Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

On the Order of Convergence and the Dynamics of Werner-King's Method

Santhosh George[1*], **Ioannis K. Argyros**[2], **Ajil Kunnarath**[1], **Padikkal Jidesh**[1]

[1]Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangaluru-575025, India
[2]Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
Email:sgeorge@nitk.edu.in

Received: 8 November 2022; **Revised:** 14 February 2023; **Accepted:** 24 February 2023

Abstract: In this paper, we present the local convergence analysis of Werner-King's method to approximate the solution of a nonlinear equation in Banach spaces. We establish the local convergence theorem under conditions on the first and second Fréchet derivatives of the operator involved. The convergence analysis is not based on the Taylor expansions as in the earlier studies (which require the assumptions on the third order Fréchet derivative of the operator involved). Thus our analysis extends the applicability of Werner-King's method. We illustrate our results with numerical examples. Moreover, the dynamics and the basins of attraction are developed and demonstrated.

***Keywords*:** Werner-King's method, Taylor expansion, Fréchet derivative, order of convergence

MSC: 47H99, 49M15, 65J15, 65D99, 65G99

1. Introduction

For approximately solving the nonlinear equation

$$f(x) = 0$$

where $f : \Omega \subset \mathbb{R}^k \to \mathbb{R}^k$ where Ω is an open convex set, Werner [1] (Also see King [2]) considered the iterative method defined for $n = 0, 1, 2, 3, \ldots$ by

DOI: https://doi.org/10.37256/cm.4120232145

given $x_0, y_0 \in \Omega$, let

$$x_{n+1} = x_n - f'(\frac{x_n + y_n}{2})^{-1} f(x_n)$$

$$y_{n+1} = x_{n+1} - f'(\frac{x_n + y_n}{2})^{-1} f(x_{n+1}). \tag{1}$$

The method is extensively studied in [1-6], and is of convergence order $1 + \sqrt{2}$. Recall that iterative method x_n is said to converge to x^* with order p, if there exist a nonzero constant C such that

$$\lim_{n\to\infty} \frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|^p} = C.$$

The convergence order of method (1) is proved in [1, 3-6] using Taylor expansion. Hence it requires assumption on the derivative of operator up to an order three. In this paper, we study the iterative method (1) for solving nonlinear equation

$$F(x) = 0, \tag{2}$$

where $F : \Omega \subset D(F) \subset X \to Y$ is a Fréchet differentiable operator between the Banach spaces X and Y and Ω is an open convex set. In fact, we consider the iterative method defined for $n = 1, 2, 3, \ldots$ by

given $x_0 \in \Omega$, let

$$y_0 = x_0 - F'(x_0)^{-1} F(x_0)$$

$$x_{n+1} = x_n - F'(\frac{x_n + y_n}{2})^{-1} F(x_n)$$

$$y_{n+1} = x_{n+1} - F'(\frac{x_n + y_n}{2})^{-1} F(x_{n+1}). \tag{3}$$

Our convergence analysis is not based on the Taylor expansion and assumptions only on the first and second derivatives of F, are used to obtain the convergence order $1 + \sqrt{2}$.

Convergence of method (3) is proved in ([7, Chapter 32]) by using assumption only on the first derivative of F. But in [7], convergence order was not proved, instead the author of [7], used the Approximate Computational Order of Convergence (ACOC) and the Computational Order of Convergence (COC). For an iterative sequence, the ACOC is defined as

$$\gamma_1 = ln(\frac{\| x_{n+1} - x_n \|}{\| x_n - x_{n-1} \|}) / ln(\frac{\| x_n - x_{n-1} \|}{\| x_{n-1} - x_{n-2} \|}).$$

and COC is defined as

$$\gamma_2 = ln(\frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|}) / ln(\frac{\| x_n - x^* \|}{\| x_{n-1} - x^* \|}).$$

Further, we extended the method (2) to the other two methods of order $\frac{3+\sqrt{17}}{2}$ and $2+\sqrt{6}$, respectively using the technique in Cordero et al. [8-9].The new methods are defined for $n = 0, 1, 2, \ldots$ as follows:

$$y_0 = x_0 - F'(x_0)^{-1} F(x_0)$$

$$z_{n+1} = x_n - F'(\frac{x_n + y_n}{2})^{-1} F(x_n)$$

$$y_{n+1} = z_{n+1} - F'(\frac{x_n + y_n}{2})^{-1} F(z_{n+1})$$

$$x_{n+1} = z_{n+1} - F'(y_n)^{-1} F(z_{n+1}) \tag{4}$$

and

$$y_0 = x_0 - F'(x_0)^{-1} F(x_0)$$

$$z_{n+1} = x_n - F'(\frac{x_n + y_n}{2})^{-1} F(x_n)$$

$$y_{n+1} = z_{n+1} - F'(\frac{x_n + y_n}{2})^{-1} F(z_{n+1})$$

$$x_{n+1} = z_{n+1} - F'(z_{n+1})^{-1} F(z_{n+1}). \tag{5}$$

Throughout the paper we have $B(x, \lambda) = \{y \in X : \|y - x\| < \lambda\}$ and $\overline{B}(x, \lambda) = \|y \in X : \|y - x\| \leq \lambda\}$.

The rest of the paper is arranged as follows. In Section 2, Section 3 and Section 4, we provide the convergence analysis of methods (3), (4) and (5), respectively. Numerical examples are given in Section 5. The dynamics and the basins of attraction are developed in Section 6. The paper ends with conclusions in Section 7.

2. Convergence analysis of (3)

Our analysis is based on the following assumptions:

(a1) x^* is a simple solution of (2) and $F'(x^*)^{-1} \in L(Y, X)$.

(a2) $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \forall x, y \in \Omega$.

(a3) $\|F'(x^*)^{-1} F''(y)\| \leq L_1, \forall y \in \Omega$.

