $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ need not be surjective for $k \geq 0$.

Certainly, the image of $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is contained in the kernel of $h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1})$. The result proved by Freudenthal in [4] (the same paper that contains the usual Freudenthal suspension theorem) says that the image of $\Sigma : \pi_{2n}(\mathbb{S}^n) \to \pi_{2n+1}(\mathbb{S}^n)$ is exactly the kernel of James-Hopf invariant $h_2 : \pi_{2n+1}(\mathbb{S}^{n+1}) \to \pi_{2n+1}(\mathbb{S}^{2n+1}) \approx \mathbb{Z}$.

In the sequel, we need the results stated in [20, Theorem B]:

Theorem 3.1 If dim $X \leq 3n - 2$ and X' is $(n - 1)$-connected with $n \geq 1$, with locally finite generated homology, then every homomorphism $\Sigma(X) \to \Sigma(X')$ is a suspension.

Thus, we may state:

Theorem 3.2 (1) If $p = 2$ and $n \geq 1$ or $p > 2$ and n odd then there is a fibration

$$\mathbb{S}^n_{(p)} \to \Omega\mathbb{S}^{n+1}_{(p)} \to \mathbb{S}^{2n+1}_{(p)}$$

which gives the long exact sequence

$$\cdots \to \pi_{n+k}(\mathbb{S}^n_{(p)}) \xrightarrow{\Sigma} \pi_{n+k+1}(\mathbb{S}^{n+1}_{(p)}) \xrightarrow{H} \pi_{n+k+1}(\mathbb{S}^{2n+1}_{(p)}) \xrightarrow{P} \pi_{n+k-1}(\mathbb{S}^n_{(p)}) \to \cdots,$$

(2) if $p > 2$ and $n = 2m - 1$ then

$$\pi_{2m+k}(\mathbb{S}^{2m}_{(p)}) \approx \pi_{2m+k-1}(\mathbb{S}^{2m-1}_{(p)}) \oplus \pi_{2m+k}(\mathbb{S}^{4m-1}_{(p)}).$$

Now, we show:

Lemma 3.3 If $n \geq 1$ then:

(1) Im$(\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1}))$ = Ker$(h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1}))$ provided:

(i) $k < 0$,

(ii) n is odd;

(2) Im$(\Sigma : \pi_{2n+k}(\mathbb{S}^n; 2) \to \pi_{2n+k+1}(\mathbb{S}^{n+1}; 2))$ = Ker$(h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}; 2) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1}; 2))$.

Proof. (1)(i): Certainly, $\mathrm{Im}(\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})) \subseteq \mathrm{Ker}(h_2 : \pi_{n+k+1}(\mathbb{S}^{n+1}) \to \pi_{n+k+1}(\mathbb{S}^{2n+1}))$. On the other hand, by Freudenthal suspension theorem [2], the sus-pension map $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is an epimorphism provided $k < 0$. Consequently, $h_2(\alpha) = 0$ for any $\alpha \in \mathrm{Im}(\Sigma) = \pi_{2n+k+1}(\mathbb{S}^{2n+1})$.

(1)(ii) and (2) follow from Theorem 3.2(1) and the proof is complete.

□

Next, we recall from [9, Exercises 5-6] the formulae:

(1) if $\alpha \in \pi_{n+1}(\mathbb{S}^{r+1})$ and $\beta \in \pi_k(\mathbb{S}^n)$ then

$$h_m(\alpha \circ \Sigma(\beta)) = h_m(\alpha) \circ \Sigma(\beta),$$

(2) if $\alpha \in \pi_n(\mathbb{S}^r)$ and $\beta \in \pi_{k+1}(\mathbb{S}^{n+1})$ then

$$h_m((\Sigma(\alpha) \circ \beta) = \Sigma\underbrace{(\alpha \wedge \cdots \wedge \alpha)}_{l} \circ h_m(\beta),$$

where l is the weight of the basic product w_{m+1}.

Applying upshots and formulae above, we state the main result:

Theorem 3.4 (1) If $k < 0$ then the suspension homomorphism $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is surjective,

(2) if $k = 0, 2, 3$ then $\mathrm{Im}(\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1}))$ for $n \geq 1$,

(3) if $k = 1$ then $\mathrm{Im}(\Sigma : \pi_{2n+1}(\mathbb{S}^n) \to \pi_{2n+2}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+2}(\mathbb{S}^{n+1}) \to \pi_{2n+2}(\mathbb{S}^{2n+1}))$ for $n \neq 2$.

The image $\mathrm{Im}(\Sigma : \pi_5(\mathbb{S}^2) \to \pi_6(\mathbb{S}^3) = \mathbb{Z}_2\{2\nu'\} \subsetneqq \mathrm{Ker}(h_2 : \pi_6(\mathbb{S}^3) \to \pi_6(\mathbb{S}^5)) = \mathbb{Z}_6\{2\nu'^{+}\}$,

(4) if $k = 4$ then $\mathrm{Im}(\Sigma : \pi_{2n+4}(\mathbb{S}^n) \to \pi_{2n+5}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+5}(\mathbb{S}^{n+1}) \to \pi_{2n+5}(\mathbb{S}^{2n+1}))$ for $n \neq 2$.

The image $\mathrm{Im}(\Sigma : \pi_8(\mathbb{S}^2) \to \pi_9(\mathbb{S}^3)) = \{0\} \subsetneqq \mathrm{Ker}(h_2 : \pi_9(\mathbb{S}^3) \to \pi_9(\mathbb{S}^5)) = \pi_9(\mathbb{S}^3)$,

(5) if $k = 5$ then $\mathrm{Im}(\Sigma : \pi_{2n+5}(\mathbb{S}^n) \to \pi_{2n+6}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+6}(\mathbb{S}^{n+1}) \to \pi_{2n+6}(\mathbb{S}^{2n+1}))$ for $n \neq 2$.

