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Research Article

Moving Averages that Preserve the Monotonicity of Quotients

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Abstract: There are operators that have the property that when applied to both the numerator and the denominator of a monotonic quotient of functions or sequences produce another monotonic quotient. Such operations have been very useful to study several problems, including various types of inequalities. Here we study when moving averages preserve the monotonicity of quotients of sequences and quotients of functions. We give results on the averages that guarantee the preservation of monotonicity and examples where monotonicity is not preserved.

***Keywords*:** moving averages, L'Hôpital's monotone rule, Gromov's theorem, monotonicity of quotients

MSC: 26A48, 26D10

1. Introduction

The well known L'Hôpital's rule is a peculiar result in that one applies the *same* operation to both the numerator and the denominator of a fraction, and the limit of the fraction is preserved. An interesting related idea is whether applying the same operation to both the numerator and the denominator of a fraction preserves monotonicity. Indeed, Gromov's theorem [1, pg. 42] says that that is the case for some primitives, while a monotone version of the L'Hôpital's rule has also been obtained [2-4]. Such ideas have found applications in many areas [5-8], including the study of the monotonicity of quotients of power series [9], inequalities for trigonometric functions [10] and generalized trigonometric and hyperbolic functions [11, 12].

In our recent article [7], we considered several operations on functions and on sequences that preserve the monotonicity of quotients. In particular, if $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}\}_{n=0}^{\infty}$ are sequences of real numbers, with $b_n > 0$, such that

$$\left\{\frac{a_n}{b_n}\right\}_{n=0}^{\infty} \tag{1}$$

is increasing (or decreasing) then so is the sequence of quotients of the moving averages of $k + 1$ terms,

$$\left\{\frac{(a_n + a_{n+1} + \cdots + a_{n+k})/k}{(b_n + b_{n+1} + \cdots + b_{n+k})/k}\right\}_{n=0}^{\infty} \tag{2}$$

The main question that we would like to consider presently is whether the same is true for more general moving averages

$$\sum_{j=0}^{k} \gamma_j a_{n+j} \tag{3}$$

where the constants γ_j satisfy $\gamma_j \geq 0$ for $0 \leq j \leq k$ and $\sum_{j=0}^{k} \gamma_j = 1$.

In Section 3 we give several examples that show that the moving average operator (3) may not preserve the monotonicity of quotients. Then in Section 4 we present several cases when the moving averages do preserve monotonicity of sequences. Finally, in Section 5 we give some results in the case of quotients of functions of a real variable.

2. Preliminaries

In this article a sequence $\{x_n\}_{n=0}^{\infty}$ is called *increasing* if $x_n \leq x_{n+1}$ for all n. When $x_n < x_{n+1}$ for all n then we say that the sequence is *strictly increasing*. Similarly for decreasing and strictly decreasing sequences.

Let $(\gamma_0, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$ with $\gamma_j \geq 0$ for all j and with $\Gamma = \sum_{j=0}^{m} \gamma_j > 0$. Then the operation on sequences

$$\{a_n\}_{n=0}^{\infty} \rightsquigarrow \left\{\frac{1}{\Gamma}\sum_{j=0}^{m} \gamma_j a_{n+j}\right\}_{n=0}^{\infty} \tag{4}$$

is the *moving average with weights* $(\gamma_0, \ldots, \gamma_m)$. We will use the notation $<\gamma_0, \ldots, \gamma_m>$ to denote this operation. Naturally $<t\gamma_0, \ldots, t\gamma_m> = <\gamma_0, \ldots, \gamma_m>$ for any $t > 0$, so the representation of the moving average by $(\gamma_0, \ldots, \gamma_m)$ is not unique; the representation becomes unique if we require the normalization $\Gamma = 1$, so that the set of moving averages with $m + 1$ weights can be considered the simplex shaped region

$$M_m = \left\{(\gamma_0, \ldots, \gamma_m) \in \mathbb{R}^{m+1} : \gamma_j \geq 0, \ \sum_{j=0}^{m} \gamma_j = 1\right\} \tag{5}$$

It would be convenient to employ representations of moving averages where Γ is not necessarily equal to 1, such representation being a kind of homogenous coordinates of the average.