(a4) $\|F'(x^*)^{-1}(F''(x) - F''(x^*)\| \leq L_2\|x - x^*\|$

Let $\varphi, \varphi_1, k, k_1 : [0, \frac{1}{L}) \to \mathbb{R}$ be a continuous nondecreasing functions define by

$$\varphi(t) = \frac{1}{4(1-Lt)}\left(\frac{LL_1}{1-Lt} + \frac{3L_2}{2}\right),$$

$$k(t) = \varphi(t)t^2 - 1,$$

$$\varphi_1(t) = \frac{1}{2(1-Lt)}\left[\frac{11L_2}{24} + \frac{3LL_1}{2(1-Lt)}\right]$$

and $k_1(t) = \varphi_1(t)t^2 - 1$. Then, since $k(0) = -1 < 0$, $k(t) \to \infty$ as $t \to \frac{1}{L}^-$, there exist a least $r_1 \in (0, \frac{1}{L})$ such that $k(r_1) = 0$. Similarly, there exists a least $r_2 \in (0, \frac{1}{L})$ such that $k_1(r_2) = 0$.

Let

$$r < \min\{1, \frac{2}{5L}, r_1, r_2\}. \tag{6}$$

Theorem 1 Suppose the conditions (a1)-(a4) hold. Then, the sequence $\{x_n\}$ defined by (3), starting from $x_0 \in B(x^*, r)$ is well defined and remains in $\bar{B}(x^*, r)$ for $n = 0, 1, 2, \ldots$ and converges to a solution x^* of (2). Moreover, we have the following estimates for $n = 1, 2, \ldots$

$$\| y_n - x^* \| \leq \frac{3L}{2(1-Lr)} \| x_n - x^* \| \; \| x_{n-1} - x^* \| \tag{7}$$

and

$$\| x_{n+1} - x^* \| \leq \varphi_1(r) \| x_n - x^* \|^2 \| x_{n-1} - x^* \|. \tag{8}$$

Proof. The proof is by induction. First, we shall prove that for $x, y \in B(x^*, r)$, $F'(\frac{x+y}{2})$ and $F'(x)$ are invertible. Note that by (a2) we get

$$\| F'(x^*)^{-1}(F'(\frac{x+y}{2}) - F'(x^*)) \| \leq L \| \frac{x+y}{2} - x^* \|$$

$$\leq \frac{L}{2}(\| x - x^* \| + \| y - x^* \|) \leq Lr < 1.$$

Hence, by Banach Lemma on invertible operators [10]

$$\| F'(\frac{x+y}{2})^{-1} F'(x^*) \| \leq \frac{1}{1-Lr}. \tag{9}$$

Similarly one can prove

$$\| F'(x)^{-1} F'(x^*) \| \leq \frac{1}{1 - L \| x - x^* \|}. \tag{10}$$

By the mean value theorem one can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + t(x_0 - x^*))dt(x_0 - x^*).$$

Thus, by (3), (10) and (a2), we obtain

$$\begin{aligned}
\| y_0 - x^* \| &\leq \| F'(x_0)^{-1} \int_0^1 [F'(x_0) - F'(x^* + t(x_0 - x^*))]dt(x_0 - x^*) \| \\
&\leq \| F'(x_0)^{-1} F'(x^*) \| \times \| \int_0^1 F'(x^*)^{-1} [F'(x_0) - F'(x^* + t(x_0 - x^*))]dt(x_0 - x^*) \| \\
&\leq \frac{L}{2(1 - L \| x_0 - x^* \|)} \| x_0 - x^* \|^2 \qquad (11) \\
&\leq \| x_0 - x^* \| < r. \qquad (12)
\end{aligned}$$

The last step follows from the relation $\frac{L}{2(1-Lr)} \| x_0 - x^* \| \leq \frac{L}{2(1-Lr)} r < \frac{3Lr}{2(1-Lr)} < 1$, and hence $y_0 \in B(x^*, r)$. Moreover, we have

$$\begin{aligned}
x_1 - x^* &= x_0 - x^* - F'\left(\frac{x_0 + y_0}{2}\right)^{-1} \int_0^1 F'(x^* + t(x_0 - x^*))dt(x_0 - x^*) \\
&= F'\left(\frac{x_0 + y_0}{2}\right)^{-1} \int_0^1 \left[F'\left(\frac{x_0 + y_0}{2}\right) - F'(x^* + t(x_0 - x^*))\right]dt(x_0 - x^*) \\
&= F'\left(\frac{x_0 + y_0}{2}\right)^{-1} \int_0^1 \int_0^1 F''\left(x^* + t(x_0 - x^*) + \theta\left(\frac{x_0 + y_0}{2} - x^* - t(x_0 - x^*)\right)\right)d\theta \\
&\quad \times \left(\frac{x_0 + y_0}{2} - x^* - t(x_0 - x^*)\right)dt(x_0 - x^*).
\end{aligned}$$

Therefore, it follows in turn

$$\begin{aligned}
\| x_1 - x^* \| &= \| F'\left(\frac{x_0 + y_0}{2}\right)^{-1} \int_0^1 \int_0^1 F''\left((x^* + t(x_0 - x^*)) + \theta\left(\frac{x_0 + y_0}{2} - x^* - t(x_0 - x^*)\right)\right)d\theta \\
&\quad \times \left(\frac{x_0 + y_0 - 2x^* - 2t(x_0 - x^*)}{2}\right)dt(x_0 - x^*) \|
\end{aligned}$$

$$= \frac{1}{2}\| F'\left(\frac{x_0+y_0}{2}\right)^{-1}\int_0^1\int_0^1 F''((x^*+t(x_0-x^*))+\theta(\frac{x_0+y_0}{2}-x^*-t(x_0-x^*)))d\theta$$

$$\times\left((1-2t)(x_0-x^*)+(y_0-x^*)\right)dt(x_0-x^*)\|$$

Let $\eta := (x^*+t(x_0-x^*))+\theta(\frac{x_0+y_0}{2}-x^*-t(x_0-x^*))$. Then

$$\| x_1-x^*\| = \frac{1}{2}\left\|\left[F'\left(\frac{x_0+y_0}{2}\right)^{-1}\left(\int_0^1\int_0^1 F''(\eta)d\theta(y_0-x^*)dt(x_0-x^*)\right.\right.\right.$$

$$\left.\left.\left.+\int_0^1\int_0^1 F''(\eta)d\theta(1-2t)dt(x_0-x^*)^2\right]\right\|\| x_0-x^*\|$$

