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Research Article

Extended Semilocal Convergence for Chebyshev-Halley-Type Schemes for Solving Nonlinear Equations under Weak Conditions

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Abstract: The application of the Chebyshev-Halley type scheme for nonlinear equations is extended with no additional conditions. In particular, the purpose of this study is two folds. The proof of the semi-local convergence analysis is based on the recurrence relation technique in the first fold. In the second fold, the proof relies on majorizing sequences. Iterates are shown to belong to a larger domain resulting in tighter Lipschitz constants and a finer convergence analysis than in earlier works. The convergence order of these methods is at least three. The numerical example further validates the theoretical results.

***Keywords*:** Chebyshev-Halley-like scheme, convergence, Banach space

MSC: 65K05, 49M37, 49M15

1. Introduction

Many applications in computational Sciences require finding a solution x^* of the nonlinear equation

$$\mathcal{E}(x) = 0, \tag{1}$$

where the operator $\mathcal{E} : D \subset X \to Y$ is acting between Banach spaces X and Y. Higher convergence order schemes have been used extensively to generate a sequence approximating x^* under certain conditions [1-4]. In particular, the third order scheme [5] has been used and is defined $\forall k = 0, 1, 2, \ldots$ by

$$x_{k+1} = x_k - T_k \mathcal{E}'(x_k)^{-1}(\mathcal{E}(x_k)), \tag{2}$$

where $a \in \mathbb{R}$, $L_{\mathcal{E}} = \mathcal{E}'(x_k)^{-1}\mathcal{E}''(x_k)\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)$, and $T_k = I + \frac{1}{2}L_{\mathcal{E}}(I - aL_{\mathcal{E}})^{-1}$. If $a = 0$, $\frac{1}{2}$, 1, then (2) reduces to the

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Chebyshev, Halley and Super-Halley Schemes [6-10], respectively.

The convergence conditions used [11-15] are:

(A1) $\|\mathcal{E}'(x_0)^{-1}\| \leq \beta$.

(A2) $\|\mathcal{E}'(x_0)^{-1}\mathcal{E}(x_0)\| \leq \eta$.

(A3) $\|\mathcal{E}''(x) \leq M_1\|$ for each $x \in D$.

(A4) $\|\mathcal{E}''(x) - \mathcal{E}''(y)\| \leq L_1\|x - y\|$ for each $x, y \in D$.

But there are even simple scalar examples where condition (A4) is not satisfied.

Example 1.1 Let $X = Y = \mathbb{R}$, $D = [-0.5, 1.5]$. Define the function λ on the interval D by

$$\epsilon(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then, we get $x^* = 1$, and

$$\epsilon'''(x) = 6\log x^2 + 60x^2 - 24x + 22.$$

But $\epsilon'''(x)$ is unbounded on D. Thus, the convergence of scheme (2) is not assured by the previous works [13-16].

That is why in reference [5] the following conditions are used:

(B3) $\|\mathcal{E}'(x)\| \leq M_1$ for each $x \in D$.

(B4) $\|\mathcal{E}''(x) - \mathcal{E}''(y)\| \leq w_1(\|x - y\|)$ for each $x, y \in D$, where $w_1(0) \geq 0$, and for $t > 0$, function w_1 is continuous and nondecreasing.

(B5) There exists $\bar{w}_1(ts) \leq \bar{w}_1(t)w_1(s)$ for $t \in [0, 1]$ and $s \in (0, +\infty)$.

Using (A1), (A2), (B3)-(B5) the Halley scheme was shown to be of $R-$order at least two [5]. In particular, if $w_1(t) = \sum_{i=1}^{j} L_i t^{q_i}$, the Halley scheme is of $R-$order at least $2 + q$, where $q = \min\{q_1, q_2, \dots q_j\}$, $q_i \in [0, 1]$. $i = 1, 2, \dots, j$.

If $a \neq 0$, scheme (2) requires the evaluation of the inverse of linear operator $I - aL_{\mathcal{E}}$ at each step. That is why to reduce the computational cost of this inversion and increase the $R-$order scheme

$$y_k = x_k - \mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)$$

$$z_k = x_k - T_k^1\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)$$

$$x_{k+1} = z_k - T_k^2\mathcal{E}'(x_k)^{-1}\mathcal{E}(y_k) \tag{3}$$

was studied in [5], where $T_k^1 = I + \frac{1}{2}L_{\mathcal{E}} + \frac{b}{2}L_{\mathcal{E}}^2 + \frac{b^2}{2}L_{\mathcal{E}}^3$, $T_k^2 = I + L_{\mathcal{E}} + cT_k^3$, $a \in [0, 1]$, $c \in [-2, 2]$ and $T_k^3 = \mathcal{E}'(x_k)^{-1}\mathcal{E}''(x_k)$ $\mathcal{E}'(x_k)^{-1}\mathcal{E}(z_k)$.

