Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

A Characterization of Associative Evolution Algebras

Desamparados Fernández Ternero, Víctor Manuel Gómez Sousa, Juan Núñez-Valdés*

Department of Geometry and Topology, University of Seville, Spain
Email: jnvaldes@us.es

Received: 16 November 2022; **Revised:** 3 February 2023; **Accepted:** 13 February 2023

Abstract: Evolution algebras are non-associative in general, that is, the binary multiplication law is not associative. However, there exist some of them that are associative. In this paper we deal with these last ones. We give an explicit classification of these algebras and show that the concept of associativity is equivalent to the existence of an unitary element for non-degenerate evolution algebras. We also explicitly describe the derivation space of these algebras.

***Keywords*:** evolution algebras, associative algebras, classification, derivations

MSC: 16H99, 47C05

1. Introduction

Since Tian introduced in 2004 evolution algebras, making them the subject of his Ph.D. Thesis [1], and later publishing an article on them jointly with Vojtechovsky in 2006 [2] and a book in 2008 [3], the number of scientific articles involving them has continued growing over the years, not only for being interesting by themselves, but also due to they are very useful tool to be used in several applications to other disciplines (an interesting historical perspective on these algebras can be consulted in [4]).

However, evolution algebras are not associative, in general, which means that some of them are. There is not much information on the latter in the literature, that is the reason for which this article can be considered novel, since it presents a result referring to the associativity of these algebras, which can serve to facilitate the study of more aspects related to this topic in general, and complements the study of other aspects of these algebras that are also rarely discussed, such as hierarchy (see [5]), the fact that they are normed (see [6]), or the links between these algebras and Markov chains (see [7]) or between the same and graphs (see [8-10], for instance).

Examples of usual topics in the literature, which have proven to be a very convenient approach, are the study of derivation spaces [11-15] and the classification of some family of evolution algebras sharing interesting properties. For example, nilpotent evolution algebras are characterized in [16-18] and power-associative evolution algebras are classified in [19], up to dimension six. Three dimensional real evolution algebras with condition $dim(E^2) = 1$ are analyzed in [20], while in [21] the authors study the case of four dimensional perfect non-simple evolution algebras. Classifications involving properties of the main operator of the algebra are dealt with in [22-23].

Following this trend, the main goals of this paper are the following. The first one is to classify those evolution

DOI: https://doi.org/10.37256/cm.4120232167

algebras that are associative, in any finite dimension. The second one is to give a characterization of associative evolution algebras in term of the existence of an unitary element for non-degenerate evolution algebras. In addition, we study the space of derivations of some associative evolution algebras.

2. Preliminaries

In this paper we will only consider finite dimensional evolution algebras. The main concepts and results on them used throughout this paper are the following.

Definition 2.1 Let $E \equiv (E, +, \cdot)$ be an algebra over a field $\mathbb{K}$. It is said that E is an evolution algebra if there exists a basis $\mathcal{B} = \{e_i : i \in \Lambda\}$ of E, where Λ is an index set, such that $e_i \cdot e_j = 0$, if $i \neq j$. The basis $\mathcal{B}$ is called a natural basis.

Since $\mathcal{B}$ is a basis, the product $e_j \cdot e_j = e_j^2$ can be written as $\Sigma_{i\in\Lambda} a_{ij} e_i$, with $a_{ij} \in \mathbb{K}$, where only a finite quantity of a_{ij}, called structure constants, are non-zero for each $j \in \Lambda$ fixed. So, the product on E is determined by the structure matrix $A = (a_{ij})$.

Definition 2.2 The evolution operator associated with $\mathcal{B}$ is the endomorphism $L : E \to E$ which maps each generator into its square, that is, $L(e_j) = e_j^2 = \Sigma_{i\in\Lambda} a_{ij} e_i$, for all $j \in \Lambda$.

Note that the matrix representation of the evolution operator with respect to the basis $\mathcal{B}$ is the structure matrix A. Furthermore, for all $x \in E$, we have that

$$L(x) = \left(\sum_{i\in\Lambda} e_i\right) \cdot x$$

so $L = M_e$, where $e = \Sigma_{i\in\Lambda} e_i$ and the endomorphism M_e is the operator of multiplication by the element e.

Definition 2.3 An evolution algebra is said to be non-degenerate if $e_j^2 \neq 0$, for all $j \in \Lambda$. On the contrary, the evolution algebra is called degenerate.

Definition 2.4 An evolution algebra E is said to be associative if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in E$.

