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Research Article

On the Net Distance Matrix of a Signed Block Graph

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Abstract: A connected signed graph $\dot{G}$, where all blocks of it are positive cliques or negative cliques (of possibly varying sizes), is called a signed block graph. Let A, N and $\tilde{D}$ be adjacency, net Laplacian and net distance matrices of a signed block graph, respectively. In this paper the formulas for the determinant of A and $\tilde{D}$ were given firstly. Then the inverse (resp. Moore-Penrose inverse) of $\tilde{D}$ is obtained if it is nonsingular (resp. singular), which is the sum of a Laplacian-like matrix and at most two matrices with rank 1.

***Keywords*:** signed block graph, net distance matrix, net Laplacian matrix, adjacency matrix, Moore-Penrose inverse

MSC: 05C50

1. Introduction

A signed graph $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple graph called the underlying graph, and $\sigma : E \rightarrow \{1, -1\}$ is the signature. An edge $e \in E$ of $\dot{G}$ is called positive (resp. negative) edge if $\sigma(e) = +1$ (resp. $\sigma(e) = -1$). The number of vertices of $\dot{G}$ is denoted by n. For a signed graph $\dot{G} = (G, \sigma)$, the degree δ_i of a vertex i of $\dot{G}$ is the number of its neighbours. The positive degree (resp. negative degree) δ_i^+ (resp. δ_i^-) is the number of positive (resp. negative) neighbours of i. The net degree of i is $\delta_i^{\pm} = \delta_i^+ - \delta_i^-$. We use K_n (resp. $-K_n$) to denote the positive (resp. negative) n-clique.

Recall that a signed hypergraph (Γ, ξ) is a hypergraph $\Gamma = (V, H)$ with a vertex-edge incidence function $\xi : V \times H \rightarrow \{-1, 0, 1\}$ and the sign of a hyper-edge h is $-\Pi_{v \in h}\xi(v, h)$. A signed hypergraph is called a signed hypertree, if it is both connected and acyclic. One can refer [1] for properties of a signed hypertree. For a signed hypertree (Γ, ξ), let $\dot{G}$ be the signed graph obtained from (Γ, ξ) by replacing each signed hyperedge with a signed clique, for example, replacing a negative hyperedge of five vertices with a negative clique of order five, then $\dot{G}$ is a signed block graph with each block a positive or negative clique, see Figure 1 for an example, where the signatures of vertices and the positive (resp. negative) hyperedges are depicted by + (resp. −), and the corresponding signed block graph with signed cliques $-K_3$, K_2, K_3 and $-K_4$, and the edges of the positive (resp. negative) clique are depicted by solid (resp. dash) lines. Let $\dot{G}$ be a signed block graph with signed cliques B_i, $1 \leq i \leq k$, here the order of B_i is b_i and the signature of B_i is η_i. In this paper, each signed clique of a signed block graph is assumed to be a positive clique or a negative clique, that is $\eta_i = 1$ if B_i is a positive clique and $\eta_i = -1$ if B_i is a negative clique.

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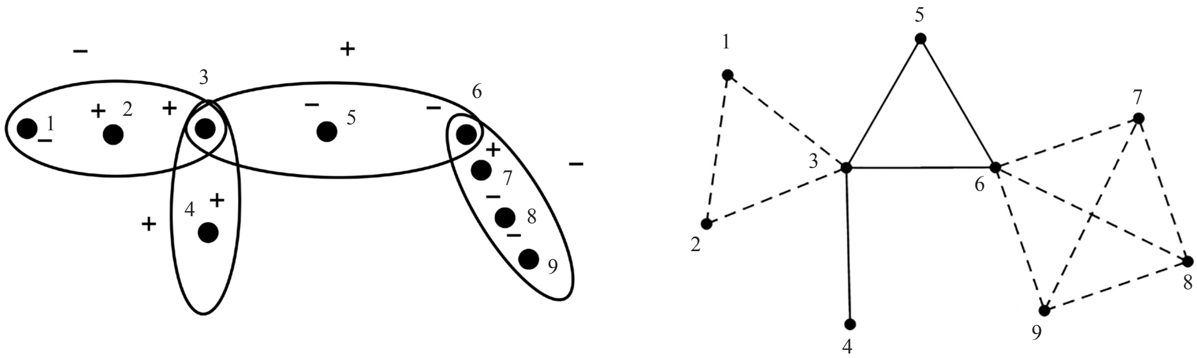

Figure 1. A signed hypertree and the corresponding signed block graph.

The adjacency matrix $A = (a_{ij})$ of a signed graph is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The net Laplacian matrix of a signed graph is defined as $N = \triangle^{\pm} - A$, where $\triangle^{\pm}$ is the diagonal matrix of vertex net degrees. For basic properties of the net Laplacian matrix, one can refer [2]. The Laplacian matrix of a unsigned graph is $L = \triangle - A$, where $\triangle$ denotes the diagonal matrix of vertex degrees. More about the Laplacian matrices of unsigned graphs see [3]. Given a signed block graph $\dot{G}$ with signed cliques B_i, $1 \le i \le k$, each B_i of $\dot{G}$ is considered as a graph on n vertices perhaps with isolated vertices and let its edge set be E_i. Let N_i and L_i be the net Laplacian matrix and the Laplacian matrix of $B_i = (V, E_i)$, respectively. Define the net Laplacian-like matrix and the Laplacian-like matrix of the signed block graph as $\hat{N} = \sum_{i=1}^{k} \frac{1}{b_i} N_i$ and $\hat{L} = \sum_{i=1}^{k} \frac{1}{b_i} L_i$, respectively.

Recall that the distance $d(i, j)$ between the vertices i and j of a graph G is the length of a shortest path from i to j. The distance matrix $D = (d_{ij})$ of G is an $n \times n$ matrix, where $d_{ij} = d(i, j)$ and $d_{ii} = 0$, $i, j = 1, 2, \ldots, n$. One can refer [4] for properties of distance matrix. For the distance matrix of a tree with n vertices, Graham and Pollak [5] showed that the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$, which depends only on the order of T, and Graham and Lovász [6] gave a formula $D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau^T$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^T$, $\tau_i = 2 - \delta_i$, $i = 1, 2, \ldots, n$. Let β be an $n \times 1$ vector with $\beta_i = \sum_{i=1}^{r} \frac{1}{b_i} - (r-1)$, where vertex i is in $r \ge 1$ cliques of size $b_1, b_2, \ldots, b_r$. Bapat and Sivasubramanian [7] showed that if G is a block graph on n vertices with distance matrix D, then the determinant of D is $\det(D) = (-1)^n \lambda \prod_{i=1}^{k} b_i$ and the inverse of D is $D^{-1} = -\hat{L} + \frac{1}{\lambda}\beta\beta^T$, where $\lambda = \sum_{i=1}^{k} \frac{b_i - 1}{b_i}$. The formula for D^{-1} establishes the relationship between the inverse of the distance matrix and the Laplacian-like matrix $\hat{L}$ of a block graph. Similar results for an odd-cycle-clique graph and a bi-block graph were given in [8-9]. More about the inverse of distance matrix and Laplacian matrix of a graph, see [10-12].

