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Some Algebraic Properties of $\ell^p(\beta)$

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Abstract: In this paper, we consider a generalized Cauchy product $\Diamond$ on $\ell^p(\beta)$ and then we characterize some Banach algebra structures for $\ell^p(\beta)$. Also, some classic properties of $\Diamond$-multiplication operator $M_{\Diamond,z}$ on $\ell^p(\beta)$ will be investigated. In particular, we obtain the form of closed ideals of $(\ell^p(\beta),\Diamond)$.
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1. Introduction

Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta(0)=1$. For $1\leq p<\infty$ we consider the space of sequences $f=\{\hat{f}(n)\}$ with $\|f\|_{\beta}^{p}=\sum_{n=0}^{\infty}|\hat{f}(n)|^p\beta(n)^p<\infty$. We shall use the formal notation $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ whether or not the series converge for any complex values of z. Let $\ell^p(\beta)$ denote the space of such formal power series. Note that if $\lim_n\frac{\beta(n+1)}{\beta(n)}$ or $\liminf_n \beta(n)^{\frac{1}{n}}=1$, then $\ell^p(\beta)$ consists of functions analytic on the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ .The Hardy, Bergman and Dirichlet spaces can be viewed in this way when p = 2 and respectively $\beta(n)=1$, $\beta(n)=(n+1)^{-1/2}$ and $\beta(n)=(n+1)^{1/2}$. Sources on formal power series include [4, 1].

Let X be a Banach space and let $A\in B(X)$, the space of all bounded linear operators on X. Then $x\in X$ is called cyclic vector for A in X if $X=\overline{span}\{A^n x:n=0,1,2\cdots\}$. Also an operator A in $B(X)$ is called Unicellular on X if the set of its invariant closed subspaces, Lat(A), is linearly ordered by inclusion.

In section 2, we define a generalized Cauchy product $\Diamond$, under certain conditions, on $\ell^p(\beta)$ and then we show that the Banach space $\ell^p(\beta)$ with the generalized Cauchy product $\Diamond$ is a Banach algebra. Then we determine invertible elements and maximal ideal space of $(\ell^p(\beta),\Diamond)$. Also, we give a suffcient condition for the $\Diamond$-multiplication operator, $M_{\Diamond,z}$ acting on $\ell^p(\beta)$ to be Unicellular. This result, as usual, leads to a description of closed ideals of the algebra $\ell^p(\beta)$ and cyclic vectors of the $\Diamond$-multiplication operator $M_{\Diamond,z}$.

2. Some banach algebra structures for $\ell^p(\beta)$

Let $\{\delta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\delta_0=1$. For $1<p<\infty$, let q be the conjugate exponent to p. For each k, $M\in\mathbb{N}\cup\{0\}$, take

$$C_o:=\sup_{n\geq 0}\sum_{k=0}^{n}\left(\frac{\delta_n\beta(n)}{\delta_k\delta_{n-k}\beta(k)\beta(n-k)}\right)^q \tag{1}$$

and

$$b_{M,k}:=\sup_{n\geq M+1}\frac{\delta_{n+k}\beta(n+k)}{\delta_n\delta_k\beta(n)\beta(k)} \tag{2}$$

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Throughout this paper, we assume that $1 < p < \theta$, $C_o < \infty$ and $\lim b_{M,k} = 0$ when $M \to \infty$. Given arbitrary two functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $g(z) \sum_{n=0}^{\infty} \hat{g}(n) z^n$ of the space $\ell^p(\beta)$, define the following generalized Cauchy product series

$$f \lozenge g = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\delta_{m+n}}{\delta_n \delta_m} \hat{f}(n) \hat{g}(m) z^{m+n} \tag{3}$$

Note that if we set $\delta_n \equiv 1$, the generalized Cauchy product $\lozenge$ will be coincided to the ordinary Cauchy product.

Let $\ell^0(\beta)$ be the set of all formal power series. For each $f \in \ell^p(\beta)$, let $M_{\lozenge,f}$: $\ell^p(\beta) \to \ell^0(\beta)$ be defined by $M_{\lozenge,f}(g) = f \lozenge g$, be its corresponding $\lozenge$-multiplication operator. It is easy to see that

$$M_{\lozenge,z}(f) = \sum_{n=0}^{\infty} \frac{\delta_{n+1}}{\delta_n \delta_1} \hat{f}(n) z^{n+1} \text{ and } M_{\lozenge,z}^N(f) = \frac{\delta_N}{\delta_1^N} z^N \lozenge f,$$

for all $N \in \mathbb{N} \cup \{0\}$ and $f \in \ell^p(\beta)$.