Let us consider the condition:

(C4) $\|\mathcal{E}''(x) - \mathcal{E}''(y)\| \leq w_2(\|x - y\|)$ for each $x, y \in D_1 \subseteq D$ where $D_1 \neq$ is a convex set, $w_2(t)$ is a continuous and nondecreasing scalar fucntion with $w_2(0) \geq 0$, and there exists non-negative real function $w_3 \in C[0, 1]$ satisfying $w_3(t) \leq 1$ and $w_2(ts) \leq w_2(t)w_2(s)$, $t \in [0, 1]$, $s \in (0, \infty)$.

Using conditions (A1)-(A3) and (C4) the $R-$order was increased. In particular, if the second derivative satisfies (A4) the $R-$order of the scheme (3) is at least five which is higher than Chebyshev's, Halley's, and Super-Halley's [1-4, 17].

In our study, we are concerned with optimization considerations. We raise the following questions.

Can we: (Q1) Increase the convergence domain?

(Q2) Weaken the suffcient semi-local convergence criteria?

(Q3) Improve the estimates on error bounds on the distances $\|x_{k+1} - x_k\|$, $\|x_k - x^*\|$?

(Q4) Improve the uniqueness information for the location of x^*.

(Q5) Use weaker conditions.

and

(Q6) Provide the results in affne invariant form.

The advantages of (Q6) are well-known [3, 17-18]. Denote this set of questions by (Q). We would like question Q to be answered positively without additional or even weaker conditions. This can be achieved by finding at least as small M_1, L_1, w_1, $\bar{w}_1$, w_2 and w_3.

In Section 2 we achieve this goal. Another concern involves conditions (A4) or (B4) or (C4). Denote the set of nonlinear equations where the operator $\mathcal{E}$ satisfies say (C4) by S1. Moreover, denote by S2 the set of nonlinear equations where the operator $\mathcal{E}'$ does not satisfy (C4). Then, S1 is a strict subset of S2. Therefore, working on S2 instead of S1 is interesting, since the applicability of scheme (3) is extended. We show how to do this by dropping condition (C4) in Section 3.

2. Semi-local convergence I

The results are presented in the affine invariant form. Therefore, the condition (A1) is dropped. In particular, the conditions (H) are used:

(H1) $\|\mathcal{E}'(x_0)^{-1}(\mathcal{E}'(x) - \mathcal{E}'(x_0))\| \le M_0\|x - x_0\|\ \forall x \in D$.

Set $D_0 = U(x_0, \dfrac{1}{M_0}) \cap D$.

(H2) $\|\mathcal{E}'(x_0)^{-1}\mathcal{E}''(x)\| \le M\ \forall x \in D_0$.

(H3) $\|\mathcal{E}'(x_0)^{-1}(\mathcal{E}''(x) - \mathcal{E}''(y))\| \le w(\|x - y\|)\ \forall x, y \in D_0$, where w is a continuous and nondecreasing function with $w(0) \ge 0$, and there exists a non-negative function $w_0 \in C[0, 1]$ such that $w_0(t) \le 1$ and $w(ts) \le w_0(t)w(s) \forall t \in [0, 1]$, $s \in (0, \infty)$.

Remark 2.1 It follows by the definition of the set D_0 that

$$D_0 \subseteq D, \tag{4}$$

so

$$M_0 \le \beta M_1, \tag{5}$$

$$M \le \beta M_1 \tag{6}$$

and

$$w(t) \le \beta w_1(t). \tag{7}$$

Notice also that using (A3) the following estimate was used in the earlier studies [3-5, 7-16]:

$$\| \mathcal{E}'(x)^{-1} \| \le \frac{\beta}{1-\beta M_1 \| x-x_0 \|}.$$

But using the weaker and needed (H1) we obtained instead the tighter estimate

$$\| \mathcal{E}'(x)^{-1}\mathcal{E}'(x_0) \| \le \frac{1}{1-M_0 \| x-x_0 \|},$$

Moreover, suppose

$$M_0 \le M.$$

Otherwise, the results that follow hold with the parameter M_0 replacing M.