For a signed graph $\dot{G}$ with n vertices and $rank(N) = n - 1$. Following [13], let $Z = (N + \frac{1}{n}J)^{-1}$, $\bar{Z} = diag(z_{11}, \ldots, z_{nn})$, where z_{ii} is the entries of the diagonal of Z, $i = 1, \ldots, n$. Call the matrix $R = (r(i, j))$, where $r(i, j) = z_{ii} + z_{jj} - 2z_{ij}$, the resistance matrix of $\dot{G}$. In a signed block graph $\dot{G}$, for two vertices i and j, $\tilde{d}(i, j) = \sum_{e \in \mathcal{P}_{i,j}} \sigma(e)$ denotes the net distance between i and j, where $\mathcal{P}_{i,j}$ is any shortest path from i to j, specially, $\tilde{d}(i, i) = 0$, $i = 1, 2, \ldots, n$. The net distance matrix of a signed block graph $\dot{G}$ is defined as $\tilde{D} = (\tilde{d}_{ij})$, where $\tilde{d}_{ij} = \tilde{d}(i, j)$, $1 \le i, j \le n$. Recall that for an $n \times n$ matrix M, the Moore-Penrose inverse (denoted by M^+) of M is the unique $n \times n$ matrix satisfying the matrix equations (i) $MM^+M = M$, (ii) $M^+MM^+ = M^+$, (iii) $(MM^+)^T = MM^+$, (iv) $(M^+M)^T = M^+M$. More for Moore-Penrose inverse of a matrix see [14].

The aim of this paper is to generalize the above results to a signed block graph. The paper is organized as follows. In Section 2, we obtain the determinants of the adjacency matrix A and the net distance matrix $\tilde{D}$ of a signed block graph and it follows that $\tilde{D}$ is nonsingular if and only if $\zeta = \sum_{i=1}^{k} \eta_i \frac{b_i - 1}{b_i} \neq 0$. In Section 3, we showed that $\tilde{D}^{-1} = -\hat{N} + \frac{1}{\zeta}\beta\beta^T$ if $\tilde{D}$ is nonsingular. In Section 4, we obtain the explicit formulas for $\hat{N}^+$ and $\tilde{D}^+$, namely, $\hat{N}^+ = -\tilde{D} - \frac{\mathbf{1}^T \tilde{D} \mathbf{1}}{n^2} J + \frac{1}{n}(\tilde{D}J + J\tilde{D})$, $\tilde{D}^+ = -\hat{N} - \frac{\beta^T \hat{N} \beta}{(\beta^T \beta)^2}\beta\beta^T + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T + \beta\beta^T \hat{N})$.

2. The determinants of A and $\tilde{D}$

Let $\mathbf{1}$ be an appropriate size column vector whose entries are ones. Let I be the identity matrix and J be the all ones matrix. A block is called a pendant block if it has only one cut vertex or if it is the only block in that component. Given a signed block graph $\dot{G}$ with signed cliques $B_1, \ldots, B_k$, if $S \subseteq \{1, \ldots, k\}$, then $\dot{G}_S$ will denote the subgraph of $\dot{G}$ induced by signed cliques B_i, $i \in S$. An isolated vertex in a signed block graph is considered to be a block of the signed block graph. The Lemma 1 in [15] also holds for signed block graphs. We state it in the following lemma and omit the proof.

Lemma 2.1 Let $\dot{G}$ be a signed block graph of order n with signed cliques $B_1, \ldots, B_k$. Let b_i and η_i be the order and the signature of B_i, respectively. Let $(\alpha_1, \ldots, \alpha_k)$ be a k-tuples of nonnegative integers satisfying the following conditions: (i) $\sum_{i=1}^{k}\alpha_i = n$, (ii) for any nonempty $S \subseteq \{1, \ldots, k\}$, $\sum_{i \in S}\alpha_i \leq |V(\dot{G}_S)|$. If B_i is a pendant block, then α_i equals either b_i or $b_i - 1$.

Given a signed block graph $\dot{G}$, for a vertex i, we use $\dot{G}\backslash\{i\}$ to denote the signed graph by deleting vertex i from $\dot{G}$. The next result gives a formula for the determinant of the adjacency matrix of a signed block graph which extend the similar result for a unsigned block graph. One can refer [15] for detail proof.

Theorem 2.2 Let $\dot{G}$ be a signed block graph on n vertices with signed cliques $B_1, \ldots, B_k$. Let η_i be the signature of B_i. Let A be the adjacency matrix of $\dot{G}$. Then

$$\det(A) = (-1)^{n-k} \sum (-1)^t (\alpha_1 - 1) \cdots (\alpha_k - 1),$$

where the summation is over all k-tuples $(\alpha_1, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions: (i) $\sum_{i=1}^{k}\alpha_i = n$ (ii) for any nonempty $S \subseteq \{1, \ldots, k\}$, $\sum_{i \in S}\alpha_i \leq |V(\dot{G}_S)|$ and $t = \sum_{i \in X}\alpha_i$, $X \subseteq \{1, \ldots, k\}$ is the index set corresponding to negative cliques.

Next we consider the signed block graph in Figure 1 to illustrate the above result.

Example 2.3 For the signed block graph in Figure 1, we get the adjacency matrix is

$$A = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

Then we have

$$\det(A) = 2 = (-1)^{9-4}((-1)^{2+3}(2-1)(2-1)(2-1)(3-1) + (-1)^{3+3}(3-1)(1-1)(2-1)(3-1)$$

$$+(-1)^{3+4}(3-1)(1-1)(1-1)(4-1) + (-1)^{2+4}(2-1)(2-1)(1-1)(4-1))$$

$$= (-1)^{n-4}\sum(-1)^t(\alpha_1 - 1)(\alpha_2 - 1)(\alpha_3 - 1)(\alpha_4 - 1),$$

where $t = \sum_{i\in X}\alpha_i$, $X \subseteq \{1, 4\}$ is the index set corresponding to negative cliques and the summation is over all 4-tuples (α_1, α_2, α_3, α_4) of nonnegative integers satisfying the conditions (i) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 9$, (ii) for any nonempty $S \subseteq \{1, 2, 3, 4\}$, $\sum_{i\in S}\alpha_i \le |V(\dot{G}_S)|$. The determinant of A not only depends on the order and the signature of each signed cliques, but also on the structure of $\dot{G}$. If we change the structure of the signed block graph in Figure 1, for example, exchange K_3 and $-K_4$, then the determinant of A will be 0.