Definition 2.5 An element $\mu \in E$ is said to be unitary if $\mu \cdot x = x$, for all $x \in E$. E is said to be an unitary evolution algebra if there exists an unitary element in E.

Definition 2.6 An evolution algebra E is said to be nonzero trivial if $e_j^2 = a_{jj} e_j$, with $a_{jj} \neq 0$, for all $j \in \Lambda$.

Note that, by definition, all nonzero trivial evolution algebras are non-degenerate.

Definition 2.7 An evolution algebra E is said to be nilpotent if there exist $n_o \in \mathbb{N}$ such that $E^{n_0} = 0$, where $E^1 = E$ and $E^n = E^{n-1} \cdot E$, for all $n \geq 2$.

Definition 2.8 We define the annihilator of an evolution algebra E as the set

$$Ann(E) = \{x \in E : x \cdot E = 0\} = span\{e_i : e_i^2 = 0\}.$$

We end these preliminaries with the following result given in [3].

Proposition 2.9 Let E be an evolution algebra. Then, the following assertions are equivalent

1. E is a nonzero trivial evolution algebra.
2. E is an unitary evolution algebra.

In this case, the unitary element of the evolution algebra is $\mu = \sum_{i\in\Lambda} \frac{1}{a_{ii}} e_i$.

3. Associative evolution algebras

We deal in this section with associative evolution algebras. In it we prove two characterizations of these algebras.

The first provides a classification of them, with reference to how the structure matrix of the algebra has to be, in a general way. The second is in terms of the existence of an unitary element for non-degenerate evolution algebras. In addition, these results also establish connections with the evolution operator of the algebra.

Theorem 3.1 Let E be an evolution algebra with natural basis $\mathcal{B} = \{e_i : i = 1, \ldots, n\}$, structure matrix $A = (a_{ij})$ and evolution operator L. Then, the following assertions are equivalent

(1) $M_a(x \cdot y) = x \cdot M_a(y)$, for all $a, x, y \in E$.

(2) $L(x \cdot y) = x \cdot L(y)$, for all $x, y \in E$.

(3) There exists a rearrangement of the basis $\mathcal{B}$ such that the structure matrix has the form

$$\begin{pmatrix} \mathcal{D}_{r\times r} & 0_{r\times s} \\ \mathcal{M}_{s\times r} & 0_{s\times s} \end{pmatrix}, \tag{$*$}$$

where $r + s = n$ and

- $\mathcal{D}_{r\times r}$ is a diagonal matrix of order r.
- $\mathcal{M}_{s\times r}$ is a $s \times r$ matrix.
- $0_{r\times s}$ and $0_{s\times s}$ are null matrices of dimension $r \times s$ and $s \times$ s, respectively.

(4) E is an associative evolution algebra.

Proof. (1) $\Rightarrow$ (2)

It is enough to take $a = e = \sum_{i=1}^{n} e_i$.

(2) $\Rightarrow$ (3)

If $i \neq j$, then

$$0 = L(0) = L(e_i \cdot e_j) = e_i \cdot L(e_j) = e_i \cdot \left(\sum_{k=1}^{n} a_{kj} e_k \right) = a_{ij} e_i^2,$$

so $a_{ij} = 0$ or $A_{*i} = 0$, for all $i \neq j$. Suppose A has s null columns. By reordering the basis, we can assume that these columns are the last ones. Let us set $r = n -$ s. Then, $a_{ij} = 0$, for all $i, j \leq r$ and $i \neq j$. Hence, assertion (3) follows.

(3) $\Rightarrow$ (4)

From the structure of A, we deduce that $e_j^2 = a_{jj} e_j + v_j$, with $v_j \in Ann(E)$. Let $x_1, x_2, x_3 \in E$, with $x_i = \sum_{j=1}^{n} x_{ij} e_j$. Then,

$$(x_1 \cdot x_2) \cdot x_3 = \left[\left(\sum_{j=1}^{n} x_{1j} e_j \right) \cdot \left(\sum_{j=1}^{n} x_{2j} e_j \right) \right] \cdot x_3 = \left(\sum_{j=1}^{n} x_{1j} x_{2j} e_j^2 \right) \cdot x_3$$

$$= \left(\sum_{j=1}^{n} a_{jj} x_{1j} x_{2j} e_j + \underbrace{x_{1j} x_{2j} v_j}_{\in Ann(E)} \right) \cdot x_3 = \left(\sum_{j=1}^{n} a_{jj} x_{1j} x_{2j} e_j \right) \cdot x_3$$

$$= \sum_{j=1}^{n} a_{jj} x_{1j} x_{2j} x_{3j} e_j^2 = \sum_{j=1}^{n} a_{jj} x_{2j} x_{3j} x_{1j} e_j^2$$

$$= (x_2 \cdot x_3) \cdot x_1 = x_1 \cdot (x_2 \cdot x_3).$$

(4) $\Rightarrow$ (1)