Next, we state the semi-local convergence result [5, Theorem 1]. But first, we consider some scalar functions. Let us define the functions p, h and φ by

$$p(u_1) = g_1(u_1) + [1+u_1+|c|u_1 g_2(u_1)]g_2(u_1),$$

$$h(u_1) = \frac{1}{1-u_1 p(u_1)},$$

$$\begin{aligned}\varphi(u_1,u_2) = {} & |c|u_1\psi(u_1,u_2)^2 + u_1^2[1+|c|\psi(u_1,u_2)]\psi(u_1,u_2) \\ & + I_2 u_2[1+u_1+|c|u_1\psi(u_1,u_2)]\psi(u_1,u_2) \\ & + \frac{u_1^2}{2}[1+\beta u_1+\beta^2 u_1^2][1+u_1+|c|u_1\psi(u_1,u_2)]\psi(U_1,u_2) \\ & + \frac{u_1}{2}[1+u_1+|c|u_1\psi(u_1,u_2)]^2\psi(u_1,u_2)^2,\end{aligned}$$

where

$$g_1(u_1) = 1+\frac{1}{2}u_1+\frac{\beta}{2}u_1^2+\frac{\beta^2}{2}u_1^3,$$

$$g_2(u_1) = \frac{u_1}{2}[1+\beta u_1+\beta^2 u_1^2+g_1(u_1)^2],$$

$$\psi(u_1,u_2) = \frac{\beta}{2}u_1^2(1+\delta u_1)+\frac{u_1^2}{2}(1+\delta u_1+\beta^2 u_1^2)+I_1 u_1+\frac{u_1^3}{8}(1+\beta u_1+\beta^2 u_1^2)^2$$

and

$$I_1 = \int_0^1 \xi(u_1)(1-u_1)d\xi,\ I_2 = \int_0^1 \xi(u_1)d\xi.$$

Moreover, define the function $g(u_1) = u_1 p(u_1) - 1$. Notice that for $u_1 > 0$ the function $g(u_1)$ is increasing. Then $g(u_1) = 0$ has non more than one root in $(0, \infty)$. Since, $g(0) = -1 < 0$, $g(\frac{1}{2}) > 0$, we know that $g(u_1) = 0$ has a root in $(0, \frac{1}{2})$. Define s^* as the positive root of equation $u_1 p(u_1) - 1 = 0$, then $s^* < \frac{1}{2}$.

Theorem 2.2 Suppose: $\mathcal{E} : D \subseteq X \to Y$ is twice Fréchet differentiable and conditions (A1)-(A3), (C4)and

$$U[x_0, R\eta] \subset D$$

hold, where $R = \frac{p(a_0)}{1-d_0}$, $a_0 = M_1\beta\eta$, $b_0 = \beta\eta w_2(\eta)$, $d_0 = h(a_0)\psi(a_0, b_0)$ satisfy $a_0 < s^*$ and $h(a_0)d_0 < 1$. Then, the following assertion holds

$$\{x_n\} \subset U[x_0, R\eta].$$

Moreover, there exists $\lim_{k\to\infty} x_k = x^* \in U[x_0, R\eta]$ so that $\mathcal{E}(x^*) = 0$,

$$\| x^* - x_k \| \le e_k = p(a_0)\eta\lambda^k \gamma^{\frac{2^k - 1}{2}} \frac{1}{1 - \lambda\gamma^{3k}},$$

where $\gamma = h(a_0)d_0$ and $\lambda = \frac{1}{h(a_0)}$ only solution of equation $\mathcal{E}(x) = 0$ in the region $U(x_0, R_1\eta) \cap D$, where $R_1 = \frac{2}{a_0} - R$.

In our case we have

Theorem 2.3 Suppose $\mathcal{E} : D \subseteq X \to Y$ is twice Fréchet differentable and the conditions (H) hold.

$$U[x_0, R_0\eta] \subset D,\ R_0 = \frac{p(\overline{a}_0)}{1 - \overline{d}_0},$$

where $\overline{a}_0 = M\beta\eta$, $\overline{b}_0 = \eta w(\eta)$, $\overline{d}_= h(\overline{a}_0)\psi(\overline{a}_0, \overline{b}_0)$ satisfy $\overline{a}_0 < s^*$ and $h(\overline{a}_0)\overline{d}_0 < 1$. Then, the following assertion holds

$$\{x_n\} \subset U[x_0, R_0\eta].$$

Moreover, there exists $\lim_{k\to\infty} x_k = x^* \in U[x_0, R_0\eta]$, such that $\mathcal{E}(x^*) = 0$,

$$\| x^* - x_k \| \le \overline{e}_k = p(\overline{a}_0)\eta\lambda_0^k \gamma_0^{\frac{2^k - 1}{2}} \frac{1}{1 - \lambda_0\gamma_0^{3k}},$$

where $\gamma_0 = h(\overline{a}_0)\overline{d}_0$ and $\lambda_0 = \frac{1}{h(\overline{a}_0)}$. Furthermore, the point x^* is the only solution of equation $\mathcal{E}(x) = 0$ in the region $U(x_0, \overline{R}_1\eta) \cap D$, where $\overline{R}_1 = \frac{2}{M_0} - R_0$.