$$\leq \frac{1}{2}\left\|F'\left(\frac{x_0+y_0}{2}\right)^{-1}F'(x^*)\right\|\left[\int_0^1\int_0^1\| F'(x^*)^{-1}F''(\eta)\| d\theta dt\| y_0-x^*\|\ \| x_0-x^*\|\right.$$

$$\left.+\left\|\int_0^1\int_0^1 F'(x^*)^{-1}(F''(\eta)-F''(x^*)+F''(x^*))(1-2t)d\theta dt(x_0-x^*)^2\right\|\right]$$

$$\leq \frac{1}{2}\frac{L_1}{1-L\| x_0-x^*\|}\left[\| y_0-x^*\|\ \| x_0-x^*\|\right.$$

$$+\left\|\int_0^1\int_0^1 F'(x^*)^{-1}(F''(\eta)-F''(x^*))(1-2t)d\theta dt(x_0-x^*)^2\right\|$$

$$\left.+\left\|\int_0^1\int_0^1 F'(x*)^{-1}F''(x^*)(1-2t)d\theta dt(x_0-x^*)^2\right\|\right]$$

$$\leq \frac{1}{2(1-L\| x_0-x^*\|)}\left[L_1\| y_0-x^*\|\ \| x_0-x^*\|\right.$$

$$\left.+L_2\int_0^1\int_0^1|1-2t|\|\eta-x^*\|\ \| x_0-x^*\|^2\ d\theta dt\right]$$

$$\leq \frac{1}{2(1-L\| x_0-x^*\|)}\left[\frac{L_1}{1-L\| x_0-x^*\|}\| x_0-x^*\|^3\right.$$

$$\left.+L_2\int_0^1\int_0^1|1-2t|(|t(1-\theta)|+|\theta|)\| x_0-x^*\|^3\ d\theta dt\right]$$

$$\le \frac{1}{4(1-L\|x_0-x^*\|)}\left[\frac{LL_1}{1-L\|x_0-x^*\|}+\frac{3L_2}{2}\right]\|x_0-x^*\|^3 \tag{13}$$

$$\le \varphi(\|x_0-x^*\|)\|x_0-x^*\|^3 . \tag{14}$$

Note that by (13), we have $\|x_1 - x^*\| \le \|x_0 - x^*\| < r$ (since $\varphi(\|x_0 - x^*\|)\|x_0 - x^*\|^2 < 1$).
Thus, the iterate $x_1 \in B(x^*, r)$.
By the mean value theorem one can write

$$F(x_1) = F(x_1) - F(x^*) = \int_0^1 F'(x^* + t(x_1 - x^*))dt(x_1 - x^*).$$

Hence, we have

$$y_1 - x^* = x_1 - x^* - F'(\frac{x_0+y_0}{2})^{-1}\int_0^1 F'(x^* + t(x_1 - x^*)dt(x_1 - x^*)$$

$$= F'(\frac{x_0+y_0}{2})^{-1}(F'(\frac{x_0+y_0}{2}) - \int_0^1 F'(x^* + t(x_0 - x^*)dt)(x_1 - x^*)$$

and

$$\begin{aligned}
\|y_1 - x^*\| &\le \|F'(\frac{x_0+y_0}{2})^{-1}(F'(\frac{x_0+y_0}{2}) - \int_0^1 F'(x^* + t(x_1 - x^*)dt)\| \ \|(x_1 - x^*)\| \\
&\le \|F'(\frac{x_0+y_0}{2})^{-1}F'(x^*)\| \times \|\int_0^1 F'(x^*)^{-1}(F'(\frac{x_0+y_0}{2}) - F'(x^* + t(x_1 - x^*)dt)\| \ \|(x_1 - x^*)\| \\
&\le \frac{1}{1-Lr} L\int_0^1 \|\frac{x_0+y_0}{2} - (x^* + t(x_1 - x^*))\| dt \|x_1 - x^*\| \\
&\le \frac{L}{2(1-Lr)}\int_0^1 \|x_0 - x^* + y_0 - x^* - 2t(x_1 - x^*)\| dt \|x_1 - x^*\| \\
&\le \frac{L(\|x_0 - x^*\| + \|y_0 - x^*\| + \|x_1 - x^*\|)}{2(1-Lr)}\|x_1 - x^*\|
\end{aligned}$$

$$\le \frac{3L\|x_0 - x^*\|}{2(1-Lr)}\|x_1 - x^*\| \tag{15}$$

$$\le \frac{3Lr}{2(1-Lr)}\|x_1 - x^*\| (\text{since}, x_0, y_0, x_1 \in B(x^*, r)) \tag{16}$$

$$\le \|x_1 - x^*\|. \tag{17}$$

The last step follows from the fact that $\frac{3Lr}{2(1-Lr)}\| x_1 - x^* \| \leq \| x_1 - x^* \|$ (since $\frac{3Lr}{2(1-Lr)} < 1$). Furthermore, consider

$$x_2 - x^* = x_1 - x^* - F'(\frac{x_1+y_1}{2})^{-1}F(x_1)$$

$$= F'(\frac{x_1+y_1}{2})^{-1}\int_0^1 (F'(\frac{x_1+y_1}{2}) - F'(x^* + t(x_1 - x^*))dt(x_1 - x^*)$$

$$= F'(\frac{x_n+y_n}{2})^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}(F'(\frac{x_1+y_1}{2}) - F'(x^* + t(x_1 - x^*))dt(x_1 - x^*)$$

$$= F'(\frac{x_1+y_1}{2})^{-1}F'(x^*)\times\int_0^1\int_0^1 F'(x^*)^{-1}F''(x^* + t(x_1 - x^*) + \theta(\frac{x_1+y_1}{2} - x^* - t(x_1 - x^*)))d\theta$$

$$\times(\frac{x_1+y_1}{2} - x^* - t(x_1 - x^*)))dt(x_1 - x^*)$$

$$= \frac{1}{2}F'(\frac{x_1+y_1}{2})^{-1}F'(x^*)\times\int_0^1\int_0^1 F'(x^*)^{-1}F''(\eta_1)d\theta[((1-2t)(x_1 - x^*)^2 + (y_1 - x^*))(x_1 - x^*)]dt$$