If Q is an $n \times n$ matrix, then we use $cof(Q)$ to denote the sum of its all algebraic cofactors. The following Theorem 2.4 is similar to the distance matrix of a graph, we omit the proof here. One can refer [16] for detail proof.

Theorem 2.4 Let $\dot{G}$ be a signed block graph with signed cliques $B_1, \ldots, B_k$. Let $\tilde{D}$ be the net distance matrix of $\dot{G}$. Then (i) $cof(\tilde{D}) = \prod_{i=1}^{k} cof(\tilde{D}_i)$, (ii) $\det(\tilde{D}) = \sum_{i=1}^{k}\det(\tilde{D}_i)\prod_{j\neq i} cof(\tilde{D}_j)$, where $\tilde{D}_i$ denotes the net distance matrix of signed clique B_i, $1 \le i \le k$.

Let $\dot{G}$ be a signed block graph with n vertices with signed cliques B_i, $1 \le i \le k$, where each B_i is a b_i-clique with signature η_i. Denoted by ζ be the constant $\zeta = \zeta_{\dot{G}} = \sum_{i=1}^{k}\frac{b_i - 1}{b_i}\eta_i$. We give the formula for the determinant of the net distance matrix of a signed block graph in the following theorem.

Theorem 2.5 Let $\dot{G}$ be a signed block graph on n vertices with signed cliques B_i, $1 \le i \le k$. Let B_i be a b_i-clique with signature η_i. Let $\tilde{D}$ be the net distance matrix of $\dot{G}$. Then

$$\det(\tilde{D}) = (-1)^{n-1}\zeta\prod_{i=1}^{k}\eta_i^{b_i-1}b_i.$$

Proof. As the net distance matrix $\tilde{D}_i$ of B_i is $\eta_i(J - I)$, we have $\det(\tilde{D}_i) = (-1)^{b_i-1}(b_i - 1)\eta_i^{b_i}$ and $cof(\tilde{D}_i) = (-1)^{b_i-1} b_i\eta_i^{b_i-1}$. Since $\sum_{i=1}^{k} b_i = n - k + 1$, then according to Theorem 2.4, it follows that

$$\det(\tilde{D}) = \sum_{i=1}^{k}\det(\tilde{D}_i)\prod_{j\neq i} cof(\tilde{D}_j) = \sum_{i=1}^{k}(-1)^{b_i-1}\eta_i^{b_i}(b_i - 1)\prod_{j\neq i}(-1)^{b_j-1}\eta_j^{b_j-1}b_j$$

$$= (-1)^{n-1}\prod_{i=1}^{k}b_i\cdot\sum_{i=1}^{k}\frac{b_i-1}{b_i}\eta_i^{b_i}\prod_{j\neq i}\eta_j^{b_j-1} = (-1)^{n-1}\prod_{i=1}^{k}b_i\cdot\sum_{i=1}^{k}\frac{b_i-1}{b_i}\eta_i\prod_{i=1}^{k}\eta_i^{b_i-1}$$

$$= (-1)^{n-1}\zeta\prod_{i=1}^{k}\eta_i^{b_i-1}b_i.$$

□

Remark 2.6 For a signed block graph $\dot{G}$, the determinant of $\tilde{D}$ only depends on the order and signature of each signed clique B_i, $1 \le i \le k$, not on the structure of $\dot{G}$, which is different from the determinant of A. According to Theorem 2.5, the net distance matrix $\tilde{D}$ of $\dot{G}$ is nonsingular if and only if $\zeta_{\dot{G}} \ne 0$.

If $\zeta_{\dot{G}} = 0$, we have $rank(\tilde{D}) = n - 1$. In fact, for any vertex i which is not a cut vertex, we always have $\zeta_{\dot{G}\setminus\{i\}} \ne 0$. In other words, the net distance matrix of the signed block graph $\dot{G}\setminus\{i\}$ is nonsingular. Hence we have the following corollary.

Corollary 2.7 Let $\tilde{D}$ be the net distance matrix of a signed block graph $\dot{G}$. Then

$$rank(\tilde{D}) = \begin{cases} n, & \text{if } \zeta \ne 0, \\ n-1, & \text{if } \zeta = 0. \end{cases}$$

Example 2.8 For the signed block graph in Figure 1, whose signed cliques are $-K_3$, K_2, K_3 and $-K_4$. We have

$$\tilde{D} = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 \end{pmatrix},$$

and $\zeta = -\frac{2}{3} + \frac{1}{2} + \frac{2}{3} - \frac{3}{4} = -\frac{1}{4}$, hence, $\det(\tilde{D}) = 18 = (-1)^8 \cdot (-\frac{1}{4}) \cdot 3 \cdot 2 \cdot 3 \cdot (-4) = (-1)^{n-1} \zeta \cdot \prod_{i=1}^{k} \eta_i^{b_i - 1} \cdot b_i$.

3. The formula for $\tilde{D}^{-1}$

For a signed block graph $\dot{G}$, let β be an $n \times 1$ vector defined as follows. Let a vertex $\upsilon \in V$ be in $r \ge 1$ cliques of size $b_1, \ldots, b_r$. We define

$$\beta_\upsilon = \sum_{i=1}^{r} \frac{1}{b_i} - (r-1).$$

For the signed block graph given in Figure 1, we get $\beta = (\frac{1}{3}, \frac{1}{3}, -\frac{5}{6}, \frac{1}{2}, \frac{1}{3}, -\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$.

In this section, we first drive the eigenvector associated with eigenvalue 0 of $\tilde{D}$ when $\tilde{D}$ is singular. Furthermore, we obtain the formula for $\tilde{D}^{-1}$ when it is nonsingular. Let us start with a lemma.

Lemma 3.1 [7, Lemma 1] Let $\dot{G}$ be a signed block graph of order n with signed blocks B_i, $1 \le i \le k$. Let β be the vector defined as above. Then

$$\sum_{\upsilon \in V(\dot{G})} \beta_\upsilon = 1.$$

For vectors f, g with the same dimension, we use $\langle f, g \rangle$ to denote the usual inner product of two vectors. The following lemma is similar to the unsigned block graph. One can refer [7] for detailed proof.

Lemma 3.2 Let $\tilde{D}$ be the net distance matrix of a signed block graph $\dot{G}$ with n vertices. Let B_i be the signed clique of $\dot{G}$ with signature η_i, $1 \le i \le k$. Let β be the vector defined as above. Then

$$\tilde{D}\beta = \zeta\mathbf{1}.$$

It follows from Lemma 3.2 that β is an eigenvector associated with eigenvalue 0 of the net distance matrix $\tilde{D}$ if $\tilde{D}$ is singular. We use $R_\upsilon(A)$ (resp. $C_\upsilon(A)$) to denote the υ-th row (resp. column) of the matrix A.