The image $\mathrm{Im}(\Sigma : \pi_8(\mathbb{S}^2) \to \pi_9(\mathbb{S}^3)) = \{0\} \subsetneqq \mathrm{Ker}(h_2 : \pi_9(\mathbb{S}^3) \to \pi_9(\mathbb{S}^5)) = \pi_{10}(\mathbb{S}^3)$,

(6) if $k = 6$ then $\mathrm{Im}(\Sigma : \pi_{2n+6}(\mathbb{S}^n) \to \pi_{2n+7}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+7}(\mathbb{S}^{n+1}) \to \pi_{2n+7}(\mathbb{S}^{2n+1}))$ for $n \neq 4$.

The image $\mathrm{Im}(\Sigma : \pi_{14}(\mathbb{S}^4) \to \pi_{15}(\mathbb{S}^5) = \mathbb{Z}_8\{\nu_5\sigma_8\} \oplus \mathbb{Z}_2\{\eta_5\mu_6\} \oplus \mathbb{Z}_3\{3\beta_1(5)\} \subsetneqq \mathrm{Ker}(h_2 : \pi_{15}(\mathbb{S}^5) \to \pi_{15}(\mathbb{S}^9)) = \pi_{15}(\mathbb{S}^5)$,

(7) if $k = 7$ then $\mathrm{Im}(\Sigma : \pi_{2n+7}(\mathbb{S}^n) \to \pi_{2n+8}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+8}(\mathbb{S}^{n+1}) \to \pi_{2n+8}(\mathbb{S}^{2n+1}))$ for $n \neq 4$.

The image $\mathrm{Im}(\Sigma : \pi_{15}(\mathbb{S}^4) \to \pi_{16}(\mathbb{S}^5)) = \mathbb{Z}_2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}_2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}_8\{2\zeta_5\} \oplus \mathbb{Z}_7\{\alpha_1(5)\} \oplus \mathbb{Z}_9\{3\alpha'_3(5)\} \subsetneqq \mathrm{Ker}(h_2 : \pi_{16}(\mathbb{S}^5) \to \pi_{16}(\mathbb{S}^9)) = \mathbb{Z}_2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}_2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}_8\{2\zeta_5\} \oplus \mathbb{Z}_7\{\alpha_1(5)\} \oplus \mathbb{Z}_9\{\alpha'_3(5)\}$,

(8) if $k = 8$ then $\mathrm{Im}(\Sigma : \pi_{2n+8}(\mathbb{S}^n) \to \pi_{2n+9}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+9}(\mathbb{S}^{n+1}) \to \pi_{2n+9}(\mathbb{S}^{n+1}))$ for $n \neq 2$.

The image $\mathrm{Im}(\Sigma : \pi_{12}(\mathbb{S}^2) \to \pi_{13}(\mathbb{S}^3)) = \mathbb{Z}_2\{2\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \subsetneqq \mathrm{Ker}(h_2 : \pi_{13}(\mathbb{S}^3) \to \pi_{13}(\mathbb{S}^5)) = \mathbb{Z}_2\{2\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3\{\alpha_1(3)\alpha_2(6)\}$,

(9) if $k = 9$ then $\mathrm{Im}(\Sigma : \pi_{2n+9}(\mathbb{S}^n) \to \pi_{2n+10}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+10}(\mathbb{S}^{n+1}) \to \pi_{2n+10}(\mathbb{S}^{2n+1}))$ for $n \neq 2$.

The image $\mathrm{Im}(\Sigma : \pi_{13}(\mathbb{S}^2) \to \pi_{14}(\mathbb{S}^3)) = \mathbb{Z}_2\{2\mu'\} \subsetneqq \mathrm{Ker}(h_2 : \pi_{14}(\mathbb{S}^3) \to \pi_{14}(\mathbb{S}^5)) = \mathbb{Z}_2\{2\mu'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3\{\alpha_1(3)\alpha_2(6)\}$.

Proof. (1) If $k < 0$ then $\dim \mathbb{S}^{n+k} = n + k \leq 2n - 2$. Thus, Lemma 3.3 implies that $\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})$ is surjective.

Next, notice that Lemma 3.3 implies

$$\mathrm{Im}(\Sigma : \pi_{2n+k}(\mathbb{S}^n) \to \pi_{2n+k+1}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+k+1}(\mathbb{S}^{n+1}) \to \pi_{2n+k+1}(\mathbb{S}^{2n+1})$$

for $n > k + 1$ with $k \geq 0$ unless n is even.

(2), $k = 0$: Then, in view of Lemma 3.3, the image

$$\mathrm{Im}(\Sigma : \pi_{2n}(\mathbb{S}^n) \to \pi_{2n+1}(\mathbb{S}^{n+1}) = \mathrm{Ker}(h_2 : \pi_{2n+1}(\mathbb{S}^{n+1}) \to \pi_{2n+1}(\mathbb{S}^{2n+1})$$

for $n \geq 1$.

$k = 2$: Then, in view of lemma 3.3, the image $\mathrm{Im}(\Sigma : \pi_{2n+2}(\mathbb{S}^n) \to \pi_{2n+3}(\mathbb{S}^{n+1})) = \mathrm{Ker}(h_2 : \pi_{2n+3}(\mathbb{S}^{n+1}) \to \pi_{2n+3}(\mathbb{S}^{2n+1})$ for $n \neq 2$.