We denote as $\mathfrak{M}_m$ the set of moving averages $<\gamma_0, \ldots, \gamma_m>$ and as $\mathfrak{M} = \cup_{m=1}^{\infty} \mathfrak{M}_m$. The set $\mathfrak{M}_m$ has a natural topology, that of the simplex shaped subset M_m of $\mathbb{R}^{m+1}$.

Suppose now that we have two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, with $b_n > 0$, and consider the sequence of quotients $\{a_n/b_n\}_{n=0}^{\infty}$. Then we shall use the notation

$$\langle \gamma_0, \ldots, \gamma_m \rangle_k \left(\left\{\frac{a_n}{b_n}\right\}_{n=0}^{\infty}\right) = \frac{\sum_{j=0}^{m} \gamma_j a_{k+j}}{\sum_{j=0}^{m} \gamma_j b_{k+j}} \tag{6}$$

for the sequence obtained by applying the moving average $<\gamma_0, \ldots, \gamma_m>$ to the numerator and the denominators of the fraction a_n/b_n. When the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are fixed we will employ the simpler notation $<\gamma_0, \ldots, \gamma_m>_k$.

Let us denote as $\mathfrak{P}_m$ the subset of $\mathfrak{M}_m$ consisting of those moving averages that preserve the monotonicity of quotients of sequences. This means that $<\gamma_0, ..., \gamma_m> \in \mathfrak{P}_m$ if and only if for any increasing sequence of quotients $\{a_n/b_n\}_{n=0}^{\infty}$, $b_n > 0$, the sequence $\{<\gamma_0, ..., \gamma_m>_k (\{a_n/b_n\}_{n=0}^{\infty})\}_{k=0}^{\infty}$ is likewise increasing and for each decreasing sequence of quotients $\{c_n/d_n\}_{n=0}^{\infty}$, $d_n > 0$, the sequence $\{<\gamma_0, ..., \gamma_m>_k (\{c_n/d_n\}_{n=0}^{\infty})\}_{k=0}^{\infty}$ is also decreasing.

Let us notice that if the moving average $<\gamma_0, ..., \gamma_m>$ preserves increasing (or decreasing) sequences then it belongs to $\mathfrak{P}_m$.

Lemma 2.1 Suppose the moving average $<\gamma_0, ..., \gamma_m>$ has the property that if $b_n > 0$, and $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) then $\{<\gamma_0, ..., \gamma_m>_k (\{a_n/b_n\})\}_{k=0}^{\infty}$ is likewise increasing (decreasing). Then $<\gamma_0, ..., \gamma_m> \in \mathfrak{P}_m$.

Proof. Suppose $<\gamma_0, ..., \gamma_m>$ preserves increasing quotients. Then if $d_n > 0$ and $\{c_n/d_n\}_{n=0}^{\infty}$ is decreasing by writing

$$\left\{\frac{\sum_{j=0}^{m}\gamma_j c_{k+j}}{\sum_{j=0}^{m}\gamma_j d_{k+j}}\right\}_{k=0}^{\infty} = \left\{-\frac{\sum_{j=0}^{m}\gamma_j\left(-c_{k+j}\right)}{\sum_{j=0}^{m}\gamma_j d_{k+j}}\right\}_{k=0}^{\infty} \tag{7}$$

It follows that $\{<\gamma_0, ..., \gamma_m>_k (\{c_n/d_n\})$ will also be decreasing. □

3. Averages that do not preserve the monotonicity of quotients

Our counterexamples are based on the ensuing construction.

Lemma 3.1 Suppose

$$Q_0 < Q_1 < Q_2 < Q_3 \tag{8}$$

There are real numbers $\{a_n\}_{n=0}^{3}$ and $\{b_n\}_{n=0}^{3}$ with $b_n > 0$ such that

$$\frac{a_n}{b_n} = Q_n, \quad 0 \le n \le 3 \tag{9}$$

but with

$$\frac{a_0 + a_2}{b_0 + b_2} > \frac{a_1 + a_3}{b_1 + b_3} \tag{10}$$

Proof. Let us first choose any real numbers $\{A_n\}_{n=0}^{3}$ and $\{B_n\}_{n=0}^{3}$ with $B_n > 0$ such that $A_n/B_n = Q_n$ for $0 \le n \le 3$. Since $A_0/B_0 < A_2/B_2$ it follows [7] that for any $x > 0$ we have