where $\eta_1 = (x^* + t(x_1 - x^*) + \theta(\frac{x_1+y_1}{2} - x^* - t(x_1 - x^*)))$. Therefore

$$\| x_2 - x^* \| \leq \frac{1}{2}\| F'(\frac{x_1+y_1}{2})^{-1}F'(x^*)\|$$

$$\times\Big[\| \int_0^1\int_0^1 F'(x^*)^{-1}[F''(\eta_1) - F''(x^*) + F''(x^*)]d\theta((1-2t)dt(x_1 - x^*)^2 \|$$

$$+\left\|\int_0^1\int_0^1 F'(x^*)^{-1}F''(\eta_1)(y_1 - x^*))(x_1 - x^*)d\theta dt\right\|\Big]$$

$$\leq \frac{1}{2}\| F'(\frac{x_1+y_1}{2})^{-1}F'(x^*)\|$$

$$\times\Big[\| \int_0^1\int_0^1 F'(x^*)^{-1}[F''(\eta_1) - F''(x^*)]d\theta((1-2t)dt(x_1 - x^*)^2 \|$$

$$+\| \int_0^1\int_0^1 F'(x^*)^{-1}F''(x^*)d\theta(1-2t)dt(x_1 - x^*)^2 \|$$

$$+\| \int_0^1\int_0^1 F'(x^*)^{-1}F''(\eta_1)(y_1 - x^*))(x_1 - x^*)d\theta dt \|\Big]$$

$$\leq \frac{1}{2(1-L\|x_0-x^*\|)}\Big[L_2\int_0^1\|\eta_1-x^*\|\|1-2t\|dt\|x_1-x^*\|^2$$

$$+L_1\|y_1-x^*\|\ \|x_1-x^*\|\Big]$$

$$\leq \frac{1}{2(1-L\|x_0-x^*\|)}\Big[L_2\int_0^1(|t+\frac{\theta}{2}(1-2t)|\|x_1-x^*\|$$

$$+|\frac{\theta}{2}|\|y_1-x^*\|)|1-2t|dt\|x_1-x^*\|^2+L_1\|y_1-x^*\|\ \|x_1-x^*\|\Big] \tag{18}$$

$$\leq \frac{1}{2(1-L\|x_0-x^*\|)}\left[\frac{11L_2}{24}+\frac{3LL_1}{2(1-L\|x_0-x^*\|)}\right]\|x_1-x^*\|^2\|x_0-x^*\|$$

$$\leq \varphi_1(r)\|x_1-x^*\|^2\|x_0-x^*\| \tag{19}$$

$$\leq \|x_1-x^*\|. \tag{20}$$

The last step follows the fact that $\varphi_1(r)\|x_1 - x^*\|^2\|x_0 - x^*\| \leq \varphi_1(r)r^2\|x_1 - x^*\| < \|x_1 - x^*\| < r$. Thus, the iterate $x_1 \in B(x^*, r)$.

Simply, replace x_1, y_1, x_2 in the previous estimate by x_n, y_n, x_{n+1} to complete the induction.

□

Remark 2 The uniqueness of the solution result can be found in [7, Chapter 32].

Theorem 3 The method (3) has convergence order $1 + \sqrt{2}$.

Proof. The proof is analogous to the proof of Theorem 3 in [11]. But for the completeness we restate the proof. Let $e_n = \|x_n - x^*\|$. Let p be maximal such that for some $C > 0$,

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^p}=C. \tag{21}$$

Then, since $e_n < r < 1$, by (8)(for large enough n), we have

$$e_{n+1}\approx\varphi_1(r)e_n^2e_{n-1}. \tag{22}$$

So, by (21) and (22), we get

$$Ce_n^p\approx\varphi_1(r)e_n^2e_{n-1},$$

or

$$e_n^{p-2}\approx\frac{1}{C}\varphi_1(r)e_{n-1},$$

or equivalently

$$e_n \approx \left(\frac{\varphi_1(r)}{C}\right)^{\frac{1}{p-2}} (e_{n-1})^{\frac{1}{p-2}}.$$

Thus, by (21), we get

$$p = \frac{1}{p-2} \text{ and } C = \left(\frac{\varphi_1(r)}{C}\right)^{\frac{1}{p-2}}.$$

Thus $p = 1 + \sqrt{2}$.

□

3. Convergence analysis of (4)

Let

$$\varphi_4(t) = \frac{L}{1-Lt}\left(\frac{3L}{2(1-Lt)} + \frac{\varphi_1(t)}{2}t\right)\varphi_1(t)$$

and $h(t) = \varphi_4(t)t^4 - 1$. Then $h(0) = -1 < 0$ and $h(t) \to \infty$ as $t \to \frac{1}{L}^-$. So $h(t) = 0$ has a smallest positive solution $\rho \in (0, \frac{1}{L})$.

Let

$$R = \min\{r, \rho\}. \tag{23}$$

Then for all $t \in (0, R)$,

$$0 \le \varphi_4(t)t^4 < 1.$$

Theorem 4 Suppose the conditions (a1)-(a4) hold. Then, the sequence $\{x_n\}$ defined by (4), starting from $x_0 \in B(x^*, R)$ is well defined and remains in (x^*, R) for $n = 0, 1, 2, \ldots$ and converges to a solution x^* of (2). Moreover, we have the following estimates for $n = 1, 2, \ldots$

$$\| y_n - x^* \| \le \frac{3L}{2(1-LR)} \| x_n - x^* \| \; \| x_{n-1} - x^* \|, \tag{24}$$

$$\| z_{n+1} - x^* \| \le \varphi_1(R) \| x_n - x^* \|^2 \| x_{n-1} - x^* \|. \tag{25}$$

and

$$\| x_{n+1} - x^* \| \le \varphi_4(R) \| x_n - x^* \|^3 \| x_{n-1} - x^* \|^2 . \tag{26}$$