Furthermore:

if $n = 2$ then $\pi_6(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2\nu'\}$ and $\Sigma(\eta_2\nu') = 0$ imply that the map $\mathrm{Im}(\Sigma : \pi_6(\mathbb{S}^2) \to \pi_7(\mathbb{S}^3))$ is trivial. Since $\pi_7(\mathbb{S}^3) = \mathbb{Z}_2\{\nu'\eta_6\}$, $h_2(\nu'\eta_6) = \eta_5^2$ and $\pi_7(\mathbb{S}^5) = \mathbb{Z}_2\{\eta_5^2\}$, the map $h_2 : \pi_7(\mathbb{S}^3) \to \pi_7(\mathbb{S}^5) = \mathbb{Z}_2\{\eta_5^2\}$ is an isomorphism.

$k = 3$: Then, in view of Lemma 3.3, the image of $\mathrm{Im}(\Sigma : \pi_{2n+3}(\mathbb{S}^n) \to \pi_{2n+4}(\mathbb{S}^{n+1}) = \{\alpha \in \pi_{2n+4}(\mathbb{S}^{n+1}); h_2(\alpha) = 0\}$ for $n \neq 2, 4$.

Furthermore:

if $n = 2$ then $\pi_7(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2\nu'\eta_6\}$ and $\Sigma(\eta_2\nu') = 0$ imply that the map $\Sigma : \pi_7(\mathbb{S}^2) \to \pi_8(\mathbb{S}^3)$ is trivial. Since $\pi_8(\mathbb{S}^3) = \mathbb{Z}_2\{\nu'\eta_6^2\}$, $h_2(\nu'\eta_6^2) = \eta_5^3 = 4\nu_5$ and $\pi_8(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5^+\}$, the map $h_2 : \pi_8(\mathbb{S}^3) \to \pi_8(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5^+\}$ is a monomorphism;

if $n = 4$ then $\pi_{11}(\mathbb{S}^4) = \mathbb{Z}\{\alpha_2(4)\} \oplus \mathbb{Z}_3\{[\iota_4; \iota_4]\alpha_1(7)\} \oplus \mathbb{Z}_5\{\alpha_1(4)\}$ and $\pi_{12}(\mathbb{S}^5) = \mathbb{Z}_2\{\sigma'''\} \oplus \mathbb{Z}_3\{\alpha_2(5)\} \oplus \mathbb{Z}_5\{\alpha_1(5)\}$ imply that $\mathrm{Im}((\Sigma : \pi_{11}(\mathbb{S}^4) \to \pi_{12}(\mathbb{S}^5)) = \mathbb{Z}_3\{\alpha_2(5)\} \oplus \mathbb{Z}_5\{\alpha_1(5)\}$. Next, for $h_2 : \pi_{12}(\mathbb{S}^5) \to \pi_{12}(\mathbb{S}^9) = \mathbb{Z}_{24}\{\nu_9^+\}$ we have $h_2(\sigma''') = 4\nu_9 \neq 0$ and $h_2(\alpha_2(5)) = h_2(\Sigma^2\alpha_2(3)) = h_2(\alpha_1(5)) = h_2(\Sigma^2\alpha_1(3)) = 0$.

(3), $k = 1$: In view of Lemma 3.3, the image $\mathrm{Im}(\Sigma : \pi_{2n+1}(\mathbb{S}^n) \to \pi_{2n+2}(\mathbb{S}^{n+1}))$ is $\{\alpha \in \pi_{2n+k+1}(\mathbb{S}^{n+1}); h_2(\alpha) = 0\}$ for $n \neq 2$.

Furthermore:

if $n = 2$ then $\pi_5(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2^3\}$, $\pi_6(\mathbb{S}^3) = \mathbb{Z}_{12}\{\nu'^+\}$ for $\nu'^+ = \nu' - \alpha_1(3)$, $\Sigma\eta_2^3 = 2\nu'$ and $\mathrm{Im}(\Sigma : \pi_5(\mathbb{S}^2) \to \pi_6(\mathbb{S}^3)) = \mathbb{Z}_2\{2\nu'\}$. Next, for $h_2 : \pi_6(\mathbb{S}^3) = \mathbb{Z}_{12}\{\nu'^+\} \to \pi_6(\mathbb{S}^5) = \mathbb{Z}_2\{\eta_5\}$ we get $h_2(2\nu') = 2\eta_5 = h_2(\alpha_1(3)) = 0$. Hence, $\mathrm{Im}(\Sigma : \pi_5(\mathbb{S}^2) \to \pi_6(\mathbb{S}^3))$ $\subsetneqq \{\alpha \in \pi_5(\mathbb{S}^3); h_2(\alpha) = 0\} = \mathbb{Z}_6\{2\nu'^+\}$.

(4), $k = 4$: Then, in view of Lemma 3.3, the image of $\Sigma : \pi_{2n+4}(\mathbb{S}^n) \to \pi_{2n+5}(\mathbb{S}^{n+1})$ is $\{\alpha \in \pi_{2n+5}(\mathbb{S}^{n+1}); h_2(\alpha) = 0\}$ for $n \neq 2, 4$.