$$\frac{A_0}{B_0} < \frac{A_0 + xA_2}{B_0 + xB_2} < \frac{A_2}{B_2} \tag{11}$$

Also $A_0/B_0 < A_1/B_1 < A_2/B_2$ and since $\lim_{x\to\infty} (A_0 + xA_2)/(B_0 + xB_2) = A_2/B_2$ we can choose x such that

$$\frac{A_1}{B_1} < \frac{A_0 + xA_2}{B_0 + xB_2} \tag{12}$$

Similarly, if $y > 0$ then

$$\frac{A_1}{B_1} < \frac{A_1 + yA_3}{B_1 + yB_3} < \frac{A_3}{B_3} \tag{13}$$

and $\lim_{y\to 0} (A_1 + yA_3)/(B_1 + yB_3) = A_1/B_1$, therefore we can choose y small enough so that

$$\frac{A_1 + yA_3}{B_1 + yB_3} < \frac{A_0 + xA_2}{B_0 + xB_2} \tag{14}$$

Then we just need to take $a_0 = A_0$, $a_1 = A_1$, $a_2 = xA_2$, $a_3 = yA_3$ and $b_0 = B_0$, $b_1 = B_1$, $b_2 = xB_2$, $b_3 = yB_3$. □

In fact, if $\gamma_0 > 0$ and $\gamma_2 > 0$, it also follows immediately from the lemma that if the Q_n satisfy (8) we can find a_n and b_n with (9) but such that

$$\frac{\gamma_0 a_0 + \gamma_2 a_2}{\gamma_0 b_0 + \gamma_2 b_2} > \frac{\gamma_0 a_1 + \gamma_2 a_3}{\gamma_0 b_1 + \gamma_2 b_3} \tag{15}$$

Therefore the moving average $\langle\gamma_0, 0, \gamma_2\rangle$ does *not* preserve the monotonicity of quotients. In fact, more is true.

Proposition 3.2 If $\gamma_0 > 0$ and $\gamma_2 > 0$ then there exists $r = r(\gamma_0, \gamma_2) > 0$ such that if $0 \leq \varepsilon < r$ then the moving average

$$\langle\gamma_0, \varepsilon, \gamma_2\rangle\left(\{a_n\}_{n=0}^{\infty}\right) = \left\{\frac{\gamma_0 a_n + \varepsilon a_{n+1} + \gamma_2 a_{n+2}}{\gamma_0 + \varepsilon + \gamma_2}\right\}_{n=0}^{\infty} \tag{16}$$

does not preserve the monotonicity of quotients, that is, there are sequences of real numbers $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$ such that $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing but

$$\left\{\frac{\gamma_0 a_n + \varepsilon a_{n+1} + \gamma_2 a_{n+2}}{\gamma_0 b_n + \varepsilon b_{n+1} + \gamma_2 b_{n+2}}\right\}_{n=0}^{\infty} \tag{17}$$

is not.

Proof. It follows at once from the Proposition 3.3 below. □

Proposition 3.3 The set $\mathfrak{P}_m$ is closed in $\mathfrak{M}_m$.

Proof. We will show that $\mathfrak{M}_m \backslash \mathfrak{P}_m$ is open in $\mathfrak{M}_m$. If $\langle\gamma_0, ..., \gamma_m\rangle \in \mathfrak{M}_m \backslash \mathfrak{P}_m$ then there are sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing but for some $k \in \mathbb{N}$,

$$\frac{\sum_{j=0}^{m} \gamma_j a_{k+j}}{\sum_{j=0}^{m} \gamma_j b_{k+j}} > \frac{\sum_{j=0}^{m} \gamma_j a_{k+j+1}}{\sum_{j=0}^{m} \gamma_j b_{k+j+1}} \tag{18}$$

Since this is a *strict* inequality, it follows that there exists $r > 0$ such that if $\langle\eta_0, ..., \eta_m\rangle \in \mathfrak{M}_m$ satisfies $\|(\eta_0, ..., \eta_m) - (\gamma_0, ..., \gamma_m)\|_{\mathbb{R}^{m+1}} < r$ then (18) is satisfied if we replace the γ_j by the η_j. Hence $\langle\eta_0, ..., \eta_m\rangle \in \mathfrak{M}_m \backslash \mathfrak{P}_m$. □

Actually, the same argument that leads to the Proposition 3.2 gives the following result.