Proof. Simply use M, w, $\overline{a}_0$, $\overline{b}_0$, $\overline{d}_0$, $R_0\overline{R}_1$, λ_0, γ_0 for M_1, w_2, a_0, b_0, d_0, R, R_1, λ, γ used in Theorem 2.2.

□

Remark 2.4 In view of (4)-(7), we have

$$a_0 < s^* \Rightarrow \bar{a}_0 < s^*,$$

$$\bar{a}_0 < a_0,$$

$$h(a_0)d_0 < 1 \Rightarrow h(\bar{a}_0)\bar{d}_0 < 1$$

$$\bar{e}_k \le e_k,$$

and

$$R_1 \le \bar{R}_1.$$

These estimates show that questions (Q) have been answered positively under our technique.

3. Semi-local convergence II

The results are also presented in an affine invariant form and the restrictive condition (C4) is dropped.

Suppose:

(H4) $\|\mathcal{E}'(x_0)^{-1}(\mathcal{E}'(x) - \mathcal{E}'(y))\| \le v(\|x - y\|)$ $\forall x, y \in D_0$, where v is a real continuous and nondecreasing function defined on the interval $[0, \infty)$.

Denote conditions (A2), (H1), (H2) and weaker (H3) or (C4) or (B4), or (A4). The semi-local convergence is based on conditions (H)$'$:

Define scalar sequences $\{t_k\}$, $\{s_k\}$ and $\{u_k\}$ $\forall k = 0, 1, 2, \ldots$ by

$$t_0 = 0,\ s_0 = \eta$$

$$u_k = s_k + \gamma_k(s_k - t_k),$$

$$t_{k+1} = u_k + \frac{\beta_k \alpha_k}{1 - M_0 t_k}$$

$$s_{k+1} = t_{k+1} + \frac{\delta_{k+1}}{1 - M_0 t_{k+1}}, \tag{8}$$

where

$$\alpha_k = \frac{1}{2}M(s_k - t_k)^2 + \frac{|b|}{2}\frac{M^2(s_k - t_k)^3}{1 - M_0 t_k} + \frac{b^2}{2}\frac{M^3(s_k - t_k)^4}{(1 - M_0 t_k)^2}$$

$$+ \int_0^1 v(\theta(u_k - t_k))d\theta(u_k - t_k),$$

$$\beta_k = 1 + \frac{M(s_k - t_k)}{1 - M_0 t_k} + |c| \frac{M\alpha_k}{(1 - M_0 t_k)^2},$$

$$\gamma_k = \frac{1}{2}\frac{M(s_k - t_k)}{1 - M_0 t_k} + \frac{|b|}{2}\left(\frac{M(s_k - t_k)}{1 - M_0 t_k}\right)^2 + \frac{b^2}{2}\left(\frac{M(s_k - t_k)}{1 - M_0 t_k}\right)^3$$

and

$$\delta_{k+1} = \frac{|c| M\alpha_k^2}{(1 - M_0 t_k)^2} + M(s_k - t_k)\left(\frac{M(s_k - t_k)}{1 - M_0 t_k} + \frac{|c| M\alpha_k}{(1 - M_0 t_k)^2}\right)^2$$

$$+2M(s_k - t_k)(t_{k+1} - u_k) + M(u_k - s_k)(t_{k+1} - u_k)$$

$$+\int_0^1 v(\theta(t_{k+1} - u_k))d\theta(t_{k+1} - u_k).$$

We shall show that these sequences are majorizing for the scheme (3).

Lemma 3.1 Suppose

$$M_0 t_k < 1 \,\forall\, k = 0, 1, 2, \ldots. \tag{9}$$

Then, the sequence $\{t_k\}$ is increasing, bounded from above by $t^{**} = \frac{1}{M_0}$ and converges to $t^* \in [0, t^{**}]$, which is its unique least upper bound.

Proof. It follows from (8) and (9) that sequence $\{t_k\}$ is such that $t_k \leq s_k \leq u_k \leq t_k{+}1 < \frac{1}{M_0}$, and as such it converges to t^*.