Proof. Note that (24) and (25) follows as in Theorem 1. To prove (26), we observe that

$$x_{n+1}-x^{*}=z_{n+1}-x^{*}-F'(y_n)^{-1}F(z_{n+1})$$

$$=F'(y_n)^{-1}\int_0^1[F'(y_n)-F'(x^{*}+t(z_{n+1}-x^{*}))]dt(z_{n+1}-x^{*})$$

$$=F'(y_n)^{-1}F'(x^{*})\times\int_0^1 F'(x^{*})^{-1}[F'(y_n)-F'(x^{*}+t(z_{n+1}-x^{*}))]dt(z_{n+1}-x^{*}).$$

Hence, by (a2), we have

$$\begin{aligned}\| x_{n+1}-x^{*}\| &\le \frac{L}{1-LR}(\| y_n-x^{*}\|+\frac{1}{2}\| z_{n+1}-x^{*}\|)\| z_{n+1}-x^{*}\|\\ &\le \frac{L}{1-LR}\left(\frac{3L}{2(1-LR)}+\frac{\varphi_1(R)}{2}\| x_n-x^{*}\|\right)\varphi_1(R)\| x_n-x^{*}\|^3\| x_{n-1}-x^{*}\|^2\\ &\le \varphi_4(R)\| x_n-x^{*}\|^3\| x_{n-1}-x^{*}\|^2 .\end{aligned}$$

Note that since $\varphi_4(R)\|x_n-x^*\|^3\|x_{n-1}-x^*\|^2\le\varphi_4(R)R^4\|x_n-x^*\|\le\|x_n-x^*\|\le R$. Therefore, the iterate $x_n+1\in B(x^*, R)$. □

Theorem 5 The method (4) has convergence order $\frac{3+\sqrt{17}}{2}$.

Proof. The proof is analogous to the proof of Theorem 3. Let $e_n=\|x_n-x^*\|$ and let q be maximal such that for some $C_1>0$,

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^q}=C_1. \tag{27}$$

Then, since $e_n<R<1$, by (26) (for large enough n), we have

$$e_{n+1}\approx\varphi_4(R)e_n^3e_{n-1}^2. \tag{28}$$

So, by (27) and (28), we get

$$C_1e_n^q\approx\varphi_4(R)e_n^3e_{n-1}^2,$$

or

$$e_n^{q-3}\approx\frac{\varphi_4(R)}{C_1}e_{n-1}^2,$$

or equivalently

$$e_n \approx \left(\frac{\varphi_4(R)}{C_1}\right)^{\frac{1}{q-3}} (e_{n-1})^{\frac{2}{q-3}}.$$

Hence, by (27), we get

$$q = \frac{2}{q-3} \text{ and } C_1 = \left(\frac{\varphi_4(R)}{C_1}\right)^{\frac{1}{q-3}}.$$

Thus, $q = \frac{3+\sqrt{17}}{2}$.

□

4. Convergence analysis of (5)

Let

$$\varphi_5(t) = \frac{L}{2(1-Lt)}\varphi_1(t)^2$$

and $h_1(t) = \varphi_5(t)t^5 - 1$. Then $h_1(0) = -1 < 0$ and $h_1(t) \to \infty$ as $t \to \frac{1}{L}^-$. So $h_1(\mathrm{t}) = 0$ has a smallest positive solution $\rho_1 \in (0, \frac{1}{L})$.

Let

$$R_1 = \min\{r, \rho_1\}. \tag{29}$$

Then, for all $t \in (0, R_1)$,

$$0 \le \varphi_5(t)t^5 < 1.$$

Theorem 6 Suppose the conditions (a1)-(a4) hold. Then, the sequence $\{x_n\}$ defined by (5), starting from $x_0 \in B(x^*, R_1)$ is well defined and remains in $\bar{B}(x^*, R_1)$ for $n = 0, 1, 2, \ldots$ and converges to a solution x^* of (2). Moreover, we have the following estimates for $n = 1, 2, \ldots$

$$\| y_n - x^* \| \le \frac{3L}{2(1-LR_1)} \| x_n - x^* \| \ \| x_{n-1} - x^* \|, \tag{30}$$

$$\| z_{n+1} - x^* \| \le \varphi_1(R_1) \| x_n - x^* \|^2 \| x_{n-1} - x^* \|. \tag{31}$$

and

$$\| x_{n+1} - x^* \| \le \varphi_5(R_1) \| x_n - x^* \|^3 \| x_{n-1} - x^* \|^2 . \tag{32}$$

Proof. Note that (30) and (31) follows as in Theorem 1. Observe that

$$x_{n+1} - x^* = z_{n+1} - x^* - F'(z_{n+1})^{-1} F(z_{n+1})$$

$$= F'(z_{n+1})^{-1} \int_0^1 [F'(z_{n+1}) - F'(x^* + t(z_{n+1} - x^*))] dt (z_{n+1} - x^*)$$

$$= F'(z_{n+1})^{-1} F'(x^*) \times \int_0^1 F'(x^*)^{-1} [F'(z_{n+1}) - F'(x^* + t(z_{n+1} - x^*))] dt (z_{n+1} - x^*).$$

Therefore, by (a2), we have

$$\begin{aligned} \| x_{n+1} - x^* \| &\leq \frac{L}{2(1 - LR_1)} \| z_{n+1} - x^* \|^2 \\ &\leq \frac{L}{2(1 - LR_1)} \varphi_1(R_1)^2 \| x_n - x^* \|^4 \| x_{n-1} - x^* \|^2 \\ &\leq \varphi_5(R_1) \| x_n - x^* \|^4 \| x_{n-1} - x^* \|^2 . \end{aligned}$$

Note that since $\varphi_5(R_1)\|x_n - x^*\|^4\|x_{n-1} - x^*\|^2 \leq \varphi_5(R_1)R_1^5\|x_n - x^*\| \leq \|x_n - x^*\| \leq R_1$. Thus, the iterate $x_{n+1} \in B(x^*, R_1)$.

□

Theorem 7 The method (5) has convergence order $2 + \sqrt{6}$.