Take $C = \sup_{n \geq 0} \frac{\beta(n+N)\delta_{n+N}}{\beta(n)\delta_n \delta_1^N}$. Since $b_{M,k} < \infty$ for each k, $M \in \mathbb{N} \cup \{0\}$, then $C < \infty$. It follows that

$$\begin{aligned} \| M_{\lozenge,z}^N f \|_\beta^p &= \sum_{n=0}^{\infty} \left(\frac{\delta_{n+N}}{\delta_n \delta_1^N}\right)^p | \hat{f}(n) |^p \ \beta(n+N)^p \\ &= \sum_{n=0}^{\infty} | \hat{f}(n) |^p \beta(n)^p \left(\frac{\delta_{n+N} \beta(n+N)}{\delta_n \beta(n) \delta_1^N}\right)^p \\ &\leq C^p \ \| f \|_\beta^p . \end{aligned}$$

Hence $\| M_{\lozenge,z}^N \| \leq C$. On the other hand, if we put $f_n(z) = z^n$, then $\| f_n \|_\beta = \beta(n)$ and $M_{\lozenge,z}^N(f_n) = \frac{\delta_{n+N}}{\delta_n \delta_1^N} z^{n+N}$ Therefore we have

$$\frac{\beta(n+N)\delta n + N}{\delta_n \delta_1^N} = \| M_{\lozenge,z}^N(f_n) \|_\beta \leq \| M_{\lozenge,z}^N \| \, \| f_n \|_\beta = \| M_{\lozenge,z}^N \| \beta(n).$$

This implies that $C \leq \| M_{\lozenge,z}^N \|$ and so $\| M_{\lozenge,z}^N \| = C$ [4]. In the next Theorem, we get that $\ell^p(\beta)$ is a unital commutative Banach algebra with respect to the generalized Cauchy product $\lozenge$.

Theorem 2.1 $(\ell^p(\beta), \lozenge)$ is a unital commutative Banach algebra.

Proof. It is easy to see that the constant function $f = 1$ is a unity for $(\ell^p(\beta), \lozenge)$. Hence if we prove that $M_{\lozenge,f}$ is a bounded operator on $\ell^p(\beta)$, then we get the result. To see this let $f, g \in \ell^p(\beta)$. Using (3), it is easy to see that

$$\widehat{(f \lozenge g)}(n) = \sum_{k=0}^{n} \frac{\delta_n}{\delta_k \delta_{n-k}} \hat{f}(k) \hat{g}(n-k) \tag{4}$$

By using Hölder inequality and (1), we have

$$\| M_{\Diamond,f}(g)\|_{\beta}^{p} = \sum_{n=0}^{\infty} |\widehat{(f \Diamond g)}(n)|^{p} \ \beta(n)^{p}$$

$$= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \frac{\delta_n}{\delta_k \delta_{n-k}} \hat{f}(k)\hat{g}(n-k) \right|^{p} \beta(n)^{p}$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} |\hat{f}(k)||\hat{g}(n-k)|\beta(k)|\beta(n-k) \right)^{p}$$

$$\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(|\hat{f}(k)|\beta(k)|\hat{g}(n-k)|\beta(n-k) \right)^{p} \right)^{\frac{p}{p}} \left(\sum_{k=0}^{n} \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k} \beta(k)\beta(n-k)} \right)^{q} \right)^{\frac{p}{q}}$$

$$\leq C_O^{\frac{p}{q}} \sum_{n=o}^{\infty} \sum_{k=0}^{n} |\hat{f}(k)|^{p} \beta(k)^{p} \, |\hat{g}(n-k)|^{p} \ \beta(n-k)^{p}$$

$$= C_O^{\frac{p}{q}} \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^{p} \ \beta(n)^{p} \right) \left(\sum_{n=0}^{\infty} |\hat{g}(n)|^{p} \ \beta(n)^{p} \right)$$

$$= C_O^{\frac{p}{q}} \ \| f \|_{\beta}^{p} \| g \|_{\beta}^{p}$$

Consequently, we get that

$\| M_{\Diamond,f}(g)\|_{\beta} = \| f \Diamond g \|_{\beta} \leq C_O^{\frac{1}{q}} \| f \|_{\beta} \| g \|_{\beta}$, and so $\| M_{\Diamond,f} \| \leq C_O^{\frac{1}{q}} \| f \| \beta$.