Furthermore:

if $n = 2$ then $\pi_8(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2\nu'\eta_6^2\}$ and $\Sigma : \pi_8(\mathbb{S}^2) \to \pi_9(\mathbb{S}^3)$ is trivial. Next, $\pi_9(\mathbb{S}^3) = \mathbb{Z}_3\{\alpha_1(3)\alpha_1(6)\}$ and $h_2(\alpha_1(3)\alpha_1(6))$ $= h_2(\alpha_1(3))\alpha_1(6) = 0$ show that $h_2 : \pi_9(\mathbb{S}^3) \to \pi_9(\mathbb{S}^5)$ is trivial. Hence, $\mathrm{Im}(\Sigma : \pi_8(\mathbb{S}^2) \to \pi_9(\mathbb{S}^3)) = \{0\} \subsetneqq \{\alpha \in \pi_9(\mathbb{S}^3; h_2(\alpha)$ $= 0\} = \pi_9(\mathbb{S}^3)$;

if $n = 4$ then $\pi_{12}(\mathbb{S}^4) = \mathbb{Z}_2\{\varepsilon_4\}$, $\pi_{13}(\mathbb{S}^5) = \mathbb{Z}_2\{\varepsilon_5\}$ and $\Sigma : \pi_{12}(\mathbb{S}^4) \to \pi_{13}(\mathbb{S}^5)$ is an isomorphism. Next, $h_2 : \pi_{13}(\mathbb{S}^5) \to$ $\pi_{13}(\mathbb{S}^9) = 0$ is trivial.

(5), $k = 5$: Then, in view of Lemma 3.3, the image of $\Sigma : \pi_{2n+5}(\mathbb{S}^n) \to \pi_{2n+6}(\mathbb{S}^{n+1})$ is $\{\alpha \in \pi_{2n+5}(\mathbb{S}^{n+1}); h_2(\alpha) = 0\}$ for $n \neq 2, 4, 6$.

Furthermore:

if $n = 2$ then $\pi_9(\mathbb{S}^2) = \mathbb{Z}_3\{\eta_2\alpha_1(3)\alpha_1(6)\}$, and $\eta_3\alpha_1(4)\alpha_1(7) = 0$ imply that $\Sigma : \pi_9(\mathbb{S}^2) \to \pi_{10}(\mathbb{S}^3)$ is trivial. Next, $\pi_{10}(\mathbb{S}^3)$ $= \mathbb{Z}_3\{\alpha_2(3)\} \oplus \mathbb{Z}_5\{\alpha_1(3)\}$ and $h_2(\alpha_2(3)\alpha_1(6)) = h_2(\alpha_1(3)) = 0$ show that $h_2 : \pi_9(\mathbb{S}^3) \to \pi_9(\mathbb{S}^5)$ is trivial. Hence, $\mathrm{Im}(\Sigma :$ $\pi_8(\mathbb{S}^2) \to \pi_9(\mathbb{S}^3)) = \{0\} \subsetneqq \{\alpha \in \pi_9(\mathbb{S}^3; h_2(\alpha) = 0\} = \pi_{10}(\mathbb{S}^3)$;

if $n = 4$ then $\pi_{13}(\mathbb{S}^4) = \mathbb{Z}_2\{\nu_4^3\} \oplus \mathbb{Z}_2\{\mu_4\} \oplus \mathbb{Z}_2\{\eta_4\varepsilon_5\}$, $\pi_{14}(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5^3\} \oplus \mathbb{Z}_2\{\mu_5\} \oplus \mathbb{Z}_2\{\eta_5\varepsilon_6\}$, and $\Sigma : \pi_{13}(\mathbb{S}^4) \to$ $\pi_{14}(\mathbb{S}^5)$ is an isomorphism. Next, $h_2 : \pi_{14}(\mathbb{S}^5) \to \pi_{14}(\mathbb{S}^9) = 0$ is trivial;

if $n = 6$ then $\pi_{17}(\mathbb{S}^6) = \mathbb{Z}_4\{\overline{\nu}_6\nu_{14}\} \oplus \mathbb{Z}_8\{\zeta_6\} \oplus \mathbb{Z}_7\{\alpha_1(4)\alpha_1(15)\} \oplus \mathbb{Z}_9\{\alpha_3'(6)\}$, $\pi_{18}(\mathbb{S}^7) = \mathbb{Z}_2\{\overline{\nu}_6\nu_{15}\} \oplus \mathbb{Z}_8\{\zeta_7\} \oplus$ $\mathbb{Z}_7\{\alpha_1(5)\alpha_1(16)\} \oplus \mathbb{Z}_9\{\alpha_3'(7)\}$ and $\Sigma : \pi_{17}(\mathbb{S}^6) \to \pi_{18}(\mathbb{S}^7)$ is a surjection. Next, $h_2 : \pi_{18}(\mathbb{S}^7) \to \pi_{18}(\mathbb{S}^{13}) = 0$ is trivial.

(6), $k = 6$: Then, in view of Lemma 3.3, the image of $\mathrm{Im}(\Sigma : \pi_{2n+6}(\mathbb{S}^n) \to \pi_{2n+7}(\mathbb{S}^{n+1}) = \{\alpha \in \pi_{2n+7}(\mathbb{S}^{n+1}); h_2(\alpha) = 0\}$ for $n \neq 2, 4, 6$.