□

Another convergence result with stronger conditions but easier to verify than (9) follows. Suppose that

$$0 \leq s_k - t_k \leq \eta,\ m < 1 \tag{10}$$

and

$$2M_0 t_k \leq 1. \tag{11}$$

Then, the following estimates hold:

$$\gamma_k \leq M\eta(1 + 2|b|M\eta + 4b^2 M62\eta^2) = \mu_3,$$

$$\alpha_k \leq (\frac{1}{2}M\eta + |b|M^2\eta^2 + 2b^2M^3\eta^3 + \int_0^1 v(\theta(1+\mu_3)\eta)d\theta(1+\mu_3))(s_k - t_k)$$

$$= \mu_0(s_k - t_k),$$

$$\frac{\beta_k \alpha_k}{1-M_0 t_k} \leq 2\mu_0 (s_k - t_k)(1 + 2M\eta + 2|c| M\mu_0 \eta)$$

$$= \mu_2 (s_k - t_k),$$

and

$$\frac{\delta_{k+1}}{1-M_0 t_k} \leq 2[4|c| M\mu_0^2 \eta + M(2M\eta + 4|c| M\mu_0 \eta)$$

$$+2M(1+\mu_2)\eta + M\mu_3(1+\mu_2)\eta$$

$$+\int_0^1 v(\theta(1+\mu_2)\eta) d\theta (1+\mu_2)](s_k - t_k)$$

$$= \mu_3 (s_k - t_k).$$

Set $m = \max\{\mu_1, \mu_2, \mu_3\}$ for $\ell_0 = \max\{\eta, u_0 - s_0, t_1 - u_0\}$ and $\ell_1 = \min\{\mu_1, \mu_2, \mu_3\}$.

Lemma 3.2 Under conditions (13) and (14) further suppose

$$0 \leq \ell_0 \leq \ell_1 \leq m < 1 - 2M_0 \eta. \tag{12}$$

Then, the following assertions hold

$$0 \leq u_k - s_k \leq m(s_k - t_k) \leq m^{k-1} \eta, \tag{13}$$

$$0 \leq s_k - t_k \leq m(s_{k-1} - t_{k-1}) \leq m^k \eta, \tag{14}$$

$$0 \leq t_{k+1} - u_k \leq m(s_k - t_k) \leq m^{k+1} \eta, \tag{15}$$

$$0 \leq t_{k+i} - t_k \leq B\eta m^{k-1} \frac{1-m^{i-1}}{1-m} \leq \frac{B\eta}{1-m} m^{k-1}, \tag{16}$$

and there exists $t^* = \lim_{k\to\infty} t_k$ such that

$$0 \leq t^* - t_k \leq \frac{B\eta}{1-m} m^{k-1} \tag{17}$$

and

$$2M_0 t_k \leq 1,$$

where $B = 1 + m + m^2$.

Proof. It follows from (8), (9), (11) and (12) that estimates (13)-(16) hold. Let $i \geq 0$ be an iteger. Then, we can write in turn that

$$0 \leq t_{k+i} - t_k = (t_{k+i} - t_{k+i-1}) + (t_{k+i-1} - t_{k+i-2}) + \ldots + (t_{k+1} - t_k)$$

$$\leq B\eta(m^{k+i-2} + \ldots + t^{k-i})$$

$$= B\eta m^{k-1}\frac{1-m^{i-1}}{1-m} \leq \frac{B\eta m^{k-1}}{1-m}, \tag{18}$$

so the assertion (17) holds. Hence, the sequence $\{t_k\}$ is complete in the Banach space X and as such it converges to some t^*. By letting $k \to \infty$ in (18), we obtain (18). Notice also that

$$2M_0 t_k \leq 2M_0 \frac{1-m^{k+1}}{1-m}\eta \leq \frac{2M_0\eta}{1-m} \leq 1,$$

by the right hand side of (12), thus $\frac{1}{1-M_0 t_k} \leq 2$.

□

Remark 3.3 The condition (12) is the sufficient convergence criterion for the sequence $\{t_k\}$. Such a criterion is standard in this type of study. It shows how close x_0 should be to the solution (i.e. how small η should be) to obtain convergence.

Notice also that each $\mu_i < 1$, $i = 1, 2, 3$ can be solved for η, which depends on M_0, M, b, c, and v, i.e. the initial data.

The following modified auxiliary result is needed from [5, Lemma 1].