Proposition 3.4 If $\gamma_0 > 0$ and $\gamma_m > 0$ then there exists $r = r(\gamma_0, \gamma_m)$ such that if $\|(\varepsilon_1, ..., \varepsilon_{m-1})\|_{\mathbb{R}^{m-1}} < r$, $\varepsilon_j \geq 0$, then the moving average $\langle\gamma_0, \varepsilon_1, ..., \varepsilon_{m-1}, \gamma_m\rangle$ does not preserve the monotonicity of quotients, that is, there are sequences of real numbers $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ with $B_n > 0$ such that $\{A_n/B_n\}_{n=0}^{\infty}$ is increasing but

$$\left\{\frac{\gamma_0 A_n + \varepsilon_1 A_{n+1} + \cdots + \varepsilon_{m-1} A_{n+m-1} + \gamma_m A_{n+m}}{\gamma_0 B_n + \varepsilon_1 B_{n+1} + \cdots + \varepsilon_{m-1} B_{n+m-1} + \gamma_m B_{n+m}}\right\}_{n=0}^{\infty} \tag{19}$$

is not.

Proof. Employing the Proposition 3.3, it is enough to see that $<\gamma_0, 0, ..., 0, \gamma_m> \in \mathfrak{M}_m \backslash \mathfrak{P}_m$. However, we can find sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$, such that $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing and

$$\frac{\gamma_0 a_0 + \gamma_m a_2}{\gamma_0 b_0 + \gamma_m b_2} > \frac{\gamma_0 a_1 + \gamma_m a_3}{\gamma_0 b_1 + \gamma_m b_3} \tag{20}$$

Then if we take new sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ by putting

$$A_0 = a_0,\ A_j = a_1, 1 \le j \le m-1,\ A_j = a_{j-m+2},\ j \ge m \tag{21}$$

$$B_0 = b_0,\ B_j = b_1, 1 \le j \le m-1,\ B_j = b_{j-m+2},\ j \ge m \tag{22}$$

we obtain that $\{A_n/B_n\}_{n=0}^{\infty}$ is increasing but

$$\frac{\gamma_0 A_0 + \gamma_m A_m}{\gamma_0 B_0 + \gamma_m B_m} > \frac{\gamma_0 A_1 + \gamma_m A_{m+1}}{\gamma_0 B_1 + \gamma_m B_{m+1}} \tag{23}$$

so that $\{<\gamma_0, 0, ..., 0, \gamma_m>_k (\{A_n/B_n\}_{n=0}^{\infty})\}_{k=0}^{\infty}$ is not increasing. □

3.1 *Addition of monotone quotients*

Another consequence of the construction given by the Lemma 3.1 is that addition does *not* preserve the monotonicity of quotients.

Proposition 3.5 Let $\rho_1 > 0$ and $\rho_2 > 0$. There are sequences of real numbers $\{c'_n\}_{n=0}^{\infty}$, $\{d'_n\}_{n=0}^{\infty}$, $\{c''_n\}_{n=0}^{\infty}$, and $\{d''_n\}_{n=0}^{\infty}$ with $d'_n > 0$ and $d''_n > 0$ such that $\{c'_n/d'_n\}_{n=0}^{\infty}$ and $\{c''_n/d''_n\}_{n=0}^{\infty}$ are increasing, but

$$\left\{\frac{\rho_1 c'_n + \rho_2 c''_n}{\rho_1 d'_n + \rho_2 d''_n}\right\}_{n=0}^{\infty} \tag{24}$$

is not.

Proof. Indeed, let $Q_0 < Q_1 < Q_2 < Q_3$ and construct a_j and b_j , for $0 \le j \le 3$ that satisfy (9) and

$$\frac{\rho_0 a_0 + \rho_2 a_2}{\rho_0 b_0 + \rho_2 b_2} > \frac{\rho_0 a_1 + \rho_2 a_3}{\rho_0 b_1 + \rho_2 b_3}$$

We just then take increasing sequences of quotients $\{c'_n/d'_n\}_{n=0}^{\infty}$ and $\{c''_n/d''_n\}_{n=0}^{\infty}$ that satisfy $c'_0 = a_0$, $c'_1 = a_1$, $c''_0 = a_2$, $c''_1 = a_3$, while $d'_0 = b_0$, $d'_1 = b_1$, $d''_0 = b_2$, $d''_1 = b_3$. □

In particular, it is possible for a sequence of the form $\{(c'_n + c''_n)/(d'_n + d''_n)\}_{n=0}^{\infty}$ to be not increasing even though $\{c'_n/d'_n\}_{n=0}^{\infty}$ and $\{c''_n/d''_n\}_{n=0}^{\infty}$ are both increasing.