Furthermore:

if $n = 2$ then $\pi_{10}(\mathbb{S}^2) = \mathbb{Z}_3\{\eta_2\alpha_2(5)\} \oplus \mathbb{Z}_5\{\eta_2\alpha_1(3)\}$ and $\pi_{11}(\mathbb{S}^3) = \mathbb{Z}_2\{\varepsilon_3\}$ imply that $\Sigma : \pi_{10}(\mathbb{S}^2) \to \pi_{11}(\mathbb{S}^3)$ is trivial. Next, $h_2(\varepsilon_3) = \nu_5^2$ yields that $h_2 : \pi_{11}(\mathbb{S}^3) \to \pi_{11}(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5^2\}$ is an isomorphism;

if $n = 4$ then $\pi_{14}(\mathbb{S}^4) = \mathbb{Z}_8\{\nu_4\sigma'\} \oplus \mathbb{Z}_4\{\Sigma\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_4\mu_5\} \oplus \mathbb{Z}_3\{\alpha_1(4)\alpha_2(7)\} \oplus \mathbb{Z}_3\{[\iota_4, \iota_4]\alpha_2(7)\} \oplus \mathbb{Z}_5\{\alpha_1(7)\}$, $\pi_{15}(\mathbb{S}^5) = \mathbb{Z}_8\{\nu_5\sigma_8\} \oplus \mathbb{Z}_2\{\eta_5\mu_6\} \oplus \mathbb{Z}_9\{\beta_1(5)\}$, $\Sigma(\nu_4\sigma') = 0$, $\Sigma^2\varepsilon' = \pm 2(\nu_5\sigma_8)$ and $\alpha_1(5)\alpha_2(8) = -3\beta_1(5)$ imply that

$$\operatorname{Im}(\Sigma : \pi_{14}(\mathbb{S}^4) \to \pi_{15}(\mathbb{S}^5)) = \mathbb{Z}_8\{\nu_5\sigma_8\} \oplus \mathbb{Z}_2\{\eta_5\mu_6\} \oplus \mathbb{Z}_3\{3\beta_1(5)\}.$$

Next, $h_2(\nu_5\sigma_8) = h_2(\eta_5\mu_6) = h_2(\beta_1(5)) = 0$ yields that $h_2 : \pi_{15}(\mathbb{S}^5) \to \pi_{15}(\mathbb{S}^9)$ is trivial. Hence,

$$\operatorname{Im}(\Sigma : \pi_{14}(\mathbb{S}^4) \to \pi_{15}(\mathbb{S}^5)) \subsetneqq \{\alpha \in \pi_{15}(\mathbb{S}^5);\, h_2(\alpha) = 0\} = \pi_{15}(\mathbb{S}^5);$$

if $n = 6$ then $\pi_{19}(\mathbb{S}^7) = 0$ imply that $\Sigma : \pi_{18}(\mathbb{S}^6) \to \pi_{19}(\mathbb{S}^7) = 0$ and $h_2 : \pi_{19}(\mathbb{S}^7) = 0 \to \pi_{19}(\mathbb{S}^{13})$ are trivial.

(7), $k = 7$: Then, in view of Lemma 3.3, the image of $\operatorname{Im}(\Sigma : \pi_{2n+7}(\mathbb{S}^n) \to \pi_{2n+8}(\mathbb{S}^{n+1}) = \{\alpha \in \pi_{2n+8}(\mathbb{S}^{n+1});\, h_2(\alpha) = 0\}$ for $n \neq 2, 4, 6, 8$.

Furthermore:

if $n = 2$ then $\pi_{11}(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2\varepsilon_3\}$ $\pi_{12}(\mathbb{S}^3) = \mathbb{Z}_2\{\mu_3\} \oplus \mathbb{Z}_2\{\eta_3\varepsilon_4\}$ imply that imply that the map $\Sigma : \pi_{11}(\mathbb{S}^2) \to \pi_{12}(\mathbb{S}^3)$ is a monomorphism. Next, $h_2(\eta_3\varepsilon_4) = 0$ and $h_2(\mu_3) = \sigma'''$;

if $n = 4$ then $\pi_{15}(\mathbb{S}^4) = \mathbb{Z}_2\{\nu_4\sigma'\eta_{14}\} \oplus \mathbb{Z}_2\{\nu_4\bar{\nu}_7\} \oplus \mathbb{Z}_2\{\nu_4\varepsilon_7\} \oplus \mathbb{Z}_2\{\varepsilon_4\nu_{12}\} \oplus \mathbb{Z}_2\{\Sigma\nu'\varepsilon_7\} \oplus \mathbb{Z}_3\{\alpha_3(4)\} \oplus \mathbb{Z}_7\{\alpha_1(4)\}$ and $\pi_{16}(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}_2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}_8\{\zeta_5\} \oplus \mathbb{Z}_7\{\alpha_1(5)\} \oplus \mathbb{Z}_9\{\alpha_3'(5)\}$. Then, $\Sigma^2\mu' = \pm 2\zeta_5$, $2\mu' = \eta_3^2\mu_5$, $\Sigma^2\nu' = 2\nu_5$, $\nu_5\Sigma\sigma'\eta_{15} + x\Sigma^2\nu'\varepsilon_8 + y\eta_5^2\mu_7 = 0$ for some integers x, y and $\alpha_3(5) = 3\alpha_3'(5)$ imply that $\operatorname{Im}(\Sigma : \pi_{15}(\mathbb{S}^4) \to \pi_{16}(\mathbb{S}^5)) = \mathbb{Z}_2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}_2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}_8\{2\zeta_5\} \oplus \mathbb{Z}_7\{\alpha_1(5)\} \oplus \mathbb{Z}_9\{3\alpha_3'(5)\}$. Certainly, $h_2(\nu_5\bar{\nu}_8) = h_2(\nu_5\varepsilon_8) = h_2(\alpha_1(5)) = h_2(\alpha_3'(5)) = 0$ and $h_2(\zeta_5) = 8\sigma_9$ implies $h_2(2\zeta_5) = 0$. Consequently, $\{\alpha \in \pi_{16}(\mathbb{S}^5);\, h_2(\alpha) = 0\} = \mathbb{Z}_2\{\nu_5\bar{\nu}_8\} \oplus \mathbb{Z}_2\{\nu_5\varepsilon_8\} \oplus \mathbb{Z}_8\{2\zeta_5\} \oplus \mathbb{Z}_7\{\alpha_1(5)\} \oplus \mathbb{Z}_9\{\alpha_3'(5)\}$ and