Here we give another condition instead of (1) and (2) under which $(\ell^{p}(\beta), \Diamond)$ is a unital commutative Banach algebra.

Remark 2.2 Suppose that there exist $N \in \mathbb{N}$ such that

$$\sum_{n,m \geq N+1} \frac{\delta_{n+m} \beta(n+m)}{\delta_n \delta_m \beta(n)\beta(m)} < \infty . \tag{5}$$

Then for every $f, g \in \ell^{1}(\beta)$, we have

$$f \Diamond g = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\delta_{m+n}}{\delta_n \delta_m} \hat{f}(n)\hat{g}(m) z^{m+n} = f(0)g + \frac{\hat{f}(1)}{\delta_1} \sum_{m=0}^{\infty} \frac{\delta_{m+1}}{\delta} g(m) z^{m+1} + \cdots$$

$$+ \frac{\hat{f}(N)}{\delta_N} \sum_{m=0}^{\infty} \frac{\delta_{m+N}}{\delta_m} \hat{g}(m) z^{m+N} + + g(0) \sum_{n=N+1}^{\infty} g(n) z^{n} + \frac{\hat{g}(1)}{\delta_1} \sum_{n=N+1}^{\infty} \frac{\delta_{n+1}}{\delta_n} \hat{f}(n) z^{n+1} + \cdots$$

$$+ \frac{\hat{g}(N)}{\delta_N} \sum_{n=0}^{\infty} \frac{\delta_{n+N}}{\delta_n} \hat{f}(n) z^{m+N} + \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{\delta_{m+n}}{\delta_n \delta_m} f(n) \hat{g}(m) z^{m+n}$$

Thus, we can write

$$f \lozenge g=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\delta_{m+n}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{m+n}=f(0)g+f(1)M_{z,\lozenge}(g)+\cdots\frac{\hat{f}(N)\delta_1^N}{\delta_N}M_{z,\lozenge}^N(g)+g(0)R_{N+1}(f)$$

$$+\hat{g}(1)R_{N+1}(M_{z,\lozenge}(f))+\cdots\frac{\hat{f}(N)\delta_1^N}{\delta_N}R_{N+1}(M_{z,\lozenge}^N(g))+\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\frac{\delta_{m+n}}{\delta_n\delta_m}\hat{f}(n)g(m)z^{m+n}$$

where $R_N(g):=\sum_{n=N}^{\infty}\hat{g}(n)z^n$. It follows that

$$\| f \lozenge g\|_\beta \le \left(2\sum_{k=0}^{N}\frac{\delta_1^k}{\beta(k)\delta_k}\| M_{z,\lozenge}^k\|+\sum_{n,m=N+1}^{\infty}\frac{\delta_{m+n}\beta(n+m)}{\delta_n\delta_m\beta(n)\delta(m)}\right)\| f\|_\beta\| g\|_\beta .$$

Thus, if we replace the condition (1) and (2) with (5), then $(\ell^1(\beta),\lozenge)$ is also a unital commutative Banach algebra. Now we give some technical lemmas that we will use them in the sequel.

Lemma 2.3a. Let $f\in\ell^p(\beta)$ and let λ be a non-zero complex number. If $\hat{f}(0)=0$, then $\lambda I-M_{\lozenge,f}$ has closed range, where *I* is the identity operator.

Proof. Let $\hat{f}(0)=0$. To show that $\lambda I-M_{\lozenge,f}$ has closed range, we only need to prove the $\lozenge$-multiplication operator $M_{\lozenge,f}$ is compact on $\ell^p(\beta)$. For $M\in\mathbb{N}$, define K_M on $\ell^p(\beta)$ by

$$K_M(g):=\sum_{m=0}^{M}\sum_{n=0}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{m+n} .$$