4. Conditions that ensure the preservation of monotonicity

We shall now consider several properties of the set $\mathfrak{P}_m$ of moving averages $<\gamma_0, \ldots, \gamma_m>$ that preserve the monotonicity of quotients. We already pointed out that $\mathfrak{P}_1 = \mathfrak{M}_1$ but $\mathfrak{P}_m \neq \mathfrak{M}_m$ for $m \geq 2$. Our first result applies to $\mathfrak{P}_2$.

Proposition 4.1 Suppose that $\alpha \leq 1$ and $\beta \leq 1$. Then $<\alpha, 1, \beta> \in \mathfrak{P}_2$.

Proof. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$ and with $\{a_n/b_n\}_{n=0}^{\infty}$ increasing be two fixed sequences. If $<\gamma_0, \gamma_1, \ldots, \gamma_m> \in \mathfrak{M}_m$ we will denote by $<\gamma_0, \ldots, \gamma_m>_k$ the quotient $\sum_{j=0}^{m} \gamma_j a_{k+j} / \sum_{j=0}^{m} \gamma_j b_{k+j}$.

Let us observe that the function

$$\langle 1, \tau \rangle_k = \frac{a_k + \tau a_{k+1}}{b_k + \tau b_{k+1}} \tag{25}$$

is an increasing function of τ for all k. Since $\beta \leq 1$ we have

$$\langle 1, \beta \rangle_k \leq \langle 1, 1 \rangle_k \tag{26}$$

while because $\alpha \leq 1$ we have

$$\langle 1, 1 \rangle_k \leq \langle 1, 1/\alpha \rangle_k = \langle \alpha, 1 \rangle_k \tag{27}$$

Also, for any γ_j, $0 \leq j \leq 2$, we have

$$\frac{a_k}{b_k} \leq \frac{\gamma_1 a_{k+1} + \gamma_2 a_{k+2}}{\gamma_1 b_{k+1} + \gamma_2 b_{k+2}} \tag{28}$$

so

$$\frac{\gamma_0 a_k + \gamma_1 a_{k+1} + \gamma_2 a_{k+2}}{\gamma_0 b_k + \gamma_1 b_{k+1} + \gamma_2 b_{k+2}} \leq \frac{\gamma_1 a_{k+1} + \gamma_2 a_{k+2}}{\gamma_1 b_{k+1} + \gamma_2 b_{k+2}} \tag{29}$$

In other words, for any γ_0,

$$\langle \gamma_0, \gamma_1, \gamma_2 \rangle_k \leq \langle \gamma_1, \gamma_2 \rangle_{k+1} \tag{30}$$

Similarly, we have that for any γ_2,

$$\langle \gamma_0, \gamma_1 \rangle_k \leq \langle \gamma_0, \gamma_1, \gamma_2 \rangle_k \tag{31}$$

If we combine these inequalities we therefore obtain

$$\langle \alpha, 1, \beta \rangle_k \leq \langle 1, \beta \rangle_{k+1} \leq \langle 1, 1 \rangle_{k+1}$$

$$\leq \langle \alpha, 1 \rangle_{k+1} \leq \langle \alpha, 1, \beta \rangle_{k+1}$$

that is, $\{<\alpha, 1, \beta>_k\}_{n=0}^{\infty}$ is increasing, which means that $<\alpha, 1, \beta> \in \mathfrak{P}_2$. □

Another type of moving averages that belong to $\mathfrak{P}_m$ are obtained as follows.