$$\operatorname{Im}(\Sigma : \pi_{15}(\mathbb{S}^4) \to \pi_{16}(\mathbb{S}^5)) \subsetneqq \{\alpha \in \pi_{16}(\mathbb{S}^5);\, h_2(\alpha) = 0\};$$

if $n = 6$ then $\pi_{19}(\mathbb{S}^6) = \mathbb{Z}_2\{\nu_6\sigma_9\nu_{16}\} \oplus \mathbb{Z}_3\{\alpha_1(6)\beta_1(9)\}$ and $\pi_{20}(\mathbb{S}^7) = \mathbb{Z}_2\{\nu_7\sigma_{10}\nu_{17}\} \oplus \mathbb{Z}_3\{\alpha_1(7)\beta_1(10)\}$. Then, the map $\Sigma : \pi_{19}(\mathbb{S}^6) \to \pi_{20}(\mathbb{S}^7)$ is an isomorphism;

if $n = 8$ then $\pi_{23}(\mathbb{S}^8) = \mathbb{Z}_2\{\sigma_8\bar{\nu}_{15}\} \oplus \mathbb{Z}_2\{\sigma_8\varepsilon_{15}\} \oplus \mathbb{Z}_2\{\Sigma\sigma'\varepsilon_{15}\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_8\} \oplus \mathbb{Z}_8\{\Sigma\rho''\} \oplus \mathbb{Z}_3\{\alpha_4(8)\} \oplus \mathbb{Z}_5\{\alpha_2(8)\}$ and $\pi_{24}(\mathbb{S}^9) = \mathbb{Z}_2\{f\sigma_9\bar{\nu}_{16}\} \oplus \mathbb{Z}_2\{\sigma_9\varepsilon_{16}\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_9\} \oplus \mathbb{Z}_8\{\rho'\} \oplus \mathbb{Z}_3\{\alpha_4(9)\} \oplus \mathbb{Z}_5\{\alpha_2(9)\}$. Then, $\Sigma^2\sigma' = 2\sigma_9$ and $\Sigma^2\rho'' = 2\rho'$ imply that $\operatorname{Im}(\Sigma : \pi_{23}(\mathbb{S}^8) \to \pi_{24}(\mathbb{S}^9)) = \mathbb{Z}_2\{\sigma_9\bar{\nu}_{16}\} \oplus \mathbb{Z}_2\{\sigma_9\varepsilon_{16}\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_9\} \oplus \mathbb{Z}_4\{2\rho'\} \oplus \mathbb{Z}_3\{\alpha_4(9)\} \oplus \mathbb{Z}_5\{\alpha_2(9)\}$. Next, $h_2(\rho') = 8\sigma_{17} \neq 0$ and $h_2(2\rho') = 8\sigma_{17} = 0$.

(8), $k = 8$: Then, in view of Lemma 3.3, the image of $\operatorname{Im}(\Sigma : \pi_{2n+8}(\mathbb{S}^n) \to \pi_{2n+8}(\mathbb{S}^{n+1}) = \{\alpha \in \pi_{2n+9}(\mathbb{S}^{n+1});\, h_2(\alpha) = 0\}$ for $n \neq 2, 4, 6, 8$.

Furthermore:

if $n = 2$ then $\pi_{12}(\mathbb{S}^2) = \mathbb{Z}_2\{\eta_2^2\varepsilon_4\} \oplus \mathbb{Z}_2\{\eta_2\mu_3\}$, $\pi_{13}(\mathbb{S}^3) = \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3\{\alpha_1(3)\alpha_2(6)\}$ and $\eta_3^2\varepsilon_5 = 2\varepsilon'$ imply that imply that the map $\operatorname{Im}(\Sigma : \pi_{12}(\mathbb{S}^2) \to \pi_{13}(\mathbb{S}^3)) = \mathbb{Z}_2\{2\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \subsetneqq \{\alpha \in \pi_{13}(\mathbb{S}^3) : h_2(\alpha) = 0\} = \mathbb{Z}_2\{2\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3\{\alpha_1(3)\alpha_2(6)\}$;

if $n = 4$ then $\pi_{16}(\mathbb{S}^4) = \mathbb{Z}_2\{\nu_4\sigma'\eta_{14}^2\} \oplus \mathbb{Z}_2\{\nu_4^4\} \oplus \mathbb{Z}_2\{\nu_4\,\mu_7\} \oplus \mathbb{Z}_2\{\nu_4\eta_7\varepsilon_8\} \oplus \mathbb{Z}_2\{\Sigma\nu'\mu_7\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\{\Sigma\nu'\eta_7\varepsilon_8\}$, $\pi_{17}(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5^4\} \oplus \mathbb{Z}_2\{\nu_5\,\mu_8\} \oplus \mathbb{Z}_2\{\nu_5\eta_8\varepsilon_9\}$, $\nu_5\sigma\sigma'\eta_{15} = \Sigma^2\nu'\varepsilon_8$ and $\Sigma^2\nu' = 2\nu_5$ imply that the map $\Sigma : \pi_{16}(\mathbb{S}^4) \to \pi_{17}(\mathbb{S}^5)$ is an epimorphism;