Lemma 3.3 Let $\dot{G}$ be a signed block graph on n vertices with signed cliques B_i, $1 \le i \le k$. Let $\tilde{D}$ be the net distance matrix of $\dot{G}$. Let $\hat{N}$ and β be the matrix and the vector as defined above, respectively. Then

$$\hat{N}\tilde{D} + I = \beta\mathbf{1}^T.$$

Proof. We prove the result by induction on k. For $k = 1$, we have $\tilde{D} = (J - I)\eta_1$, $\hat{N} = (I - \frac{1}{n}J)\eta_1$ and $\beta = \frac{1}{n}\mathbf{1}$. Then it is easy to verify that $\hat{N}\tilde{D} + I = \beta\mathbf{1}^T$. So let $k \ge 2$ and assume the result to be true for signed block graphs with less than k signed cliques.

Next considering a signed block graph $\dot{G}$ with k signed cliques, we may assume, without loss of generality, that B_1 is a pendant signed clique and its cut vertex is vertex c. Let $\dot{G}^* = \dot{G}\backslash(B_1\backslash\{c\})$ and $\hat{N}_{\dot{G}^*} = \sum_{i=2}^{k}\frac{1}{b_i}N_i$. Let e_c be the $|V(\dot{G}^*)|$ dimensional column vector with 1 in position c and zeros elsewhere. Let α denote the c-th column of $\tilde{D}_{\dot{G}^*}$, *i.e.* $\alpha = \tilde{D}_{\dot{G}^*}e_c$. Let f_i be the $|V(B_1\backslash\{c\})|$dimensional column vector with 1 in position i and zeros elsewhere, where the index set of it is $V(B_1\backslash\{c\})$. Then we have

$$\hat{N} = \begin{pmatrix} \hat{N}_{\dot{G}^*} + \frac{b_1 - 1}{b_1}\eta_1 e_c e_c^T & -\frac{\eta_1}{b_1}e_c\mathbf{1}^T \\ -\frac{\eta_1}{b_1}\mathbf{1}e_c^T & \frac{\eta_1}{b_1}(b_1 I - J) \end{pmatrix}, \tilde{D} = \begin{pmatrix} \tilde{D}_{\dot{G}^*} & (\alpha + \eta_1\mathbf{1})\mathbf{1}^T \\ \mathbf{1}(\alpha^T + \eta_1\mathbf{1}^T) & (J - I)\eta_1 \end{pmatrix}$$

Thus it follows that

$$\hat{N}\tilde{D} + I = \begin{pmatrix} \hat{N}_{\dot{G}^*} + \frac{b_1 - 1}{b_1}\eta_1 e_c e_c^T & -\frac{\eta_1}{b_1}e_c\mathbf{1}^T \\ -\frac{\eta_1}{b_1}\mathbf{1}e_c^T & \frac{\eta_1}{b_1}(b_1 I - J) \end{pmatrix}\begin{pmatrix} \tilde{D}_{\dot{G}^*} & (\alpha + \eta_1\mathbf{1})\mathbf{1}^T \\ \mathbf{1}(\alpha^T + \eta_1\mathbf{1}^T) & (J - I)\eta_1 \end{pmatrix} + I.$$

Next we need to prove that $(\hat{N}\tilde{D} + I)_{ij} = \beta_i$, for all $1 \le i, j \le n$.

Case I: For row $i \in \dot{G}^*\backslash\{c\}$.

For such a row i and for columns $j \in \dot{G}^*$, by an induction assumption, we have $(\hat{N}_{\dot{G}^*}\tilde{D}_{\dot{G}^*} + I)_{ij} = (\beta_{\dot{G}^*})_i$. Since $\beta_i = (\beta_{\dot{G}^*})_i$ for $i \in \dot{G}^*\backslash\{c\}$, we are done for all columns in $\dot{G}^*$. While for columns $j \in B_1\backslash\{c\}$, we have

$$(\hat{N}\tilde{D} + I)_{ij} = \langle R_i(\hat{N}_{\dot{G}^*}),\ \alpha + \eta_1\mathbf{1}\rangle = \langle R_i(\hat{N}_{\dot{G}^*}),\ \alpha\rangle + \langle R_i(\hat{N}_{\dot{G}^*}),\ \eta_1\mathbf{1}\rangle = (\hat{N}_{\dot{G}^*}\tilde{D}_{\dot{G}^*})_{ic} = \beta_i.$$

Case II: For row $i \in B_1 \backslash \{c\}$.

Since for any row $i \in B_1 \backslash \{c\}$, it is easy to note that $\beta_i = \frac{1}{b_1}$. For columns $j \in B_1 \backslash \{c\}$, if $i \neq j$, then we get

$$(\hat{N}\tilde{D}+I)_{ij} = \langle -\frac{\eta_1}{b_1} e_c^T, \alpha + \eta_1 \mathbf{1} \rangle + \langle \frac{\eta_1}{b_1}(b_1 f_i^T - \mathbf{1}^T), \eta_1 (\mathbf{1} - f_j) \rangle$$

$$= -\frac{1}{b_1} + \langle \eta_1 f_i^T, \eta_1 \mathbf{1} \rangle + \langle \eta_1 f_i^T, -\eta_1 f_j \rangle + \langle -\frac{\eta_1}{b_1} \mathbf{1}^T, \eta_1 \mathbf{1} \rangle + \langle -\frac{\eta_1}{b_1} \mathbf{1}^T, -\eta_1 f_j \rangle$$

$$= -\frac{1}{b_1} + 1 - \frac{1}{b_1}(b_1 - 1) + \frac{1}{b_1} = \frac{1}{b_1} = \beta_i,$$

and it is easy to see that when $i = j$,

$$(\hat{N}\tilde{D}+I)_{ij} = \frac{1}{b_1} - 1 + 1 = \beta_i.$$

For the column $j = c$, it is simple to note that

$$(\hat{N}\tilde{D}+I)_{ic} = \langle -\frac{\eta_1}{b_1} e_c^T, \alpha \rangle + \langle \frac{\eta_1}{b_1}(b_1 f_i^T - \mathbf{1}^T), \eta_1 \mathbf{1} \rangle$$