Since

$$\sum_{m=0}^{M}\sum_{n=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{n+m}=\sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}\left(\sum_{n=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_1^m}f(n)z^{n+m}\right)$$

$$=\sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}R_{M+1}(M_{\lozenge,z}^m(f)),$$

then we have

$$K_M(g)=\sum_{m=0}^{M}\sum_{n=0}^{M}\frac{\delta_{m+n}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{n+m}+\sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}R_{M+1}(M_{\lozenge,z}^m(f)),$$

and so K_M is a bounded and finite-rank operator on $\ell^p(\beta)$. Also, it is easy to verify that

$$\sum_{n=M+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{g}(n)\hat{f}(m)z^{n+m}=\sum_{n=2M+2}^{\infty}\left(\sum_{k=M+1}^{n-M-1}\frac{\delta n}{\delta_k\delta_{n-k}}g(k)f(n-k)\right)z^n .$$

Therefore

$$\|M_{\lozenge,f}(g)-K_M(g)\|_\beta = \|f \lozenge g-K_M(g)\|_\beta$$

$$=\left\|\sum_{n=M+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{g}(n)\hat{f}(m)z^{n+m}+\sum_{n=0}^{M}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}f(n)g(m)z^{n+m}\right\|_\beta$$

$$\leq\left\|\sum_{n=2M+2}^{\infty}\left(\sum_{k=M+1}^{n-M-1}\hat{g}(k)\hat{f}(n-k)\right)z^n\right\|_\beta+\frac{|\hat{f}(1)|}{\delta_1}\left(\sum_{m=M+1}^{\infty}\frac{\delta_{m+1}^p}{\delta_m^p}|g(m)|^p\,\beta(m+1)^p\right)^{\frac{1}{p}}.$$

Since

$$\frac{|\hat{f}(k)|}{\delta_k}\left(\sum_{m=M+1}^{\infty}\frac{\delta_{m+k}^p}{\delta_m^p}|\hat{g}(m)|^p\,\beta(m+k)^p\right)^{\frac{1}{p}}=|\hat{f}(k)|\,\beta(k)\left(\sum_{m=M+1}^{\infty}\frac{\delta_{m+k}^p\beta(m+k)^p}{\delta_m^p\delta_k^p\beta(k)^p\beta(m)^p}|\hat{g}(m)|^p\,\beta(m)^p\right)^{\frac{1}{p}}$$

$$\leq\|f\|_\beta\left(\sup_{m\geq M+1}\frac{\delta_{m+k}\beta(m+k)}{\delta_m\delta_k\beta(m)\beta(k)}\right)\left(\sum_{m=1}^{\infty}|\hat{g}(m)|^p\,\beta(m)^p\right)^{\frac{1}{p}}$$

$$\leq\|f\|_\beta\|g\|_\beta\,b_{M,k},$$

holds for every $0\leq k\leq M$, then we get that

$$\|M_{\lozenge,f}(g)-K_M(g)\|_\beta\leq\left(\sum_{n=2M+2}^{\infty}\left(\sum_{k=M+1}^{n-M-1}\frac{\delta_n\beta(n)}{\delta_k\delta_{n-k}\beta(k)\beta(n-k)}|\hat{g}(k)|\,\beta(k)\,|\hat{f}(n-k)|\,\beta(n-k)\right)^p\right)^{\frac{1}{p}}$$

$$+\|f\|_\beta\|g\|_\beta\,(b_{M,1}+b_{M,2}+\cdots b_{M,M})\overset{\text{Hölder}}{\leq}\left(\sum_{n=2M+2}^{\infty}\left(\sum_{k=M+1}^{n-M-1}|\hat{f}(k)|^p\,\beta(k)^p\,|\hat{g}(n-k)|^p\,\beta(n-k)^p\right)\right.$$

$$\left.\times\left(\sum_{k=M+1}^{n-M-1}\left(\frac{\delta_n\beta(n)}{\delta_k\delta_{n-k}\beta(n-k)\beta(k)}\right)^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}+\|f\|_\beta\|g\|_\beta\,(b_{M,1}+b_{M,2}+\cdots+b_{M,M})$$

$$\leq C_O^{\frac{1}{q}}\|g\|_\beta\left(\sum_{n=M}^{\infty}|\hat{f}(n)|^p\,\beta(n)^p\right)^{\frac{1}{p}}+\|f\|_\beta\|g\|_\beta\,(b_{M,1}+b_{M,2}+\cdots b_{M,M})$$

Hence by (2), $\|M_{\lozenge,f}-K_M\|_\beta\to 0$, when $M\to\infty$. This implies that $M_{\lozenge,f}$ is the norm limit of a sequence of finite-rank operators and therefore compact.