Proof. The proof is analogous to the proof of Theorem 3. Let $e_n = \|x_n - x^*\|$ and let s be maximal such that for some $C_2 > 0$,

$$\lim_{n \to \infty} \frac{e_{n+1}}{e_n^s} = C_2. \tag{33}$$

Then, since $e_n < R_1 < 1$ by (32) (for large enough n), we have

$$e_{n+1} \approx \varphi_5(R_1) e_n^4 e_{n-1}^2. \tag{34}$$

Hence, by (33) and (34), we get

$$C_2 e_n^s \approx \varphi_5(R_1) e_n^4 e_{n-1}^2,$$

or

$$e_n^{s-4} \approx \frac{\varphi_5(R_1)}{C_2} e_{n-1}^2,$$

or equivalently

$$e_n \approx \left(\frac{\varphi_5(R_1)}{C_2}\right)^{\frac{1}{s-4}} (e_{n-1})^{\frac{2}{s-4}}.$$

So, by (33), we get

$$s = \frac{2}{s-4} \text{ and } C_2 = \left(\frac{\varphi_5(R_1)}{C_2}\right)^{\frac{1}{s-4}}.$$

Thus, $s = 2 + \sqrt{6}$.

□

5. Examples

Three examples are presented in this section.

Example 8 Let $X = Y = C[0, 1]$, the space of continuous functions defined on [0, 1] and be equipped with the max norm. Let $D = \bar{B}(0, 1)$. Define function F on D by

$$F(\vartheta)(x) = \vartheta(x) - 5\int_0^1 x\theta\vartheta(\theta)^3 \, d\theta.$$

We have that

$$F'(\vartheta(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\vartheta(\theta)^2 \xi(\theta) d\theta, \text{ for each } \xi \in D.$$

Then, we get that for $x^* = 0$, $L = 15$ and $L_1 = 31$, $L_2 = 8.5$. Then the parameters are:

$$r = R = \frac{2}{5L} = 0.0267, r_1 = 0.0386, r_2 = 0.0296, \rho = 0.0387, \rho_1 = 0.0204 = R_1.$$

Example 9 Let $X = Y = \mathbb{R}_3$, $D = \bar{B}(0, 1)$, $x^* = (0, 0, 1)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (sinx, \frac{y^2}{5} + y, z)^T.$$

Then, the Fréchet-derivatives are given by

$$F'(v) = \begin{bmatrix} cosx & 0 & 0 \\ 0 & \frac{2y}{5}+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and

$$F''(v) = \left[\begin{array}{ccc|ccc|ccc} -\sin x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using conditions (a1)-(a4), we have that $L = L_2 = 1$ and $L_1 = \frac{2}{5}$.
Then the parameters are:

$$r = R = R_1 = \frac{2}{5L} = 0.4,\ r_1 = 0.6808,\ r_2 = 0.6161,\ \rho = 0.5217,\ \rho_1 = 0.6016.$$

Example 10 Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define F on D by

$$F(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then, we have $t_* = 1$ and $L = L_1 = L_2 = 44.4234$. Then the parameters are:

$$r = R = \frac{2}{5L} = 0.0090,\ r_1 = 0.0197,\ r_2 = 0.0120\ \rho = 0.0155,\ \rho_1 = 0.0073 = R_1.$$

6. Basins of attractions

In this section, we study the basin of attraction (i.e. the collection of all initial points from which the iterative method converges to a solution of a given equation) or Fatou sets and Julia sets (i.e. the complement of Fatou sets) corresponding to the method (3), (4) and (5). We consider three examples to provide the basins of attractions:

Example 11 Let $(x, y) \in \mathbb{R}^2$, consider the system of equations

$$\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases}$$

with solutions $\{(-1, -1), (0, 0), (1, 1)\}$.

Example 12 Let $(x, y) \in \mathbb{R}^2$, consider the system of equations

$$\begin{cases} 3x^2 y - y^3 = 0 \\ x^3 - 3xy^2 - 1 = 0 \end{cases}$$

with solutions $\{(-\frac{1}{2}, -\frac{\sqrt{3}}{2}), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (1, 0)\}$,

Example 13 Let $(x, y) \in \mathbb{R}^2$, consider the system of equations

$$\begin{cases} x^2 + y^2 - 4 = 0 \\ 3x^2 + 7y^2 - 16 = 0 \end{cases}$$

with solutions $\{(\sqrt{3}, 1), (-\sqrt{3}, 1), (\sqrt{3}, -1), (-\sqrt{3}, -1)\}$.

A rectangular region is considered for generating the basin of attraction: $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, -2 \leq y \leq 2\}$. The region contains all the roots of test problems under consideration. Equidistant grid of 401×401 points in $\mathcal{R}$ are used as the initial guess x_0, for the methods (3), (4) and (5). A fixed tolerance 10^{-8} and a maximum of 50 iterations are used for all the cases. Each attracting basin corresponding to a root is assigned a color. If the desired tolerance is not attained within a fixed number of iterations, then a conclusion is made that, the iterative method starting at x_0 does not converges to any of the roots. Such points are assigned a black color. Eventually the basins of attraction is distinguished by the respective colors of distinct roots of the method.

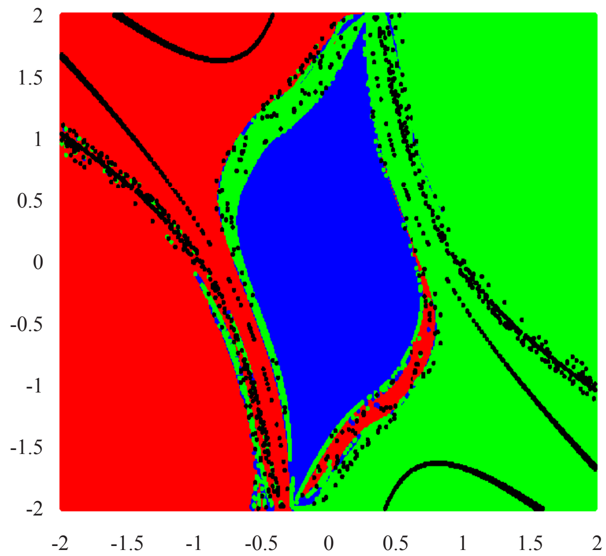

Figure 1. Dynamical plane of the methods (3) with basins of attraction for the Example 11