Proposition 4.2 Suppose $\langle\gamma_0, \gamma_1, ..., \gamma_m\rangle \in \mathfrak{P}_m$. Then for all $\tau > 0$ we also have

$$\left\langle \gamma_0, \tau\gamma_1 \ldots, \tau^m \gamma_m \right\rangle \in \mathfrak{P}_m \tag{32}$$

Proof. Since $\langle\gamma_0, \gamma_1, ..., \gamma_m\rangle$ preserves the monotonicity of quotients, for each pair of real sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$ and with $\{a_n/b_n\}_{n=0}^{\infty}$ increasing we have that

$$\frac{\sum_{j=0}^{m} \gamma_j a_{n+j}}{\sum_{j=0}^{m} \gamma_j b_{n+j}} \le \frac{\sum_{j=0}^{m} \gamma_j a_{n+j+1}}{\sum_{j=0}^{m} \gamma_j b_{n+j+1}} \tag{33}$$

for all $n \ge 0$. If $\tau > 0$, we can apply this inequality to the two sequences $\{a_n\tau^n\}_{n=0}^{\infty}$ and $\{b_n\tau^n\}_{n=0}^{\infty}$ since $\{a_n\tau^n/b_n\tau^n\}_{n=0}^{\infty} = \{a_n/b_n\}_{n=0}^{\infty}$ is increasing. Thus,

$$\frac{\sum_{j=0}^{m} \gamma_j a_{n+j} \tau^{n+j}}{\sum_{j=0}^{m} \gamma_j b_{n+j} \tau^{n+j}} \le \frac{\sum_{j=0}^{m} \gamma_j a_{n+j+1} \tau^{n+j+1}}{\sum_{j=0}^{m} \gamma_j b_{n+j+1} \tau^{n+j+1}} \tag{34}$$

and consequently,

$$\frac{\sum_{j=0}^{m} \gamma_j \tau^j a_{n+j}}{\sum_{j=0}^{m} \gamma_j \tau^j b_{n+j}} \le \frac{\sum_{j=0}^{m} \gamma_j \tau^j a_{n+j+1}}{\sum_{j=0}^{m} \gamma_j \tau^j b_{n+j+1}} \tag{35}$$

and it follows that $\langle\gamma_0, \tau\gamma_1 ..., \tau^m\gamma_m\rangle$ preserves the monotonicity of quotients. □

We would like to point out other conditions that guarantee that moving averages preserves the monotonicity of quotients.

Proposition 4.3 Let $\langle\gamma_0, ..., \gamma_m\rangle \in \mathfrak{M}_m$ with $\gamma_0 > 0$ and $\gamma_m > 0$. If all the zeros of the polynomial

$$p(z) = \sum_{j=0}^{m} \gamma_j z^j \tag{36}$$

are real then $\langle\gamma_0, ..., \gamma_m\rangle \in \mathfrak{P}_m$.

Proof. Since $(\gamma_0, ..., \gamma_m)$ are homogeneous coordinates, we may assume that $\gamma_m = 1$. Let T be the basic moving operator $\langle 0, 1\rangle$, $T(\{a_n\}_{n=0}^{\infty}) = \{a_{n+1}\}_{n=0}^{\infty}$. Notice that the moving average can then be written as $\langle\gamma_0, ..., \gamma_m\rangle = p(T)$. If all the zeros of p are real, they are all negative, so we can then write $p(z) = (z + \eta_1) \cdots (z + \eta_m)$, where the η_j are positive. Therefore

$$\left\langle \gamma_0, \ldots, \gamma_m \right\rangle = \left\langle \eta_1, 1 \right\rangle \circ \cdots \circ \left\langle \eta_m, 1 \right\rangle \tag{37}$$

so that $\langle\gamma_0, ..., \gamma_m\rangle$ is the composition of m moving average operators that belong to $\mathfrak{P}_1$, and since clearly the composition of operators of $\mathfrak{P}_q$ and $\mathfrak{P}_k$ belongs to $\mathfrak{P}_{k+q}$, it follows that $\langle\gamma_0, ..., \gamma_m\rangle \in \mathfrak{P}_m$. □

The result of the Proposition 4.3 does not characterize all the moving averages that preserve monotonicity. In fact, the moving average $\langle 1, 1, 1\rangle$ belongs to $\mathfrak{P}_2$ but the zeros of the polynomial $p(z) = 1 + z + z^2$ are not real. Observe that if $\langle\gamma_0, \gamma_1, \gamma_2\rangle \in \mathfrak{M}_2$ the Proposition 4.3 says that if $4\gamma_0\gamma_2 < \gamma_1^2$ then $\langle\gamma_0, \gamma_1, \gamma_2\rangle \in \mathfrak{P}_2$; a stronger result is obtained if we combine the Propositions 4.1 and 4.2.