$$= 1 - \frac{\eta_1}{b_1} \eta_1 (b_1 - 1) = \frac{1}{b_1} = \beta_i.$$

For columns $j \notin B_1$, we obtain

$$(\hat{N}\tilde{D}+I)_{ij} = \langle -\frac{\eta_1}{b_1} e_c^T, C_j(\tilde{D}_{\dot{G}^*}) \rangle + \langle \frac{\eta_1}{b_1}(b_1 f_i^T - \mathbf{1}^T), (\tilde{d}_{jc} + \eta_1)\mathbf{1} \rangle$$

$$= -\frac{\eta_1}{b_1} \tilde{d}_{jc} + \langle \frac{\eta_1}{b_1} b_1 f_i^T, \tilde{d}_{jc} \mathbf{1} \rangle + \langle \frac{\eta_1}{b_1} b_1 f_i^T, \eta_1 \mathbf{1} \rangle + \langle -\frac{\eta_1}{b_1} \mathbf{1}^T, \tilde{d}_{jc} \mathbf{1} \rangle + \langle -\frac{\eta_1}{b_1} \mathbf{1}^T, \eta_1 \mathbf{1} \rangle$$

$$= -\frac{\eta_1}{b_1} \tilde{d}_{jc} + \eta_1 \tilde{d}_{jc} + 1 - \frac{\eta_1}{b_1} \tilde{d}_{jc}(b_1 - 1) - \frac{1}{b_1}(b_1 - 1)$$

$$= -\frac{\eta_1}{b_1} \tilde{d}_{jc} + \eta_1 \tilde{d}_{jc} + 1 - \eta_1 \tilde{d}_{jc} + \frac{\eta_1}{b_1} \tilde{d}_{jc} - 1 + \frac{1}{b_1} = \frac{1}{b_1} = \beta_i.$$

Case III: For the row c.

For the column c, we have

$$(\hat{N}\tilde{D}+I)_{cc} = \langle R_c(\hat{N}_{\dot{G}^*}), C_c(\tilde{D}_{\dot{G}^*})\rangle + \langle \frac{b_1-1}{b_1}\eta_1 e_c^T, C_c(\tilde{D}_{\dot{G}^*})\rangle + \langle -\frac{\eta_1}{b_1}\mathbf{1}^T, \eta_1\mathbf{1}\rangle + 1$$

$$= (\hat{N}_{\dot{G}^*}\tilde{D}_{\dot{G}^*})_{cc} - \frac{1}{b_1}(b_1-1)+1 = (\beta_{\dot{G}^*})_c - 1 - 1 + \frac{1}{b_1} + 1$$

$$= (\beta_c + 1 - \frac{1}{b_1}) - 1 + \frac{1}{b_1} = \beta_c.$$

For columns $j \in \dot{G}^*\backslash\{c\}$, we get

$$(\hat{N}\tilde{D}+I)_{cj} = \langle R_c(\hat{N}_{\dot{G}^*}), C_j(\tilde{D}_{\dot{G}^*})\rangle + \langle \frac{b_1-1}{b_1}\eta_1 e_c^T, C_j(\tilde{D}_{\dot{G}^*})\rangle + \langle -\frac{\eta_1}{b_1}\mathbf{1}^T, (\tilde{d}_{jc}+\eta_1)\mathbf{1}\rangle$$

$$= (\beta_{\dot{G}^*})_c + \frac{b_1-1}{b_1}\eta_1\tilde{d}_{jc} - \frac{\eta_1}{b_1}\tilde{d}_{jc}(b_1-1) - \frac{1}{b_1}(b_1-1)$$

$$= (\beta_c + 1 - \frac{1}{b_1}) - 1 + \frac{1}{b_1} = \beta_c.$$

And for columns $j \in B_1\backslash\{c\}$, we drive that

$$(\hat{N}\tilde{D}+I)_{cj} = \langle R_c(\hat{N}_{\dot{G}^*}), \alpha + \eta_1\mathbf{1}\rangle + \langle \frac{b_1-1}{b_1}\eta_1 e_c^T, \alpha + \eta_1\mathbf{1}\rangle + \langle -\frac{\eta_1}{b_1}\mathbf{1}^T, \eta_1(1-f_j)\rangle$$

$$= \langle R_c(\hat{N}_{\dot{G}^*}), C_c(\tilde{D}_{\dot{G}^*})\rangle + \frac{b_1-1}{b_1} - \frac{1}{b_1}(b_1-1) + \frac{1}{b_1}$$

$$= (\hat{N}_{\dot{G}^*}\tilde{D}_{\dot{G}^*})_{cc} + \frac{1}{b_1} = (\beta_{\dot{G}^*})_c - 1 + \frac{1}{b_1}$$

$$= (\beta_c + 1 - \frac{1}{b_1}) - 1 + \frac{1}{b_1} = \beta_c.$$

Therefore, we completed the proof of $\hat{N}\tilde{D}+I = \beta\mathbf{1}^T$. □

We are now in a position to give a formula for the inverse of the net distance matrix when it is nonsingular.

Theorem 3.4 Let $\tilde{D}$ be the net distance matrix of a signed block graph $\dot{G}$. Let $\hat{N}$, ζ, β be defined as above. If $\zeta \neq 0$, then

$$\tilde{D}^{-1} = -\hat{N} + \frac{1}{\zeta}\beta\beta^T.$$

Proof. According to Lemmas 3.2 and 3.3, it is easy to verify that

$$(-\hat{N}+\frac{1}{\zeta}\beta\beta^T)\tilde{D}=-\hat{N}\tilde{D}+\frac{1}{\zeta}\beta\beta^T\tilde{D}=I-\beta\mathbf{1}^T+\frac{1}{\zeta}\beta\mathbf{1}^T\zeta=I.$$

Therefore, we get $\tilde{D}^{-1}=-\hat{N}+\frac{1}{\zeta}\beta\beta^T$. □

For the signed block graph in Figure 1, we also have $\beta=(\frac{1}{3},\frac{1}{3},-\frac{5}{6},\frac{1}{2},\frac{1}{3},-\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4})^T$ and

$$\hat{N}=\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{12} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

Then we obtain $\tilde{D}^{-1}=-\hat{N}+\frac{1}{\zeta}\beta\beta^T$.

In particular, for a signed block graph, if its all signed cliques are $\pm K_2$, then we have the following corollary from Theorem 3.4.

Corollary 3.5 Let $\tilde{D}$ be the net distance matrix of a signed tree $\dot{T}$ on n vertices with p positive edges and q negative edges, where $p \neq q$. Let N be the net Laplacian matrix of $\dot{T}$ and τ be an $n \times 1$ vector with $\tau_i = 2 - \delta_i$, where δ_i is the degree of vertex i. Then

$$\tilde{D}^{-1}=-\frac{1}{2}N+\frac{1}{2(p-q)}\tau\tau^T.$$

4. The Moore-Penrose inverses of N, $\hat{N}$, $\tilde{D}$ and R

It is clear to see that $\hat{N}\mathbf{1}=0$, $N\mathbf{1}=0$ and $\tilde{D}\beta=0$ when $\zeta=0$. Thus we know that $\hat{N}$, N and $\tilde{D}$ are singular when $\zeta=0$. In this section, the rank of $\hat{N}$ and N of a signed block graph is obtained. And the explicit formulas for the Moore-Penrose inverse of $\hat{N}$ and $\tilde{D}$ are given. Let us start with a lemma.