If E is associative, then

$$M_a(x\cdot y)=(x\cdot y)\cdot a=x\cdot\big(y\cdot a\big)=x\cdot M_a(y).$$

□

The previous result provides us with a classification of associative evolution algebras, up to basis permutations. However, some of these algebras are isomorphic. The following result will allow us to classify these algebras up to isomorphism.

Proposition 3.2 Let E be a finite-dimensional associative evolution algebra. Then, $E = T \oplus N$, where

• T is a nonzero trivial evolution algebra with structure matrix the identity.

• N is a nilpotent evolution algebra with $N^3 = 0$.

Proof. Let $\mathcal{B} = \{e_1, ..., e_n\}$ be a natural basis of E such that the structure matrix $A = (a_{ij})$ relative to this basis is like $(*)$. By reordering the basis, we can assume that $a_{ii} \neq 0$ if and only if $i \leq t$, for some $t \in \{0, \dots, r\}$. Let us consider the following elements of E

$$\eta_i=\frac{1}{a_{ii}}\Big(e_i+\sum_{k=r+1}^{n}\frac{a_{ki}}{a_{ii}}e_k\Big),$$

for $i \leq t$. Then,

$$\eta_i\cdot\eta_j=\frac{1}{a_{ii}}\frac{1}{a_{jj}}\Big(e_i+\underbrace{\sum_{k=r+1}^{n}\frac{a_{ki}}{a_{ii}}e_k}_{\in Ann(E)}\Big)\cdot\Big(e_j+\underbrace{\sum_{k=r+1}^{n}\frac{a_{kj}}{a_{jj}}e_k}_{\in Ann(E)}\Big)$$

$$=\frac{1}{a_{ii}a_{jj}}e_i\cdot e_j=\delta_{ij}\frac{1}{a_{ii}^2}e_i^2=\delta_{ij}\frac{1}{a_{ii}^2}\Big(a_{ii}e_i+\sum_{k=r+1}^{n}a_{ki}e_k\Big)$$

$$=\delta_{ij}\frac{1}{a_{ii}}\Big(e_i+\sum_{k=r+1}^{n}\frac{a_{ki}}{a_{ii}}e_k\Big)=\delta_{ij}\eta_i,$$

where δ_{ij} is the Kronecker delta function. We define $T = span(\eta_1, ..., \eta_t)$ and $N = span(e_{t+1}, ..., e_n)$. From the above equality it follows that T is a nonzero trivial evolution algebra with identity structure matrix relative to the natural basis $\{\eta_1, \dots, \eta_t\}$. Furthermore, N is a nilpotent evolution algebra, because $N^2 \subseteq Ann(E)$ and then $N^3 = 0$.

□

From the previous result it follows that to classify associative evolution algebras, up to isomorphisms, it is enough to classify nilpotent evolution algebras N with $N^3 = 0$, which is a well-studied problem. Details can be found in [16-18], where nilpotent evolution algebras of dimension up to five are classified.

Collorary 3.3 Let E be a non-degenerate evolution algebra with evolution operator L. Then, the following assertions are equivalent

(1) $M_a(x \cdot y) = x \cdot M_a(y)$, for all $a, x, y \in E$.

(2) $L(x \cdot y) = x \cdot L(y)$, for all $x, y \in E$.

(3) E is a nonzero trivial evolution algebra.

(4) E is an unitary evolution algebra.

(5) E is an associative evolution algebra.

Proof. From assertion (3) of Theorem 3.1, we get $A = \mathcal{D}_{n\times n}$, that is, E is a nonzero trivial evolution algebra. This is equivalent to E being an unitary evolution algebra (Proposition 2.9).

□

4. Derivation space

In this section we provide a result on the space of derivations of evolution algebras, in the case of these being associative, that can be added to those already obtained regarding this concept in the case of evolution algebras in general (see [11-15] in this respect, for instance).