Lemma 3.4 Suppose that the iterates $\{x_k\}$ are well defined by the scheme (6). Then, the following Ostrowski-type relationship [3-4] holds

$$\begin{aligned}\mathcal{E}(x_{k+1}) = &-c\mathcal{E}''(x_k)(\mathcal{E}'(x_k)^{-1}\mathcal{E}(z_k))^2 \\ &+\mathcal{E}''(x_k)\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)(L_{\mathcal{E}} + cT_k^3)\mathcal{E}'(x_k)^{-1}\mathcal{E}(z_k) \\ &-\mathcal{E}''(x_k)(y_k - x_k)(x_{k+1} - z_k) \\ &+\int_0^1 \mathcal{E}''(y_k + \theta(z_k - y_k))d\theta(z_k - y_k)(x_{k+1} - z_k) \\ &+\int_0^1 [\mathcal{E}'(z_k + \theta(x_{k+1} - z_k)) - \mathcal{E}'(z_k)]d\theta(x_{k+1} - z_k),\end{aligned}$$

and

$$\mathcal{E}(z_k) = -\frac{1}{2}\mathcal{E}''(x_k)(\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k))^2$$

$$-\frac{b}{2}\mathcal{E}''(x_k)\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)L_{\mathcal{E}}\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)$$

$$-\frac{b^2}{2}\mathcal{E}''(x_k)\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)L_{\mathcal{E}}^2\mathcal{E}'(x_k)^{-1}\mathcal{E}(x_k)$$

$$+\int_0^1(\mathcal{E}'(x_k+\theta(z_k-x_k))-\mathcal{E}'(x_k))d\theta(z_k-x_k).$$

Next, we present the second semi-local convergence result for the scheme (3).

Theorem 3.5 Under the conditions (H)', further suppose $U[x_0, t^*] \subset D$, if conditions of Lemma 3.1 or Lemma 3.2 hold. Then, the following assertions hold $\{x_k\} \subset U[x_0, t^*]$ and there exists $\lim_{k\to\infty} x_k = x^* \in U[x_0, t^*]$ so that

$$\| x^* - x_k \| \le t^* - t_k.$$

Proof. Assertions

$$\| y_n - x_n \| \le s_n - t_n, \tag{19}$$

$$\| z_n - y_n \| \le u_n - s_n, \tag{20}$$

and

$$\| x_{n+1} - z_n \| \le t_{n+1} - u_n,$$

shall be shown using induction. By (8) and the first substep of method (3), we have

$$\| y_0 - x_0 \| = \| \mathcal{E}'(x_0)^{-1}\mathcal{E}(x_0) \| \le \eta = s_0 - t_0 < t^*,$$

thus, the iterat $y_0 \in U[x_0, t^*]$ and (19) holds for $n = 0$. Let $u \in U[x_0, t^*]$. Using (H1), we get

$$\| \mathcal{E}'(x_0)^{-1}(\mathcal{E}'(u) - \mathcal{E}'(x_0)) \| \le M_0 \| u - x_0 \| \le M_0 t^* < 1,$$

so $\mathcal{E}'(u)^{-1} \in L(Y, X)$ and

$$\| \mathcal{E}'(u)^{-1}\mathcal{E}'(x_0) \| \le \frac{1}{1 - M_0 \| u - x_0 \|}$$

follow from the Banach lemma on linear operators with inverses [1, 3-4,17]. Some estimates are obtained using the definitions, (H) and Lemma 3.2

$$\| L_n \| \le \frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|},$$

$$\| T_n^1 - I \| \le \frac{1}{2} \frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|} + \frac{|b|}{2} \left(\frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|} \right)^2$$

$$+ \frac{b^2}{2} \left(\frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|} \right)^3 = \overline{\gamma}_n,$$

$$z_n = x_n - \mathcal{E}'(x_n)^{-1} \mathcal{E}(x_n) + \mathcal{E}'(x_n)^{-1} \mathcal{E}(x_n) - T_n^1 \mathcal{E}'(x_n)^{-1} \mathcal{E}(x_n)$$

$$= y_n - (T_n^1 - I) \mathcal{E}'(x_n)^{-1} \mathcal{E}(x_n)$$

$$= y_n + (T_n^1 - I)(y_n - x_n),$$

thus,

$$\| z_n - y_n \| \le \overline{\gamma}_n \| y_n - x_n \| \le \gamma_n (s_n - t_n) \le u_n - s_n,$$

$$\| \mathcal{E}'(x_0)^{-1} \mathcal{E}(z_n) \| \le \frac{1}{2} M \| y_n - x_n \|^2$$

$$+ \frac{|b|}{2} M \| y_n - x_n \| \frac{M \| y_n - x_n \|^2}{1 - M_0 \| x_n - x_0 \|}$$

$$+ \frac{b^2}{2} M \| y_n - x_n \| \left(\frac{M \| y_n - x_n \|^2}{1 - M_0 \| x_n - x_0 \|} \right)^2 \| y_n - x_n \|$$