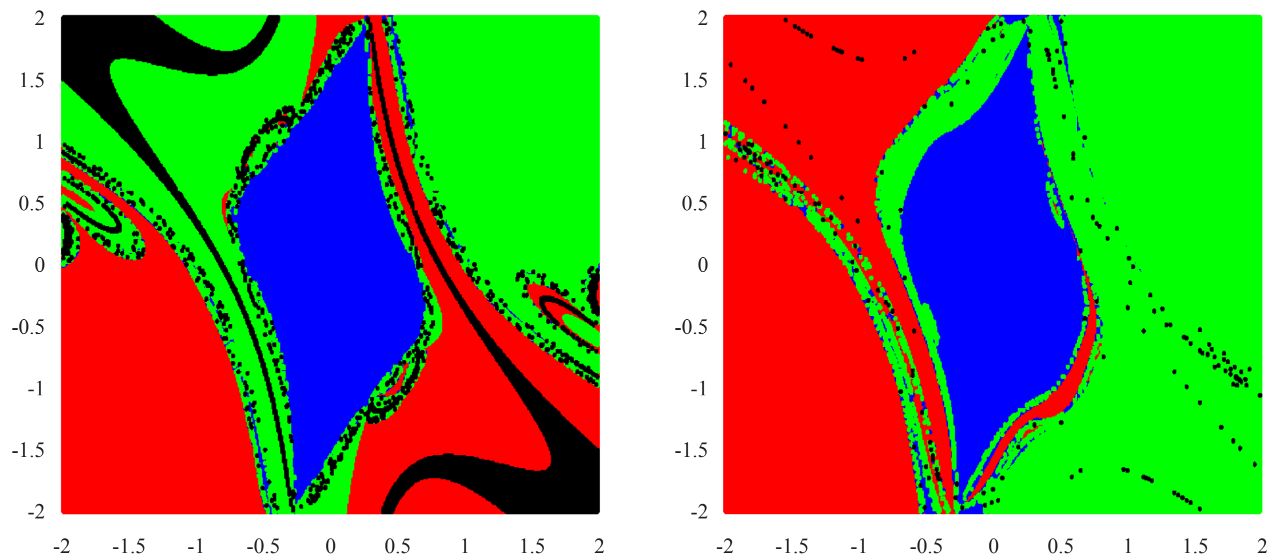

Figure 2. Dynamical plane of the methods (4)(left) and (5)(right) with basins of attraction for the Example 11

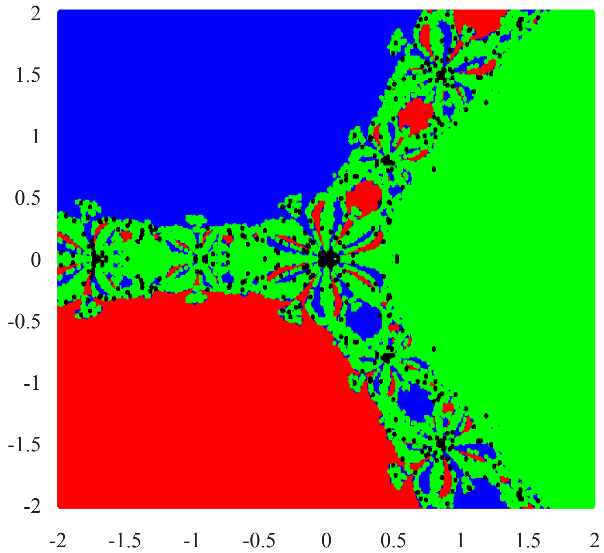

Figure 3. Dynamical plane of the methods (3) with basins of attraction for the Example 12

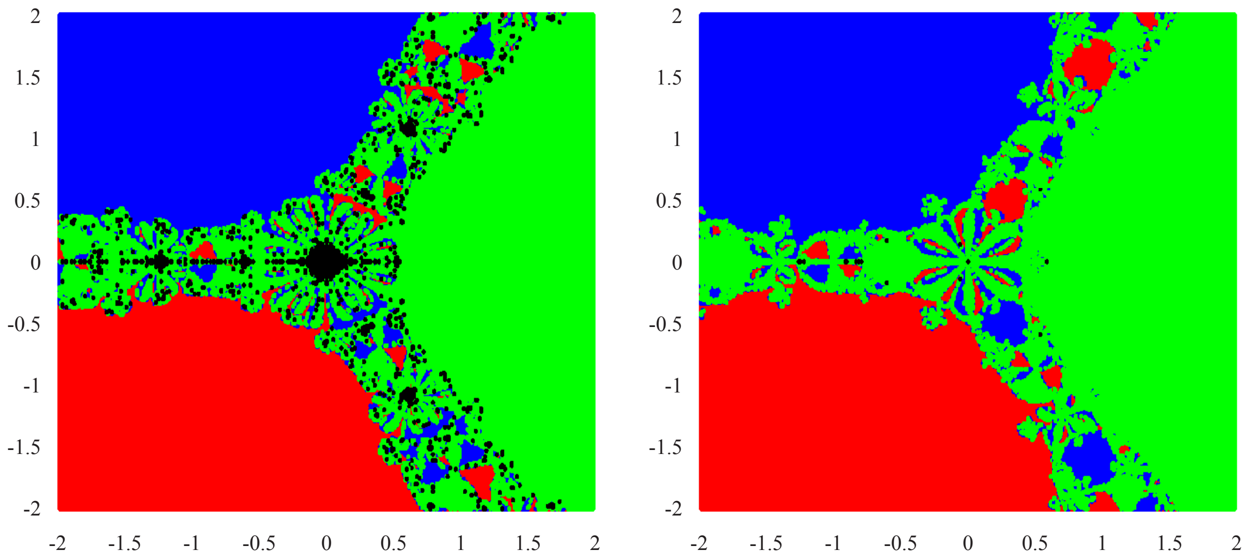

Figure 4. Dynamical plane of the methods (4)(left) and (5)(right) with basins of attraction for the Example 12

Figure 1-Figure 6 demonstrates the basin of attraction corresponding to each root of above Examples (Example 11-Example 13) for the methods (3), (4) and (5). The black regions commonly known as Julia sets which contains all initial points from which the iterative methods do not converge any of the roots are easily identifiable from these figures.

The experimentation is performed on a 16-core 64 bit Windows machine with Intel Core i7-10700 CPU 2.90GHz using MATLAB.