Proposition 4.4 If

$$\gamma_0\gamma_2 < \gamma_1^2 \tag{38}$$

then $<\gamma_0, \gamma_1, \gamma_2> \in \mathfrak{P}_2$.

Proof. Indeed, the Proposition 4.1 gives that $<1, 1, \gamma_0\gamma_2/\gamma_1^2> \in \mathfrak{P}_2$, and thus Proposition 4.2 yields that $<1, \tau, \tau^2\gamma_0\gamma_2/\gamma_1^2> \in \mathfrak{P}_2$ for any $\tau > 0$. If we take $\tau = \gamma_1/\gamma_0$ we obtain that

$$\left\langle 1, \frac{\gamma_1}{\gamma_0}, \left(\frac{\gamma_1}{\gamma_0}\right)^2 \frac{\gamma_0\gamma_2}{\gamma_1^2} \right\rangle = \left\langle 1, \frac{\gamma_1}{\gamma_0}, \frac{\gamma_2}{\gamma_0} \right\rangle = \langle \gamma_0, \gamma_1, \gamma_2 \rangle \in \mathfrak{P}_2 \tag{39}$$

□

5. The continuous case

In this section we give two counterexamples involving moving averages of smooth functions.

First, let us recall [7] that if u and v are two locally Lebesgue integrable functions over $[a, b)$, with v strictly positive at all points of this interval, and u/v increasing, then

$$w(x) = \frac{\int_0^c u(x+t)\mathrm{d}t}{\int_0^c v(x+t)\mathrm{d}t} \tag{40}$$

is increasing in $(a, b - c)$. We will construct an example of such functions u and v that, additionally, are smooth in $\mathbb{R}$ as well as a positive smooth function φ with compact support such that

$$w_\varphi(x) = \frac{\int_{-\infty}^{\infty} u(x+t)\varphi(t)\mathrm{d}t}{\int_{-\infty}^{\infty} v(x+t)\varphi(t)\mathrm{d}t} \tag{41}$$

is not increasing. In other words, if χ_E denotes the characteristic function of a set E, the moving average $<\chi_{(0,c)}>$ given by

$$\left\langle \chi_{(0,c)} \right\rangle \{u\}(x) = \frac{1}{c}\int_0^c u(x+t)\mathrm{d}t \tag{42}$$

preserves the monotonicity of quotients, but in general if φ is a positive continuous function with compact support the moving average

$$\langle \varphi \rangle \{u\}(x) = \frac{\int_{-\infty}^{\infty} u(x+t)\varphi(t)\mathrm{d}t}{\int_{-\infty}^{\infty} \varphi(t)\mathrm{d}t} \tag{43}$$

does not, not even if φ is smooth.

We will need the following extension result.

Lemma 5.1 Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$ and with $\{a_n/b_n\}_{n=0}^{\infty}$ increasing. Let $I_n = [x_n, y_n]$ be a sequence of disjoint intervals with $\{x_n\}_{n=0}^{\infty}$ increasing to infinity. There are C^∞ functions u and v defined in $\mathbb{R}$ with v strictly positive such that u/v is increasing and such that

$$u(x) = a_n, \, v(x) = b_n, \, x \in I_n \tag{44}$$

Proof. It is just enough to see that if $\{c_n\}_{n=0}^{\infty}$ is an increasing sequence, then there exists a smooth *increasing* function $w \in C^{\infty}(\mathbb{R})$ with $w(x) = c_n$ for $x \in I_n$, and then use this to define $w = u/v$. □

Let us take a positive test function φ, with supp $\varphi = [0, 1/2] \cup [2, 5/2]$ such that

$$\int_0^{1/2} \varphi(t)\mathrm{d}t = \int_2^{5/2} \varphi(t)\mathrm{d}t = 1 \tag{45}$$