Lemma 2.3b Let condition (5) be satisfied. If $f\in\ell^1(\beta)$ and λ is a nonzero complex number and if, $\hat{f}(0)=0$ then $\lambda I-M_{\lozenge,f}$ has closed range, where *I* is the identity operator.

Proof. Let $\hat{f}(0)=0$. To show that $\lambda I-M_{\lozenge,f}$ has closed range, we only need to prove the $\lozenge$-multiplication operator $M_{\lozenge,f}$ is compact on $\ell^1(\beta)$. For $M\in\mathbb{N}$ define K_M on $\ell^1(\beta)$ by

$$K_M(g)=\sum_{m=0}^{M}\sum_{n=0}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{n+m}.$$

Since

$$\sum_{m=0}^{M}\sum_{n=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{n+m} = \sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}\left(\sum_{n=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_1^m}\hat{f}(n)z^{n+m}\right)$$
$$= \sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}R_{M+1}(M_{\lozenge,z}^m(f)),$$

then

$$K_M(g) = \sum_{m=0}^{M}\sum_{n=0}^{M}\frac{\delta_{m+n}}{\delta_n\delta_m}\hat{f}(n)\hat{g}(m)z^{n+m} + \sum_{m=0}^{M}\frac{\hat{g}(m)\delta_1^m}{\delta_m}R_{M+1}(M_{\lozenge,z}^m(f)),$$

and so K_M is a bounded and finite-rank operator on $\ell^1(\beta)$.

Therefore we have

$$\| M_{\lozenge,f}(g) - K_M(g)\|_\beta = \| f \lozenge g - K_M(g)\|_\beta$$
$$= \left\| \sum_{n=M+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}\hat{g}(n)\hat{f}(m)z^{n+m} + \sum_{n=1}^{M}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}}{\delta_n\delta_m}f(n)g(m)z^{n+m}\right\|_\beta$$
$$\leq \left(\sum_{n=M+1}^{\infty}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}\beta(n+m)}{\delta_n\delta_m\beta(n)\beta(m)} + \sum_{n+1}^{M}\sum_{m=M+1}^{\infty}\frac{\delta_{n+m}\beta(n+m)}{\delta_n\delta_m\beta(n)\beta(m)}\right)\| f\|_\beta \| g\|_\beta .$$

Hence by (5), $\| M_{\lozenge,f} - K_M\|_\beta \to 0$ when $M \to \infty$. Thus, $M_{\lozenge,f}$ is the limit in the norm of a sequence of the finite-rank operators and therefore compact. Here we provide some suffcient conditions under which $M_{\lozenge,f}$ is one-to-one.

Lemma 2.4 Let $f \in \ell^p(\beta)$ and $\hat{f}(0) \neq 0$. Then $M_{\lozenge,f}$ is one to one.

Proof. Let $g \in \ell^p(\beta)$ and $M_{\lozenge,f}(g) = f \lozenge g = 0$. Then $\widehat{(f \lozenge g)}(n) = 0$, for all $n \in \mathbb{N} \cup \{0\}$. Hence, by (4) we get that

$$\widehat{(f \lozenge g)}(0) = \hat{f}(0)\hat{g}(0) = 0 \Rightarrow g(0) = 0$$

$$\widehat{(f \lozenge g)}(1) = \hat{f}(0)\hat{g}(1) + f(1)g(0) = 0 \Rightarrow g(1) = 0,$$

and so on. Thus, we get $\hat{g}(0) = g(1) = g(2) = \cdots = 0$, and so $g = 0$.

Now we can get an equivalent condition to invertibility of elements of $\ell^p(\beta)$ with respect to $\lozenge$.

Theorem 2.5 If $f \in \ell^p(\beta)$ then f is $\lozenge$-invertible if and only if $\hat{f}(0) \neq 0$.

Proof. Suppose that $\hat{f}(0) \neq 0$. Put $h = f - \hat{f}(0)$. $M_{\lozenge,f} = \hat{f}(0)I + M_{\lozenge,h}$ with $\hat{h}(0) = 0$. By the above lemmas and the open mapping theorem, $M_{\lozenge,f}$: $M_{\lozenge,f}(\ell^p(\beta)) \to \ell^p(\beta)$ is bounded. On the other hand, since $M_{\lozenge,f}$ is compact, then the residual spectrum of $M_{\lozenge,f}$ is empty, and so $M_{\lozenge,f}^{-1} \in B(\ell^p(\beta))$. Conversely, suppose that f is invertible. Then there exists $g \in \ell^p(\beta)$ such that $f \lozenge g = 1$ and so $\hat{f}(0)\hat{g}(0) = \widehat{(f \lozenge g)}(0) = 1$. This implies that $\hat{f}(0) \neq 0$.