An endomorphism $d : E \to E$ is called a derivation if $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$, for all $x, y \in E$. In [3], Tian shows that this is equivalent to

$$\begin{cases} d_{ji}a_{kj} + d_{ij}a_{ki} = 0, \text{ for all } i, j, k \text{ with } i \neq j, \\ 2a_{ji}d_{ii} = \sum_k a_{ki}d_{jk}, \text{ for all } i, j, k. \end{cases}$$

where $D = (d_{ij})$ is the matrix associated with endomorphism d. The set of all derivations is denoted by $Der(E)$ and has a Lie algebra structure.

Theorem 4.1 Let E be an associative evolution algebra with natural basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ such that its structure matrix $A = (a_{ij})$ is like (∗), with $a_{ii} \neq 0$, for all $i \leq r$. Then, $d \in Der(E)$ if and only if the first r rows of D are null and $DA = 0$. As a consequence, $Der(E)$ is a s^2-dimensional Lie algebra.

Proof. We consider the matrices A and D as block matrices

$$A = \begin{pmatrix} \mathcal{D} & 0 \\ \mathcal{M} & 0 \end{pmatrix}, D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where the blocks have the same size as those of (∗). The endomorphism d is a derivation if and only if

$$\begin{cases} d_{ji}A_{*j} + d_{ij}A_{*i} = 0, \text{ for all } i \neq j, \\ DA = 2ADiag(D). \end{cases}$$

The first condition implies that D_{11} is diagonal and $D_{12} = 0$. The second one gives

$$\begin{cases} D_{11}\mathcal{D} = 2\mathcal{D}D_{11}, \\ D_{21}\mathcal{D} + D_{22}\mathcal{M} = 2\mathcal{M}D_{11}. \end{cases}$$

Now, the first of these conditions implies $D_{11} = 0$ and the second one is equivalent to $DA = 0$.

□

5. Conclusions

This paper deals with two aspects of evolution algebras that at first do not seem to be very related to each other and about which there is not much information in the literature. They are the associativity and the evolution operator of the algebra. Authors show the first known explicit classification of associative algebras and a characterization in terms of the existence of an unitary element for non-degenerate evolution algebras. They also explicitly describe the derivation space of some associative evolution algebras.

Regarding the associativity of the algebra and as already mentioned in the Introduction, evolution algebras are not associative in general and only a small number of them have this property. This might be the reason why the topic of the associativity of the algebras of evolution has been scarcely dealt with by researchers until now, since there are quite few references in the literature about it: [19] and [24] are the most representative. In the first of them, authors deal with power-associative evolution algebras, which are those evolution algebras verifying the property $(x \cdot x) \cdot (x \cdot x) = ((x \cdot x) \cdot x) \cdot x$, $\forall x \in E$ (and the same occurs performing e_i by itself several times). In the second one, authors introduce the concept of commutative associative universal faithful representation and describe it (when it exists) for all evolution algebras of dimension two.

And with respect to the evolution operator of evolution algebras, there are not too many contributions in the literature. Apart from some last works [22] and [23] by the authors of this paper, [25] can also be consulted. In that last short paper of 6 pages, authors review previous results for discrete-time dynamical systems and evolution algebras of sex linked inheritance and discuss some open problems related to such inheritance.

In general, it is difficult to find any connection between the associativity of an algebra and the existence of an unitary element, since they are concepts that seem unrelated and to start from different points of interest. Despite this, we have seen that these two concepts are equivalent in non-degenerate evolution algebras. Moreover, we can affirm that the existence of an unitary element in the algebra implies the associative property, in a general way.

In addition to this, the equivalence between items (3) and (4) of Theorem 3.1 gives us a classification of associative evolution algebras, which is a novel result. Furthermore, we have seen that the dimension of the space $Der(E)$ of some associative evolution algebras is s^2, which sets out a connection with the dimension of $Ann(E)$. Moreover, in these cases, we can calculate $Der(E)$ by solving the equation $DA = 0$, which in turn is equivalent to solve r linear equations to obtain $null(A^t)$ and can be done in an efficient way, especially taking into account the structure of A.

Aknowledgement

This paper is partially supported by autonomic P.A.I. Group FQM-326.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Tian JP. *Evolution algebra theory*. Riverside: University of California; 2004.

[2] Tian JP, Vojtechovsky P. Mathematical concepts of evolution algebras in nonmendelian genetics. *Quasigroups and Related Systems*. 2006; 14(1): 111-122.

[3] Tian JP. Evolution algebras and their applications. *Lecture Notes in Mathematics 1921*. Berlin: Springer-Verlag; 2008.

[4] Ceballos M, Falcón RM, Núñez-Valdés J, Tenorio AF. A historical perspective of Tian's evolution algebras. *Expositiones Mathematicae*. 2022; 40(3): 819-843.