Then choose, following the construction of Section 3, sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ with $b_n > 0$, with $\{a_n/b_n\}_{n=0}^{\infty}$ increasing, and with

$$\frac{a_0 + a_2}{b_0 + b_2} > \frac{a_1 + a_3}{b_1 + b_3} \tag{46}$$

Finally, using the lemma, find u, $v \in C^{\infty}(\mathbb{R})$, with v strictly positive, such that u/v is increasing and such that for $n \in \mathbb{N}$

$$u(x) = a_n, \, v(x) = b_n, \, n \le x \le n + 1/2 \tag{47}$$

Then

$$\int_{-\infty}^{\infty} u(n+t)\varphi(t)\mathrm{d}t = a_n + a_{n+2}, \; \int_{-\infty}^{\infty} v(n+t)\varphi(t)\mathrm{d}t = b_n + b_{n+2} \tag{48}$$

Therefore the quotient $w_{\varphi}(x) = \int_{-\infty}^{\infty} u(x+t)\varphi(t)\mathrm{d}t \Big/ \int_{-\infty}^{\infty} v(x+t)\varphi(t)\mathrm{d}t$ will not be increasing, since

$$w_{\varphi}(0) > w_{\varphi}(1) \tag{49}$$

If $<\varphi(x)>$ does not preserve the monotonicity of quotients, then for $\lambda > 0$ the moving average $<\varphi(\lambda x)>$ does not either. Therefore we may summarize our first construction as follows.

Proposition 5.2 Let $c > 0$. Then there exists a smooth function ϕ defined in $\mathbb{R}$ and whose support is contained in $[0, c]$ such that the moving average $<\phi>$ does not preserve the monotonicity of quotients.

Our second example concerns the addition of monotonic quotients.

Proposition 5.3 There are C^{∞} functions u_1, u_2, v_1, and v_2 defined in $\mathbb{R}$ with v_1 and v_2 strictly positive such that u_1/v_1 and u_2/v_2 are increasing but such that

$$\frac{u_1 + u_2}{v_1 + v_2} \tag{50}$$

is not.

Proof. According to Proposition 3.5 we can find sequences $\{c_n'\}_{n=0}^{\infty}$, $\{d_n'\}_{n=0}^{\infty}$, $\{c_n''\}_{n=0}^{\infty}$, and $\{d_n''\}_{n=0}^{\infty}$ with $d_n' > 0$ and $d_n'' > 0$ such that $\{c_n'/d_n'\}_{n=0}^{\infty}$ and $\{c_n''/d_n''\}_{n=0}^{\infty}$ are increasing, but

$$\left\{ \frac{c_n' + c_n''}{d_n' + d_n''} \right\}_{n=0}^{\infty} \tag{51}$$

is not. On the other hand, Lemma 5.1 allows us to find C^∞ functions u_1, u_2, v_1, and v_2 with v_1 and v_2 strictly positive such that u_1/v_1 and u_2/v_2 are increasing and such that

$$u_1(n) = c'_n,\ u_2(n) = c''_n,\ v_1(n) = d'_n,\ v_2(n) = d''_n \tag{52}$$

For these smooth functions the quotient (51) is not increasing. □

Our last result is a continuous analog of the Proposition 4.2.

Proposition 5.4 Let φ be a positive integrable function with compact support such that the moving average $<\varphi>$ preserves the monotonicity of quotients. For $\tau > 0$, let $\varphi_\tau(x) = \varphi(x)\tau^x$. Then the moving average $<\varphi_\tau>$ also preserves the monotonicity of quotients.

Proof. Let u and v two locally Lebesgue integrable functions defined in $\mathbb{R}$ with v strictly positive at all points and u/v increasing. Then $u(x)\tau^x/v(x)\tau^x$ is also increasing, so

$$\frac{\int_{-\infty}^{\infty} u(x+t)\left(\varphi(t)\tau^t\right)\mathrm{d}t}{\int_{-\infty}^{\infty} v(x+t)\left(\varphi(t)\tau^t\right)\mathrm{d}t} = \frac{\int_{-\infty}^{\infty}\left(u(x+t)\tau^{x+t}\right)\varphi(t)\mathrm{d}t}{\int_{-\infty}^{\infty}\left(v(x+t)\tau^{x+t}\right)\varphi(t)\mathrm{d}t} \tag{53}$$

is also increasing. □

This last result also holds if we replace the function φ by a positive measure.

Conflicts of interest

The author declares no competing financial interest.

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