By the above observations, we obtain the maximal ideal space of $(\ell^p(\beta), \lozenge)$.

Corollary 2.6 The maximal ideal space of $(\ell^p(\beta), \lozenge)$ consists of one homomorphism $\varphi(f) = \hat{f}(0)(f \in \ell^p(\beta))$.

Proof. Let $\mathfrak{M}(\ell^p(\beta))$ be the maximal ideal space of $\ell^p(\beta)$ with generalized Cauchy product $\lozenge$. Recall that for each $f \in \ell^p(\beta)$, $\lambda \in \sigma(f)$, the spectrum of f, if and only if $f - \lambda$ is not invertible. By Theorem 2.5, $\lambda \in \sigma(f)$ if and only if $\widehat{(f-\lambda)}(0) = 0$ or equivalently $\hat{f}(0) = \lambda$. Since $\sigma(f) = \{\varphi(f) : \varphi \in \mathfrak{M}(\ell^p(\beta))\}$, thus $\varphi \in \mathfrak{M}(\ell^p(\beta))$ if and only if $\varphi(f) = \hat{f}(0)$, for each $f \in \ell^p(\beta)$.

Yousefi in [3] gives, some suffcient conditions for the usual multiplication operator M_z on $\ell^p(\beta)$ to be Unicellular. For study the related topics, see [2, 5]. In the following we give a suffcient condition for the $\lozenge$-multiplication operator $M_{\lozenge,z}$ acting on $\ell^p(\beta)$ to be Unicellular.

Let $\ell_0^p(\beta) = \ell^p(\beta)$, $\ell_\infty^p(\beta) = \{0\}$ and let for $i \in \mathbb{N} \cup \{0\}$, $\ell_i^p(\beta) = \{\sum_{n \geq i} c_n z^n \in \ell^p(\beta)\}$. Given two functions $f(z) = \sum_{n=i}^{\infty}\hat{f}(n)z^n$ and $g(z) = \sum_{n=i}^{\infty}\hat{g}(n)z^n$ of the subspace $\ell_i^p(\beta)$. Let $\lozenge_i$ be the restriction of generalized Cauchy product $\lozenge$ on $\ell_i^p(\beta)$ defined as follows:

$$(f \lozenge_i g)(z) := \sum_{n,m \geq i} \frac{\delta_{n+m-i}}{\delta_n \delta_m} \hat{f}(n) \hat{g}(m) z_{n+m-i}.$$

For each n, k, $M \in \mathbb{N} \cup \{0\}$, define

$$C_i := \sup_{n \geq i} \sum_{k=i}^{n} \left(\frac{\delta_n \beta(n)}{\delta_k \delta_{n-k+i} \beta(k) \beta(n-k+i)} \right)^q,$$

and

$$b^i_{M,k} := \sup_{n \geq M+i+1} \frac{\delta_{n+k} \beta(n+k)}{\delta_n \delta_{k+i} \beta(n) \beta(k+i)}.$$

Note that when $i = 0$, these are coincide with (1) and (2) respectively. For $f \in \ell^p_i(\beta)$, put

$$M_{\lozenge_i}, f(g) := f \lozenge_i g, \quad g \in \ell^p_i(\beta).$$

Then for each $n \in \mathbb{N} \cup \{0\}$, we have

$$M^n_{\lozenge,z}(f) = \frac{\delta_n}{\delta_1^n} z^n \lozenge f = \frac{\delta_{n+i}}{\delta_1^n} z^{n+i} \lozenge_i f = M_{\lozenge_i,f}\left(\frac{\delta_{n+i}}{\delta_1^n} z^{n+i}\right) \tag{6}$$

It follows that for each $i \in \mathbb{N} \cup \{0\}$, $\ell^p_i(\beta)$ is an invariant subspace for $M_{\lozenge,z}$. Let $\|\cdot\|_{\beta,i}$ be the restriction of $\|\cdot\|_\beta$ on $\ell^p_i(\beta)$. According to the procedure used in the proof of the Theorem 2.1, we get that

$$\widehat{(f \lozenge_i g)}(n) = \sum_{k=i}^{n} \frac{\delta_n}{\delta_k \delta_{n-k+i}} \hat{f}(k) \hat{g}(n-k+i),$$

and $\| f \lozenge_i g \|_{\beta,i} \leq C_i^{\frac{1}{q}} \| f \|_{\beta}, {}_i\| g \|_{\beta,i}$. Thus $\ell^p_i(\beta)$ is a Banach algebra with multiplication $\lozenge_i$ and unity $\delta_i z^i$.