Lemma 4.1 Let $\dot{G}$ be a signed block graph on n vertices with signed cliques B_i, $1 \le i \le k$. Let η_i be the signature of B_i, $1 \le i \le k$. Let N be the net Laplacian matrix of $\dot{G}$. Let $\hat{N}$ be the matrix corresponding to $\dot{G}$ defined as above. Then

$$rank(\hat{N}) = rank(N) = n-1.$$

Proof. We prove $rank(N) = n - 1$ by induction on k. For $k = 1$, we have $\tilde{D} = (J - I)\eta_1$ and $N = (nI - J)\eta_1$. Then we get $rank(N) = n - 1$. So let $k \geq 2$ and assume the result to be true for signed block graphs with less than k signed cliques.

Next considering a signed block graph $\dot{G}$ with k signed cliques, we may assume, without loss of generality, that B_1 is a pendant signed clique and its cut vertex is vertex b_1. Let $\dot{G}^* = \dot{G} \backslash (B_1 \backslash \{b_1\})$. Let e_{b_1} be the $|V(\dot{G}^*)|$ dimensional column vector with 1 in position b_1 and zeros elsewhere. Then the net Laplacian matrix of $\dot{G}$ is given by

$$N = \begin{pmatrix} N_{\dot{G}^*} + (b_1 - 1)\eta_1 e_{b_1} e_{b_1}^T & -\eta_1 e_{b_1} \mathbf{1}^T \\ & \\ -\eta_1 \mathbf{1} e_{b_1}^T & \eta_1 (b_1 I - J) \end{pmatrix},$$

where $N_{\dot{G}^*}$ is the net Laplacian matrix of $\dot{G}^*$. In N, plus the columns corresponding to the index set $V(B_1 \backslash \{b_1\})$ to the column b_1 and plus the rows corresponding to the index set $V(B_1 \backslash \{b_1\})$ to the row b_1. The resulting matrix is

$$N_1 = \begin{pmatrix} N_{\dot{G}^*} & 0 \\ 0 & \eta_1 (b_1 I - J) \end{pmatrix}.$$

We get, by an induction assumption, $rank(N) = rank(N_{\dot{G}^*}) + (b_1 - 1) = \sum_{i=1}^{k} (b_1 - 1) = n - 1$. Similarly, the rank of the matrix $\hat{N}$ of a signed block graph is $n - 1$. □

Next we give the formulas for the Moore-Penrose inverse of $\hat{N}$ and $\tilde{D}$ of a signed block graph.

Theorem 4.2 Let $\dot{G}$ be a signed block graph on n vertices with net distance matrix $\tilde{D}$. Let $\hat{N}$ be the matrix corresponding to $\dot{G}$ as defined above. Then the Moore-Penrose inverse of $\hat{N}$ is

$$\hat{N}^+ = -\tilde{D} - \frac{\mathbf{1}^T \tilde{D} \mathbf{1}}{n^2} J + \frac{1}{n} (\tilde{D} J + J \tilde{D}).$$

Proof. Let $H = -\tilde{D} - \frac{\mathbf{1}^T \tilde{D} \mathbf{1}}{n^2} J + \frac{1}{n} (\tilde{D} J + J \tilde{D})$. We prove that H is the Moore-Penrose inverse of $\hat{N}$ by the definition of Moore-Penrose inverse. In fact,

$$\hat{N} H = -\hat{N} \tilde{D} - \frac{\mathbf{1}^T \tilde{D} \mathbf{1}}{n^2} \hat{N} J + \frac{1}{n} (\hat{N} \tilde{D} J + \hat{N} J \tilde{D})$$

$$= I - \beta \mathbf{1}^T + \frac{1}{n} (\beta \mathbf{1}^T - I) J = I - \frac{1}{n} J.$$

Similarly, $H\hat{N} = I - \frac{1}{n} J$. Thus both $\hat{N}H$ and $H\hat{N}$ are symmetric. Hence

$$\hat{N} H \hat{N} = (I - \frac{1}{n} J) \hat{N} = \hat{N}.$$

Finally,

$$H\hat{N}H = (I - \frac{1}{n}J)(-\tilde{D} - \frac{\mathbf{1}^T \tilde{D}\mathbf{1}}{n^2}J + \frac{1}{n}(\tilde{D}J + J\tilde{D}))$$

$$= H + \frac{1}{n}J\tilde{D} + \frac{\mathbf{1}^T \tilde{D}\mathbf{1}}{n^3}JJ - \frac{1}{n^2}J\tilde{D}J - \frac{1}{n^2}JJ\tilde{D}$$

$$= H + \frac{1}{n}J\tilde{D} + \frac{\mathbf{1}^T \tilde{D}\mathbf{1}}{n^2}J - \frac{\mathbf{1}^T \tilde{D}\mathbf{1}}{n^2}J - \frac{1}{n}J\tilde{D} = H.$$

Therefore, H is the Moore-Penrose inverse of $\hat{N}$, i.e. $\hat{N}^+ = -\tilde{D} - \frac{\mathbf{1}^T \tilde{D}\mathbf{1}}{n^2}J + \frac{1}{n}(\tilde{D}J + J\tilde{D})$. □

Theorem 4.3 Let $\dot{G}$ be a signed block graph on n vertices with $\zeta = 0$. Let $\tilde{D}$ be the net distance matrix of $\dot{G}$. Let $\hat{N}$ and β be the matrix and the vector defined as above, respectively. Then

$$\tilde{D}^+ = -\hat{N} - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T + \beta\beta^T \hat{N}).$$

Proof. Let $H = -\hat{N} - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T + \beta\beta^T \hat{N})$. We prove the Moore-Penrose inverse of $\tilde{D}$ by the definition of the Moore-Penrose inverse. In fact, we have

$$H\tilde{D} = -\hat{N}\tilde{D} - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T \tilde{D} + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T \tilde{D} + \beta\beta^T \hat{N}\tilde{D})$$

$$= -\beta\mathbf{1}^T + I + \frac{1}{\beta^T \beta}\beta\beta^T(\beta\mathbf{1}^T - I) = I - \frac{1}{\beta^T \beta}\beta\beta^T.$$