if $n = 6$ then $\pi_{20}(\mathbb{S}^6) = \mathbb{Z}_4\{\sigma''\sigma_{13}\} \oplus \mathbb{Z}_2\{\bar{\nu}_6\nu_{14}^2\} \oplus \mathbb{Z}_3\{\alpha\}$ and $\pi_{21}(\mathbb{S}^7) = \mathbb{Z}_8\{\sigma'\sigma_{14}\nu_{17}\} \oplus \mathbb{Z}_4\{\kappa_7\} \oplus \mathbb{Z}_3\{\Sigma\alpha\}$. But, $\Sigma\sigma'' = 2\sigma'$ and $2\kappa_7 = \bar{\nu}_7\nu_{15}^2 + 2x\Sigma\sigma''\sigma_{14}$ with $x = 0$ or 1. Hence, $\mathrm{Im}(\Sigma : \pi_{20}(\mathbb{S}^6) \to \pi_{21}(\mathbb{S}^7)) = \mathbb{Z}_4\{2\sigma'\sigma_{14}\} \oplus \mathbb{Z}_2\{2\kappa_7\} \oplus \mathbb{Z}_3\{\Sigma\alpha\}$. Next, $h_2(\sigma'\sigma_{14}) = \eta_{13}\sigma_{14} = n\bar{u}_{13} + \varepsilon_{13} \neq 0$ and $h_2 : \pi_{21}(\mathbb{S}^7) \to \pi_{21}(\mathbb{S}^{13}) = \mathbb{Z}_2\{\bar{\nu}_{13}\} \oplus \mathbb{Z}_2\{\varepsilon_{13}\}$ is an epimorphism. Hence, $h_2(\kappa_7) = \bar{\nu}_{13} \neq 0$ or $\varepsilon_{13} \neq 0$ and so $\mathrm{Im}(\Sigma : \pi_{20}(\mathbb{S}^6) \to \pi_{21}(\mathbb{S}^7)) = \{\alpha \in \pi_{21}(\mathbb{S}^7);\, h_2(\alpha) = 0\}$;

if $n = 8$ then $\pi_{24}(\mathbb{S}^8) = \mathbb{Z}_2\{\Sigma_8\nu_{15}^3\} \oplus \mathbb{Z}_2\{\sigma_8\mu_{15}\} \oplus \mathbb{Z}_2\{\sigma_8\eta_{15}\varepsilon_{15}\} \oplus \mathbb{Z}_2\{\Sigma\sigma'\mu_{15}\} \oplus \mathbb{Z}_2\{\Sigma^2\zeta'\} \oplus \mathbb{Z}_2\{\mu_8\sigma_{15}\} \oplus \mathbb{Z}_2\{\eta_8\bar{\varepsilon}_9\} = \pi_{24}^8$ and $\pi_{25}(\mathbb{S}^9) = \mathbb{Z}_2\{\sigma_9\nu_{16}^3\} \oplus \mathbb{Z}_2\{\sigma_9\mu_{16}\} \oplus \mathbb{Z}_2\{\sigma_9\eta_{16}\varepsilon_{17}\} \oplus \mathbb{Z}_2\{\mu_9\sigma_{18}\} = \pi_{25}^9$. Then, by [13, (10.17)], the map $\Sigma : \pi_{24}(\mathbb{S}^8) \to \pi_{25}(\mathbb{S}^9)$ is an epimorphism.

(9), $k = 9$: Then, in view of Lemma 3.3, the image of $\mathrm{Im}(\Sigma : \pi_{2n+9}(\mathbb{S}^n) \to \pi_{2n+10}(\mathbb{S}^{n+1}) = \{\alpha \in \pi_{2n+9}(\mathbb{S}^{n+1});\, h_2(\alpha) = 0\}$ for $n \neq 2, 4, 6, 8, 10$.

Furthermore:

if $n = 2$ then $\pi_{13}(\mathbb{S}^2) = \mathbb{Z}_4\{\eta_2\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_2^2\mu_4\} \oplus \mathbb{Z}_3\{\eta_2\alpha_1(3)\alpha_2(6)\}$, $\pi_{14}(\mathbb{S}^3) = \mathbb{Z}_4\{\mu'\} \oplus \mathbb{Z}_2\{\varepsilon_3\nu_{11}\} \oplus \mathbb{Z}_2\{\nu'\varepsilon_6\} \oplus \mathbb{Z}_3\{\alpha_3(5)\} \oplus \mathbb{Z}_7\{\alpha_1(5)\}$, $\eta_3\alpha_1(4)\alpha_2(7) = 0$ and $\eta_3^2\mu_5 = 2\mu'$ imply that imply that $\mathrm{Im}(\Sigma : \pi_{13}(\mathbb{S}^2) \to \pi_{14}(\mathbb{S}^3)) = \mathbb{Z}_2\{2\mu'\}$.

Next, $h_2(\mu') = \mu_5 \neq 0$, $h_2(\varepsilon_3\nu_{11}) = h_2(\varepsilon_3)\nu_{11} = \nu_5^3 \neq 0$, $h_2(\nu'\varepsilon_6) = h_2(\nu')\varepsilon_6 = \eta_5\varepsilon_6 \neq 0$ and $h_2(\alpha_3(5)) = h_2(\alpha_1(5)) = 0$ lead to $\mathrm{Im}(\Sigma : \pi_{13}(\mathbb{S}^2) \to \pi_{14}(\mathbb{S}^3)) = \mathbb{Z}_2\{2\mu'\} \subsetneqq \{\alpha \in \pi_{14}(\mathbb{S}^3) : h_2(\alpha) = 0\} = \mathbb{Z}_2\{2\mu'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3\{\alpha_1(3)\alpha_2(6)\}$;