$$+ \int_0^1 v(\theta \| z_n - x_n \|) d\theta \| z_n - x_n \| \le \overline{\alpha}_n,$$

$$\| x_{n+1} - z_n \| \le \left(1 + \frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|} + \frac{|c| M \overline{\alpha}_n}{1 - M_0 \| x_n - x_0 \|)^2} \right) \frac{\overline{\alpha}_n}{1 - M_0 \| x_n - x_0 \|}$$

$$\le t_{n+1} - u_n,$$

$$\| y_{n+1} - x_{n+1} \| \le \frac{1}{1 - M_0 \| x_{n+1} - x_0 \|} \left[\frac{|c| M \bar{\alpha}_n^2}{(1 - M_0 \| x_n - x_0 \|)^2} \right.$$

$$+ M \| y_n - x_n \| \left(\frac{M \| y_n - x_n \|}{1 - M_0 \| x_n - x_0 \|} + \frac{|c| M \bar{\alpha}_n}{(1 - M_0 \| x_n - x_0 \|)^2} \right)$$

$$+ 2M \| y_n - x_n \| \ \| x_{n+1} - z_n \| + M \| z_n - y_n \| \ \| x_{n+1} - z_n \|$$

$$\left. + \int_0^1 v(\theta \| x_{n+1} - z_n \|) d\theta \| x_{n+1} - z_n \| \right]$$

$$\le \frac{\bar{\delta}_{n+1}}{1 - M_0 \| x_{n+1} - x_0 \|}$$

$$\le \frac{\bar{\delta}_{n+1}}{1 - M_0 t_{n+1}} \le s_{n+1} - t_{n+1}, \tag{21}$$

where we also used $\| y_n - x_n \| \le s_n - t_n$, $\| z_n - y_n \| \le u_n - s_n$, $\| x_{n+1} - z_n \| \le t_{n+1} - u_n$,

$$\| z_n - x_0 \| \le \| z_n - y_n \| + \| y_n - x_0 \| \le u_n - s_n + s_n - t_0 = u_n < t^*,$$

$$\| x_{n+1} - x_0 \| \le \| x_{n+1} - z_n \| + \| z_n - x_0 \| \le t_{n+1} - u_n + u_n - t_0 = t_{n+1} < t^*,$$

$$\| x_n + \theta(y_n - x_n) - x_0 \| \le (1 - \theta) \| x_n - x_0 \| + \theta \| y_n - x_0 \|$$

$$< (1 - \theta) t^* + \theta t^* = t^*,$$

$$\| z_n + \theta(x_{n+1} - z_n) - x_0 \| \le (1 - \theta) \| z_n - x_0 \| + \theta \| x_{n+1} - x_0 \|$$

$$< (1 - \theta) t^* + \theta t^* = t^*,$$

and

$$\| y_{n+1} - x_0 \| \le \| y_{n+1} - x_{n+1} \| + \| x_{n+1} - x_0 \|$$

$$\le s_{n+1} - t_{n+1} + t_{n+1} - t_0 = s_{n+1} < t^*,$$

so, $z_n, x_{n+1}, x_n + \theta(y_n - x_n), z_n + \theta(x_{n+1} - z_n), y_{n+1} \in U(x_0, t^*)$ and the induction for estimates is completed.

It follows that sequence $\{t_n\}$ is complete in X and as such it converges to some $x^* \in U[x_0, t^*]$. By letting $n \to \infty$ in the estimation (see (21))

$$\| \mathcal{E}'(x_0)^{-1}\mathcal{E}(x_{n+1}) \| \leq \delta_{n+1}$$

and using the continuity of $\mathcal{E}$ we obtain $\mathcal{E}(x^*) = 0$. Moreover, see (20) for the proof of (26).

□

Remark 3.6 The condition $U[x_0, t^*] \subset D$ can be replaced by $U[x_0, \frac{1}{M_0}] \subset D$ if conditions of Lemma 3.1 hold or $U[x_0, \frac{\eta}{1-m} \subset D]$ under conditions of Lemma 3.2 where $\frac{1}{M_0}$ and $\frac{\eta}{1-m}$ are given in closed form in contrast to t^*.

The uniqueness of the solution x^* result follows without necessarily using conditions of Theorem 2.2 or Theorem 2.3 or Theorem 3.5.

Proposition 3.7 Suppose $x^* \in U(x_0, \xi_0) \subset D$ is a simple solution of equation $\mathcal{E}(x) = 0$; Condition (H2) holds and there exists $\xi \geq \xi_0$ such that

$$M_0(\xi_0 + \xi) < 2. \tag{22}$$

Set $G = U[x_0, \xi] \cap D$. Then, the point x^* is the only solution of equation $\mathcal{E}(x) = 0$ in the set G.