It is similar to show that $\tilde{D}H = I - \frac{1}{\beta^T \beta}\beta\beta^T$. Thus both $H\tilde{D}$ and $\tilde{D}H$ are symmetric matrices. We further drive that

$$\tilde{D}H\tilde{D} = \tilde{D}(I - \frac{1}{\beta^T \beta}\beta\beta^T) = \tilde{D}$$

and

$$H\tilde{D}H = (-\hat{N} - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T + \beta\beta^T \hat{N}))(I - \frac{1}{\beta^T \beta}\beta\beta^T)$$

$$= H + \frac{1}{\beta^T \beta}\hat{N}\beta\beta^T + \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^3}\beta\beta^T\beta\beta^T - \frac{1}{(\beta^T \beta)^2}\hat{N}\beta\beta^T\beta\beta^T - \frac{1}{(\beta^T \beta)^2}\beta\beta^T \hat{N}\beta\beta^T$$

$$= H + \frac{1}{\beta^T \beta}\hat{N}\beta\beta^T + \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T - \frac{1}{\beta^T \beta}\hat{N}\beta\beta^T - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T = H.$$

Hence, H is the Moore-Penrose inverse of $\tilde{D}$, i.e. $\tilde{D}^{+} = -\hat{N} - \frac{\beta^T \hat{N}\beta}{(\beta^T \beta)^2}\beta\beta^T + \frac{1}{\beta^T \beta}(\hat{N}\beta\beta^T + \beta\beta^T \hat{N}).$ □

Since the rank of net Laplacian matrix N of a signed block graph with n vertices is $n - 1$, next we turn to the Moore-Penrose inverse of N. Ali, Atik and Bapat [13] showed that $R^{-1} = -\frac{1}{2}N + \frac{1}{\rho^T R\rho}\rho\rho^T$ if $\rho^T R\rho \neq 0$, and $rank(R) = n - 1$ if $\rho^T R\rho = 0$, where $\rho = (\rho_1, \ldots, \rho_n)^T$, $\rho_i = 2 - \sum_{j\sim i} r_{ij}\sigma(ij)$. And $R^{+} = -\frac{1}{2}N + \frac{1}{2}(R^{+}\mathbf{1}\rho^T + \rho\mathbf{1}^T R^{+}) - \frac{\mathbf{1}^T R^{+}\mathbf{1}}{2}\rho\rho^T$, which is a relationship between R^{+} and N. The following theorem give the Moore-Penrose inverses of N and R if R is singular. Since the proofs are similar to $\tilde{D}^{+}$ and $\hat{N}^{+}$, we omit the proofs here.

Theorem 4.4 Let $\dot{G}$ be a signed block graph on n vertices. Let N be the net Laplacian matrix and R the resistance matrix of $\dot{G}$, respectively. Then

$$N^{+} = -\frac{1}{2}R - \frac{\mathbf{1}^T R\mathbf{1}}{2n^2}J + \frac{1}{2n}(RJ + JR),$$

$$R^{+} = -\frac{1}{2}\hat{N} - \frac{\rho^T N\rho}{2(\rho^T \rho)^2}\rho\rho^T + \frac{1}{2\rho^T \rho}(N\rho\rho^T + \rho\rho^T N).$$

Next we illustrate the above results by an example.

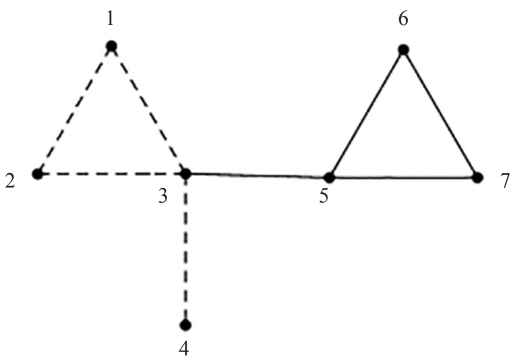

Figure 2. *A* signed block graph of order seven with signed blocks $-K_3$, $-K_2$, K_2 and K_3.

Example 4.5 Consider the signed block graph $\dot{G}$ in Figure 2. Then we have $\zeta = 0$, $\beta = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, \frac{1}{3})^T$, $\rho = (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}, 1, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3})^T$, $\rho^T R\rho = 0$,

$$\tilde{D}=\begin{pmatrix}0&-1&-1&-2&0&1&1\\-1&0&-1&-2&0&1&1\\-1&-1&0&-1&1&2&2\\-2&-2&-1&0&0&1&1\\0&0&1&0&0&1&1\\1&1&2&1&1&0&1\\1&1&2&1&1&1&0\end{pmatrix}, \hat{N}=\begin{pmatrix}-\frac{2}{3}&\frac{1}{3}&\frac{1}{3}&0&0&0&0\\\frac{1}{3}&-\frac{2}{3}&\frac{1}{3}&0&0&0&0\\\frac{1}{3}&\frac{1}{3}&-\frac{2}{3}&\frac{1}{2}&-\frac{1}{2}&0&0\\0&0&\frac{1}{2}&-\frac{1}{2}&0&0&0\\0&0&-\frac{1}{2}&0&\frac{7}{6}&-\frac{1}{3}&-\frac{1}{3}\\0&0&0&0&-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\0&0&0&0&-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix},$$

$$N=\begin{pmatrix}-2&1&1&0&0&0&0\\1&-2&1&0&0&0&0\\1&1&-2&1&-1&0&0\\0&0&1&-1&0&0&0\\0&0&-1&0&3&-1&-1\\0&0&0&0&-1&2&-1\\0&0&0&0&-1&-1&2\end{pmatrix} \text{ and } R=\begin{pmatrix}0&-\frac{2}{3}&-\frac{2}{3}&-\frac{5}{3}&\frac{1}{3}&1&1\\-\frac{2}{3}&0&-\frac{2}{3}&-\frac{5}{3}&\frac{1}{3}&1&1\\-\frac{2}{3}&-\frac{2}{3}&0&-1&1&\frac{5}{3}&\frac{5}{3}\\-\frac{5}{3}&-\frac{5}{3}&-1&0&0&\frac{2}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}&1&0&0&\frac{2}{3}&\frac{2}{3}\\1&1&\frac{5}{3}&\frac{2}{3}&\frac{2}{3}&0&\frac{2}{3}\\1&1&\frac{5}{3}&\frac{2}{3}&\frac{2}{3}&\frac{2}{3}&0\end{pmatrix}.$$