By these observations, we obtain the Lat$(M_{\lozenge,z})$.

Theorem 2.7 Let $C_i < \infty$ and let $\lim b^i_{M,k} = 0$ when $M \to \infty$. Then, Lat$(M_{\lozenge,z}) = \{\ell^p_i(\beta) : i \geq 0\}$, and so $M_{\lozenge,z}$ is an Unicellular operator on $(\ell^p(\beta), \lozenge)$.

Proof. Note that the set $\{\ell^p_i(\beta) : i \geq 0\}$ is linearly ordered by inclusion.

For $f \in \ell^p(\beta)$, put

$$E(f) = \overline{span}\{f, M_{\lozenge,z}(f), M^2_{\lozenge,z}(f), M^3_{\lozenge,z}(f), \ldots\}.$$

Since $E(f)$ is an invariant subspace for $M_{\lozenge,z}$, the operator $M_{\lozenge,z}$ is Unicellular in the Banach algebra $\ell^p(\beta)$ if and only if for all nonzero $f \in \ell^p(\beta)$, $E(f)$ is equal to $\ell^p_i(\beta)$ for some $i = 0, 1, 2, \ldots i(f)$.

Now, we show that equality of $E(f)$ with $\ell^p_i(\beta)$ is equivalent to the condition $\hat{f}(i) \neq 0$. To see this note that by (6) we have

$$E(f) = \overline{span}\{M^n_{\lozenge,z}(f) : n \geq 0\} = \overline{span}\left\{M_{\lozenge,f}\left(\frac{\delta_{n+i}}{\delta_1^n} z^{n+i}\right) : n \geq 0\right\} = \overline{M_{\lozenge_i,f}(\ell^p_i(\beta))},$$

and so

$$E(f) = \ell^p_i(\beta) \Leftrightarrow \overline{M_{\lozenge_i,f}(\ell^p_i(\beta))} = \ell^p_i(\beta).$$

Now, we show that

$$\overline{M_{\lozenge_i,f}(\ell^p_i(\beta))} = \ell^p_i(\beta) \Leftrightarrow \hat{f}(i) \neq 0.$$

Indeed, $\overline{M_{\Diamond_{i,f}}(\ell_i^p(\beta))} = \ell_i^p(\beta)$ then there exists a sequence of $\{f_n\} \subseteq \ell_i^p(\beta)$ such that $f \Diamond_i f_n \to \delta_i z^i$ and so $\frac{1}{\delta_i}\hat{f}(i) f_n \to \delta_i$ as $n \to \infty$, which implies that $\hat{f}(i) \neq 0$.

Conversely, if $\hat{f}(i) \neq 0$, then it is suffcient prove that $M_{\Diamond_{i,f}}$ is an invertible operator in $\ell_i^p(\beta)$ which will imply that $\overline{M_{\Diamond_{i,f}}(\ell_i^p(\beta))} = \ell_i^p(\beta)$, as desired. Put $h = f - \hat{f}(i)z^i$. Then $M_{\Diamond_{i,f}} = \frac{\hat{f}(i)}{\delta_i} I + M_{\Diamond_{i,h}}$.By the same argument in the proof of the Lemmas 2.3 and 2.4, it is suffcient to show that $M_{\Diamond_{i,h}}$ and $M_{\Diamond_{i,f}}$ are compact and one-to-one operators respectively in $\ell_i^p(\beta)$. For this purpose define

$$K_M^i(g) = \sum_{m=i}^{i+M}\sum_{n=i}^{i+M} \frac{\delta_{m+n-i}}{\delta_n \delta_m} \hat{h}(n)\hat{g}(m) z^{n+m-i} + \sum_{m=i}^{i+M} \frac{\hat{g}(m)\delta_1^{m-i}}{\delta_m} R_{M+1}(M_{\Diamond,z}^{m-i}(h)),$$