Remark 14 From the figures it is easy to understand that method (5) has a larger basins of attraction compared to methods (3) and (4).

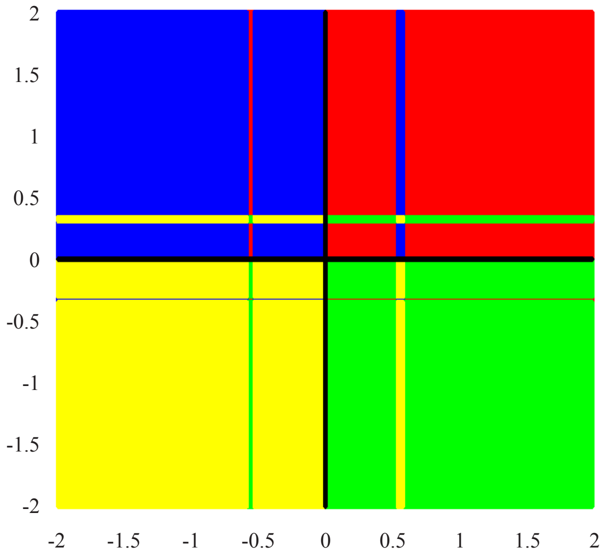

Figure 5. Dynamical plane of the methods (3) with basins of attraction for the Example 13

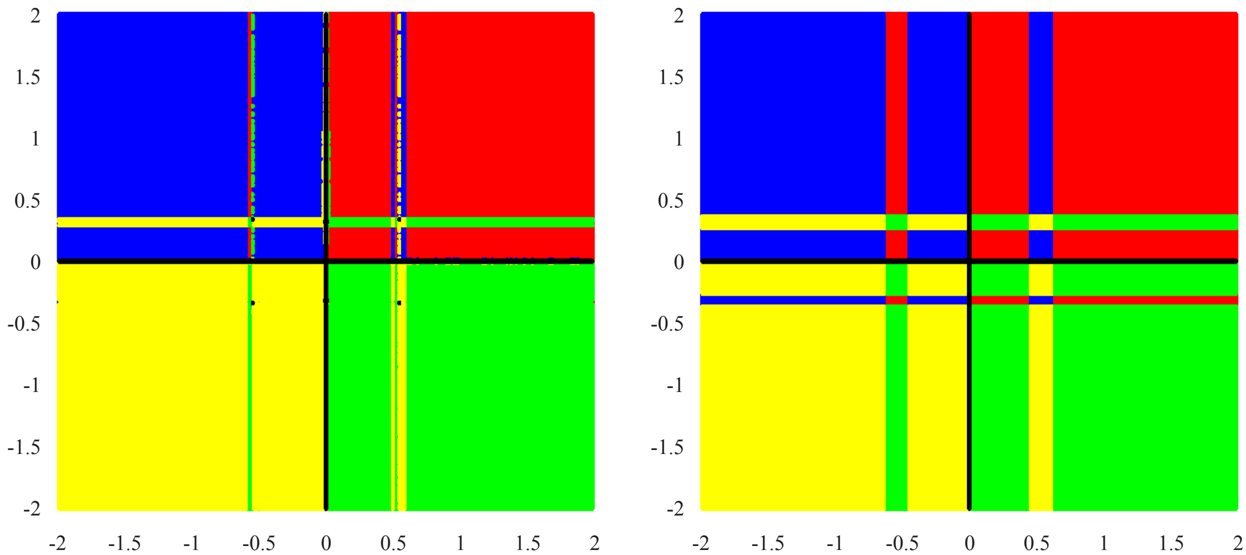

Figure 6. Dynamical plane of the methods (4)(left) and (5)(right) with basins of attraction for the Example 13

7. Conclusions

A process is developed to determine the convergence order of the methods (3), (4) and (5). The analysis involves only the first and the second derivative in contrast to the earlier works which use the third derivatives [1]. Moreover, the computable error distances are also provided which are not given before [1]. Hence, the applicability of these methods stand extended. The new process is independent of these methods. Therefore, this process can also be used to extend the usage of other higher-order methods using the inverses of linear operators. This is the topic of our future research.

Acknowledgement

The authors Santhosh George, Jidesh P and Ajil K wish to thank the SERB, Govt. of India for the financial support under Project Grant No. CRG/2021/004776.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Werner W. Some supplementary results on the $1 + \sqrt{2}$ order method for the solution of nonlinear equations. *Numer. Math*. 1982; 38(3): 383-392.
[2] King RF. Tangent methods for nonlinear equations. *Numer. Math*. 1971; 18(4): 298-304.
[3] Argyros IK, Ren H. On the convergence of efficient king-wernertype methods of order $1 + \sqrt{2}$. *Journal of Computational and Applied Mathematics*. 2015; 285: 169-180.
[4] King RF. A family of fourth-order methods for nonlinear equations. *SIAM Numer. Anal*. 1973; 10: 876-879.
[5] Sharmaa JR. Guhaa RK, Gupt P. Improved King's methods with optimal order of convergence based on rational approximations. *Applied Mathematics Letters*. 2013; 26(4): 473-480.
[6] Werner W. Überein Verfahren der Ordnung $1 + \sqrt{2}$ zur Nullstellenbestimmung. *Numer. Math*. 1979; 32(3): 333-342.
[7] Argyros CI, Regmi S, Argyros IK, George S. *Contemporary Algorithms: Theory and Applications Volume II.* Nova, USA; 2022.
[8] Cordero A, Hueso JL, Martínez E, Torregrosa JR. Increasing the convergence order of an iterative method for nonlinear systems. *Appl. Math. Lett*. 2012; 25: 2369-2374.
[9] Cordero A, Martínez E, Toregrossa JR. Iterative methods of order four and five for systems of nonlinear equations. *J. Comput. Appl. Math*. 2012; 231: 541-551.
[10] Argyros IK. *The theory and applications of iteration methods*. 2nd ed. CRC Press, Taylor and Francis Group; 2022.
[11] Cárdenas E, Castro R, Sierra W. A newton-type midpoint method with high efficiency index. *Journal of Mathematical Analysis and Applications*. 2020; 491(2): 124381.