Hence we obtain

$$\hat{N}^{+}=\frac{1}{49}\begin{pmatrix}-40&9&37&51&-5&-26&-26\\9&-40&37&51&-5&-26&-26\\37&37&16&30&-26&-47&-47\\51&51&30&-54&-12&-33&-33\\-5&-5&-26&-12&30&9&9\\-26&-26&-47&-33&9&86&37\\-26&-26&-47&-33&9&37&86\end{pmatrix}=-\tilde{D}-\frac{\mathbf{1}^T\tilde{D}\mathbf{1}}{n^2}J+\frac{1}{n}(\tilde{D}J+J\tilde{D}),$$

$$\tilde{D}^{+}=\begin{pmatrix} 248/441 & -193/441 & -13/441 & -53/294 & -17/882 & 17/441 & 17/441 \\ -193/441 & 248/441 & -13/441 & -53/294 & -17/882 & 17/441 & 17/441 \\ -13/441 & -13/441 & -58/441 & 1/294 & 433/882 & 8/441 & 8/441 \\ -53/294 & -53/294 & 1/294 & 19/98 & -5/294 & 5/147 & 5/147 \\ -17/882 & -17/882 & 433/882 & -5/294 & -989/882 & 107/441 & 107/441 \\ 17/441 & 17/441 & 8/441 & 5/147 & 107/441 & -214/441 & 227/441 \\ 17/441 & 17/441 & 8/441 & 5/147 & 107/441 & 227/441 & -214/441 \end{pmatrix}$$

$$=-\hat{N}-\frac{\beta^{T}\hat{N}\beta}{(\beta^{T}\beta)^{2}}\beta\beta^{T}+\frac{1}{\beta^{T}\beta}(\hat{N}\beta\beta^{T}+\beta\beta^{T}\hat{N}),$$

$$N^{+}=\frac{1}{147}\begin{pmatrix} -32 & 17 & 45 & 66 & -18 & -39 & -39 \\ 17 & -32 & 45 & 66 & -18 & -39 & -39 \\ 45 & 45 & 24 & 45 & -39 & -60 & -60 \\ 66 & 66 & 25 & -81 & -18 & -39 & -39 \\ -18 & -18 & -39 & -18 & 45 & 24 & 24 \\ -39 & -39 & -60 & -39 & 24 & 101 & 52 \\ -39 & -39 & -60 & -39 & 24 & 52 & 101 \end{pmatrix}=-\frac{1}{2}R-\frac{\mathbf{1}^{T}R\mathbf{1}}{2n^{2}}J+\frac{1}{2n}(RJ+JR),$$

and

$$R^{+}=\begin{pmatrix} 362/441 & -599/882 & -41/882 & -65/294 & -47/882 & 31/882 & 31/882 \\ -599/882 & 362/441 & -41/882 & -65/294 & -47/882 & 31/882 & 31/882 \\ -41/882 & -41/882 & -43/441 & 25/294 & 493/882 & 11/441 & 11/441 \\ -65/294 & -65/294 & 25/294 & 47/196 & -61/588 & 59/588 & 59/588 \\ -47/882 & -47/882 & 493/882 & -61/588 & -2473/1764 & 599/1764 & 599/1764 \\ 31/882 & 31/882 & 11/441 & 59/588 & 599/1764 & -331/441 & 661/882 \\ 31/882 & 31/882 & 11/441 & 59/588 & 599/1764 & 661/882 & -331/441 \end{pmatrix}$$

$$=-\frac{1}{2}\hat{N}-\frac{\rho^{T}N\rho}{2(\rho^{T}\rho)^{2}}\rho\rho^{T}+\frac{1}{2\rho^{T}\rho}(N\rho\rho^{T}+\rho\rho^{T}N).$$

Remark 4.6 *A* symmetric matrix is called centered symmetric if it has zero row sums and a symmetric matrix is called hollow symmetric if it has zero diagonal elements. Kurata and Bapat [13] showed that there is a one to one correspondence between the classes of hollow symmetric matrices and centered symmetric matrices. For a signed block graph $\dot{G}$ with n vertices, $\tilde{D}$ (resp. R) is a hollow symmetric matrix and corresponding centered symmetric matrix is $\hat{N}^{+}$ (resp. N), moreover, if $\zeta_{\dot{G}}\neq 0$, (resp. $\rho^{T}R\rho=0$) then matrices $\tilde{D}$ (resp. R) and $\hat{N}^{+}$ (resp. N) have the same rank $n-1$.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Sivasubramanian S. q-Analogs of distance matrices of 3-hypertrees. *Linear Algebra Appl*. 2009; 431: 1234-1248.

[2] Stanić Z. Some properties of the eigenvalues of the net Laplacian matrix of a signed graph. *Discuss. Math. Graph Theory*. 2022; 42: 893-903.

[3] Molitierno JJ. *Applications of combinatorial matrix theory to Laplacian matrices of graphs*. CRC Press, Boca Raton. FL; 2012.

[4] Bapat RB. *Graphs and matrices*. 2nd ed. Hinduston Book Agency, New Delhi; 2018.

[5] Graham RL, Pollak HO. On the addressing problem for loop switching. *Bell System Tech. J*. 1971; 50: 2495-2519.

[6] Graham RL, Lovász L. Distance matrix polynomials of trees. *Adv. Math*. 1978; 29(1): 60-80.

[7] Bapat RB, Sivasubramanian S. Inverse of distance matrix of a block graph. *Linear and Multilinear Algebra*. 2011; 59(12): 1393-1397.

[8] Hou Y, Fang A, Sun Y. Inverse of the distance matrix of cycle-clique graph. *Linear Algebra Appl*. 2015; 485: 33-46.

[9] Hou Y, Sun Y. Inverse of the distance matrix of a bi-block graph. *Linear and Multilinear Algebra*. 2016; 64(8): 1509-1517.

[10] Balaji R, Bapat RB, Goel S. An inverse formula for the distance matrix of a wheel graph with even number of vertices. *Linear Algebra Appl*. 2021; 610: 274-292.

[11] Bapat RB, Kirkland SJ, Neumann M. On distance matrices and Laplacians. *Linear Algebra Appl*. 2005; 401: 193-209.

[12] Kurata H. Bapat RB. Moore-Penrose inverse of a hollow symmetic matrix and a predistance matrix. *Spec. Matrices.* 2016; 4(1): 270-282.

[13] Ali P, Atik F, Bapat RB. Identities for minors of the Laplacian, resistance and distance matrices of graphs with arbitrary weights. *Linear and Multilinear Algebra*. 2018; 68(2): 323-336.

[14] Bapat RB, Manjunatha PK. The generalized Moore-Penrose inverse. *Linear Algebra Appl*. 1992; 165: 59-69.

[15] Bapat RB, Roy S. On the adjacency matrix of a block graph. *Linear and Multilinear Algebra*. 2014; 62: 406-418.

[16] Graham RL, Hoffman AJ, Hosoya H. On the distance matrix of a directed graph. *J. Graph Theory*. 1977; 1(1): 85-88.