for every integer $M \in \mathbb{N}$ and $g \in \ell_i^p(\beta)$. By an argument similar to the proof of the Lemma 2.3, K_M^i is a finite-rank operator and we can show that

$$\| M_{\Diamond_{i,h}}(g) - K_M^i(g) \|_{\beta,i} =$$

$$\leq C_i^{\frac{1}{q}} \| g \|_{\beta,i} \left(\sum_{n=M+i+1}^{\infty} | \hat{h}(n) |^p \beta(n)^p \right)^{\frac{1}{p}} + \| h \|_{\beta,i} \| g \|_{\beta,i} (b_{M,1}^i + b_{M,2}^i + \cdots + b_{M,M}^i)$$

Thus, $\| M_{\Diamond_{i,h}} - K_M^i \| \to 0$ when $M \to \infty$, and hence $M_{\Diamond_{i,h}}$ is a compact operator. Now, let $g \in \ell_i^p(\beta)$ and $M_{\Diamond_{i,f}}(g) = f \Diamond_i g = 0$. Then, for all $n \geq i$, $\widehat{(f \Diamond_i g)}(n) = 0$. Hence we get that

$$\widehat{(f \Diamond_i g)}(i) = \frac{1}{\delta_i} \hat{f}(i)\hat{g}(i) = 0 \Rightarrow g(i) = 0$$

$$\widehat{(f \Diamond_i g)}(i+1) = \frac{1}{\delta_i} \hat{f}(i)\hat{g}(i+1) + \frac{1}{\delta_i} f(i+1) g(i) = 0 \Rightarrow g(i+1) = 0,$$

and so on. It follows that $\hat{g}(i) = g(i+1) = g(i+2) = \cdots = 0$; i.e., $g = 0$. This completes the proof of the theorem.

Corollary 2.8 If we set $\{\delta_n = 1\}$ in the last theorem, then we will have a suffcient condition for multiplication operator M_z on $\ell^p(\beta)$ to be unicellular. This condition is different from given conditions by B. Yousefi in [3].

Corollary 2.9 Let $f \in \ell^p(\beta)$. Then *f* is a cyclic vector for $M_{\Diamond,z}$ if and only if $\hat{f}(0) \neq 0$.

Proof. Let $f \in \ell^p(\beta)$. Then we have

$$\overline{span}\left\{ M_{\Diamond,z}^n(f) : n \geq 0 \right\} = \overline{span}\left\{ M_{\Diamond,f}\left(\frac{\delta_n}{\delta_1^n} z^n\right) : n \geq 0 \right\}.$$

If $\hat{f}(0) \neq 0$, then by Theorem 2.5, $M_{\Diamond,f}$ is an invertible operator on $\ell^p(\beta)$, and $\overline{span}\left\{ M_{\Diamond,z}^n(f) : n \geq 0 \right\} = \ell^p(\beta)$ which implies that *f* is a cyclic vector for $M_{\Diamond,z}$. Conversely, suppose *f* is a cyclic vector for $M_{\Diamond,z}$. Then there exists sequence $\{f_n\} \subseteq \ell^p(\beta)$ such that $\| f_n \Diamond f - 1 \|_\beta \to 0$. This implies that $\hat{f}_n(0) f(0) \to 1$, and so $\hat{f}(0) \neq 0$.

In the following theorem, we characterize the form of all closed ideals of the Banach algebra $(\ell^p(\beta), \Diamond)$.

Theorem 2.10 Let $i \in \mathbb{N} \cup \{0\}$, $C_i < \infty$ and let $\lim b_{M,k}^i = 0$ when $M \to \infty$. Then the closed ideals of $(\ell^p(\beta), \Diamond)$ are exactly of the form $\ell^p(\beta)$.

Proof. For any $i \in \mathbb{N} \cup \{0\}$, it is easy to see that $\ell^p(\beta)$ is a closed ideal of $(\ell^p(\beta), \Diamond)$. Let *K* be an arbitrary closed ideal of $(\ell^p(\beta), \Diamond)$. Then for each $f \in K$, $z \Diamond f \in K$ and so $M_{\Diamond,z} \subseteq K$. Now, by Theorem 2.7, $K = \ell_i^p(\beta)$ for some $i \in \mathbb{N} \cup \{0\}$. This completes the proof.

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