Proof. Let $y^* \in G$ with $\mathcal{E}(y^*) = 0$. Define linear operator $Q = \int_0^1 \mathcal{E}'(x^* + \theta(y^* - x^*))d\theta$. By using (H2) and (22) we get in turn

$$\| \mathcal{E}'(x_0)^{-1}(Q - \mathcal{E}'(x_0)) \| \leq M_0 \int_0^1 ((1-\theta) \| x^* - x_0 \| + \theta \| y^* - x_0 \|) d\theta$$

$$\leq \frac{M_0}{2}(\xi_0 + \xi) < 1.$$

Therefore, $x^* = y^*$ is implied since $Q^{-1} \in L(Y, X)$ and $Q(y^* - x^*) = \mathcal{E}(y^*) - \mathcal{E}(x^*) = 0$.

□

4. Numerical applications

Three numerical examples further validate the theory.

Example 4.1 Let us consider a scalar function ϵ defined on the set $\Omega = [x_0 - (1 - p), x_0 + 1 - p]$ for $p \in (0, 1)$ by

$$\epsilon(x) = x^3 - p.$$

Choose $x_0 = r = r_1 = 1$. Then, we obtain the estimates $\eta = \frac{1-p}{3}$,

$$|\epsilon'(x) - \epsilon'(x_0)| = 3|x^2 - x_0^2|$$

$$\leq 3|x + x_0||x - x_0| \leq 3(|x - x_0||2|x_0|)|x - x_0|$$

$$= 3(1 - p + 2)|x - x_0| = 3(3 - p)|x - x_0|,$$

for each $x \in \Omega$, so $L_0 = (3 - p)$, $\Omega_0 = U(x_0, \frac{1}{L_0}) \cap \Omega = U(x_0, \frac{1}{L_0})$,

$$
\begin{aligned}
|\epsilon'(y)-\epsilon'(x)| &= 3|y^2-x^2| \\
&\leq 3|y+x||y-x| \leq 3(|y-x_0+x-x_0+2x_0)|y-x| \\
&= 3(|y-x_0|+|x-x_0|+2|x_0|)|y-x| \\
&\leq 3(\frac{1}{L_0}+\frac{1}{L_0}+2)|y-x| = 6(1+\frac{1}{L_0})|y-x|,
\end{aligned}
$$

for each $x, y \in \Omega$ and so $L_1 = 3(1+\frac{1}{L_0})$ and $L = \frac{1}{L_0^2}$.

The following Table 1, shows that the condition (4) is satisfied.

Hence, the convergence to $x^* = \sqrt{p}$ is established under weaker conditions than in [5].

Table 1. Sequence (4)

n	1	2	3	4	5	6
t_{n+1}	0.1995	0.2307	0.2324	0.2324	0.2324	0.2324
$L_0\, t_{n+1}$	0.5168	0.5974	0.6018	0.6018	0.6018	0.6018

Example 4.2 We already saw in Example 1.1 that the older conditions (A4), (B4), and (C4) do not hold. Hence, the convergence of the scheme (3) to the solution $x^* = 1$ cannot be assured with the previous approaches. However, the scheme (2) for $x_0 = 0.7$ converges to x^* after three iterations.

Example 4.3 Let us consider the nonlinear system

$$\epsilon(z) = 0,$$

where $z = (z_1, z_2)^T$

$$\epsilon(z) = (2z_1 - \frac{1}{9}z_1^2 - z_2, -z_1 + 2z_2 - \frac{1}{9}z_2^2)^T.$$

The derivatives are

$$\epsilon'(z) = \begin{pmatrix} 2-\frac{2}{9}z_1 & 1 \\ -1 & 2-\frac{2}{9}z_2 \end{pmatrix}$$

and

$$\epsilon''(z) = \left(\begin{array}{cc|cc} 2-\frac{2}{9}z_1 & 1 & 0 & 0 \\ -1 & 2-\frac{2}{9}z_2 & 0 & -\frac{2}{9} \end{array} \right).$$

Choose $z_0 = (11.4, 11.4)^T$. Then, the solution $z^* = (9, 9)^T$ is obtained after three iterations.

5. Conclusion

A two-fold finer semilocal convergence analysis for scheme (3) is presented with advantages as already stated in the introduction. Hence, the applicability of the scheme (3) is extended with the same or weaker conditions than before. This technique can be used to extend the applicability of other schemes analogously.

Conflict of interest

The authors declare no competing financial interest.

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