if $n = 4$ then $\pi_{17}(\mathbb{S}^4) = \mathbb{Z}_8\{\nu_4^2\sigma_{10}\} \oplus \mathbb{Z}_2\{\nu_4\eta_7\mu_8\} \oplus \mathbb{Z}_2\{\Sigma\nu'\mu_7\mu_8\} \oplus \mathbb{Z}_3\{\alpha_1(4)\beta_1(7)\} \oplus \mathbb{Z}_3\{[\iota_4, \iota_4]\beta_1(7)\}$ and $\pi_{18}(\mathbb{S}^5) = \mathbb{Z}_2\{\nu_5\sigma_8\nu_{15}\} \oplus \mathbb{Z}_2\{\nu_5\eta_8\mu_9\} \oplus \mathbb{Z}_3\{\alpha_1(5)\beta_1(8)\}$. But, by [13, (7.19)], we $x(\nu_4^2\sigma_{10}) = 2\nu_5\sigma_8\nu_{15} = 0$ for an odd x so $\nu_4^2\sigma_{10} = 0$. Next, $h_2(\nu_5\sigma_8\nu_{15}) = \Sigma\nu_5\sharp\nu_5)h_2(\sigma_8\nu_{15}) = \nu_{15}^3 \neq 0$. Hence, $\mathrm{Im}(\Sigma : \pi_{17}(\mathbb{S}^4) \to \pi_{18}(\mathbb{S}^5)) = \mathbb{Z}_2\{\nu_5\eta_8\mu_9\} \oplus \mathbb{Z}_3\{\alpha_1(5)\beta_1(8)\}\{\alpha \in \pi_{18}(\mathbb{S}^5);\, h_2(\alpha) = 0\}$;

if $n = 6$ then $\pi_{21}(\mathbb{S}^6) = \mathbb{Z}_4\{\rho'''\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_6\} \oplus \mathbb{Z}_3\{\alpha_4(6)\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6]\beta_1(11)\} \oplus \mathbb{Z}_5\{\alpha_2(6)\}$, $\pi_{22}(\mathbb{S}^7) = \mathbb{Z}_8\{\rho''\} \oplus \mathbb{Z}_2\{\sigma'\bar{\nu}_{14}\} \oplus \mathbb{Z}_2\{\bar{\varepsilon}_7\} \oplus \mathbb{Z}_3\{\alpha_4(7)\} \oplus \mathbb{Z}_5\{\alpha_2(7)\}$ and $\Sigma\rho''' = 2\rho''$. Next, $h_2(\rho'') = \mu_{13}$, $h_2(\sigma'\bar{\nu}_{14}) = \eta_{13}\bar{\nu}_{14} \neq 0$ and $h_2(\sigma'\varepsilon_{14}) = \eta_{13}\varepsilon_{14} \neq 0$.

if $n = 8$ then $\pi_{25}(\mathbb{S}^8) = \mathbb{Z}_2\{\sigma_8\eta_{15}\mu_{16}\} \oplus \mathbb{Z}_2\{\Sigma\sigma'\eta_{15}\mu_{16}\} \oplus \mathbb{Z}_2\{\nu_8\kappa_{11}\} \oplus \mathbb{Z}_2\{\bar{\mu}_8\} \oplus \mathbb{Z}_2\{\eta_8\mu_9\sigma_{18}\} \oplus \mathbb{Z}_2\{\mu_8\sigma_{15}\} \oplus \mathbb{Z}_3\{[\iota_8, \iota_8]\beta_1(15)\}$ and $\pi_{26}(\mathbb{S}^9) = \mathbb{Z}_2\{\sigma_9\eta_{16}\mu_{17}\} \oplus \mathbb{Z}_2\{\nu_9\kappa_{12}\} \oplus \mathbb{Z}_2\{\bar{\nu}_9\} \oplus \mathbb{Z}_2\{\eta_9\mu_{10}\sigma_{19}\}$. This implies that the map $\Sigma : \pi_{25}(\mathbb{S}^8) \to \pi_{26}(\mathbb{S}^9)$ is an epimorphism;

if $n = 10$ then $\pi_{29}(\mathbb{S}^{10}) = \mathbb{Z}_8\{\bar{\xi}_{10}\} \oplus \mathbb{Z}_2\{\bar{\sigma}_{10}\} \oplus \mathbb{Z}_3\{\alpha_5(10)\} \oplus \mathbb{Z}_3\{[\iota_{10}, \iota_{10}]\beta_1(9)\} \oplus \mathbb{Z}_{11}\{\alpha_2(9)\}$ and $\pi_{30}(\mathbb{S}^{11}) = \mathbb{Z}_2\{\lambda'\eta_{29}\} \oplus \mathbb{Z}_2\{\xi'\eta_{29}\} \oplus \mathbb{Z}_2\{\bar{\xi}_{11}\} \oplus \mathbb{Z}_2\{\bar{\sigma}_{11}\}$. Next, $h_2(\lambda'\eta_{29}) \equiv \varepsilon_{21}\eta_{29}$ $(\mathrm{mod}(\bar{\nu}_{21}\eta_{29} + \varepsilon_{21}\eta_{29}) = (\nu_{21}^3 + \eta_{21}\varepsilon_{29}))$ and $h_2(\lambda'\eta_{29}) = \bar{\nu}_{21}\eta_{29} + \varepsilon_{21}\eta_{29} = \nu_{21}^3 + \eta_{21}\varepsilon_{29}$. Hence, $h_2(\lambda'\eta_{29})$, $h_2(\xi'\eta_{29}) \neq 0$ and the proof is complete.

□

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Conflict of interest

The author declares that there is no personal or organizational conflict of interests with this work.

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