[5] Cruz FJ, del Valle A, Núñez-Valdés J, Pena M. The concept of hierarchy of algebras and graphs. *Journal of Applied Mathematics and Computing*. 2021; 67: 233-255.

[6] González-López R, Núñez-Valdés J. A note on evolution algebras as Banach algebras. *Contemporary Mathematics*. 2021; 2(3): 210-214.

[7] López-Blázquez F, Núñez-Valdés J, Recacha S, Villar MT. *Connecting statistics, probability, algebra and discrete mathematics*. arXiv:2104.02470. Cornell University; 2021. Available from: https://doi.org/10.48550/arXiv.2104.02470.

[8] González-López R, Núñez-Valdés J. Endowing evolution algebras with properties of discrete structures. *Portugaliae Mathematica*. 2020; 77(3-4): 423-443.

[9] Ceballos M, Núñez-Valdés J, Tenorio AF. Finite dimensional evolution algebras and pseudodigraphs. *Mathematical Methods in the Applied Sciences*. 2022; 45(4): 2424-2442.
[10] Elduque A, Labra A. Evolution algebras and graphs. *Journal of Algebra and Its Applications.* 2015; 14(7): 1550103. Available from: https://doi.org/10.1142/S0219498815501030.
[11] Alsarayreh A, Qaralleh I, Ahmad MZ. Derivation of three-dimensional evolution algebra. *JP Journal of Algebra, Number Theory and Applications*. 2017; 39(4): 25-44.
[12] Camacho LM, Gómez JR, Omirov BA, Turdibaev RM. The derivations of some evolution algebras. *Linear and Multilinear Algebra*. 2013; 61: 309-322.
[13] Reis T, Cadavid P. Derivations of evolution algebras associated to graphs over a field of any characteristic. *Linear and Multilinear Algebra*. 2022; 70(15): 2884-2897.
[14] Casado YC, Cadavid P, Montoya MLR, Rodriguez PM. On the characterization of the space of derivations in evolution algebras. *Annali di Matematica Pura ed Applicata*. 2021; 200: 737-755. Available from: https://doi.org/10.1007/s10231-020-01012-2.
[15] Cadavid P, Montoya MLR, Rodriguez PM. Characterization theorems for the spaces of derivations of evolution algebras associated to graphs. *Linear and Multilinear Algebra*. 2018; 68(7): 1340-1354.
[16] Casas JM, Ladra M, Omirov BA, Rozikov UA. On evolution algebras. *Algebra Colloquium*. 2014; 21(2): 331-342.
[17] Hegazi AS, Abdelwahab H. Nilpotent evolution algebras over arbitrary fields. *Linear Algebra and its Applications*. 2015; 486: 345-360.
[18] Elduque A, Labra A. On nilpotent evolution algebras. *Linear Algebra and its Applications*. 2016; 505: 11-31.
[19] Ouattara M, Savadogo S. Power-Associative Evolution Algebras. In: Molina MS, El Kaoutit L, Louzari M, Ben Yakoub L, Benslimane M. (eds.) *MAMAA 2018: Associative and Non-Associative Algebras and Applications*. Springer Proceedings in Mathematics and Statistics. Springer, Cham; 2020. p.23-49. Available from: https://doi.org/10.1007/978-3-030-35256-1_2.
[20] Imomkulov AN. Classification of a family of three dimensional real evolution algebras. *TWMS Journal of Pure and Applied Mathematics*. 2019; 10: 225-238.
[21] Casado YC, Kanuni M, Molina MS. *Classification of four dimensional perfect non-simple evolution algebras*. arXiv:1801.03856. Cornell University; 2018. Available from: https://doi.org/10.48550/arXiv.1801.03856.
[22] Fernández-Ternero D, Gómez-Sousa VM, Núñez-Valdés J. Using the evolution operator to classify evolution algebras. *Mathematical and Computational Applications*. 2021; 26(3): 57.
[23] Fernández-Ternero D, Gómez-Sousa VM, Núñez-Valdés J. Evolution algebras whose evolution operator is a homomorphism. *Computational and Mathematical Methods*. 2021; 3(6): 1-14.
[24] Gonçalves MIC, Gonçalves D, Barquero DM, González CM, Molina MS. Squares and associative representations of two-dimensional evolution algebras. *Journal of Algebra and its Applications*. 2021; 20(6): 38.
[25] Rozikov UA. Evolution operators and algebras of sex linked inheritance. *Asia Pacific Math Newsletter*. 2013; 3(1): 6-11.