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Research Article

The Sharp Bound of the Third Hankel Determinant for the Inverse of Bounded Turning Functions

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Abstract: The objective of this paper is to estimate the best possible upper bound to the third Hankel determinant for the inverse of functions with normalized conditions $f(0) = 0$, $f'(0) = 1$ in the unit disc whose derivative has positive real part.

***Keywords*:** Holomorphic function, Hankel determinant, inverse of bounded turning function, Carathéodory function

MSC: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ represent family of mappings f of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, denotes the open unit disc and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. Every $f \in \mathcal{S}$, has an inverse f^{-1} given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, |w| < r_o(f); \left(r_o(f) \geq \frac{1}{4}\right). \tag{2}$$

When $f(\mathbb{D})$ is a convex region, Libera et al. [1] obtained a linkage between the coefficients of f and f^{-1} for every f of the form (1). On the other hand, Kapoor, and Mishra [2] extended on the findings of Krzyz et al. [3] who studied bounds on initial coefficients of inverse of starlike functions of order α. Furthermore, Ali [4] investigated sharp bounds on the early coefficients of inverse functions when the function is a member of the class of strongly starlike functions.

Pommerenke [5] characterized the r^{th}- Hankel determinant of order n, for f with $r, n \in \mathbb{N} = \{1, 2, 3, ...\}$, namely

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$$H_{r,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix} (a_1=1). \tag{3}$$

The Fekete-Szegö functional is obtained for $r = 2$ and $n = 1$ in (3), denoted by $H_{2,1}(f)$.

In recent years, research on estimation of an upper bound to the third Hankel determinant, namely $H_{3,1}(f)$ obtained for $r = 3$ and $n = 1$ in (3), as follows:

$$H_{3,1}(f)=\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \tag{4}$$

which has been focused on by many authors [6-13]. The sharp bounds of $|H_{3,1}(f)|$ for the functions namely, analytic, bounded turning, starlike and convex functions, which are the familiar subfamilies of $\mathcal{S}$, symbolized as $\mathcal{T}$, $\Re$, $\mathcal{S}^*$ and $\mathcal{K}$ respectively fulfilling the analytic conditions $\operatorname{Re}\frac{f(z)}{z} > 0$, $\operatorname{Re} f'(z) > 0$, $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0$ in the unit disc $\mathbb{D}$, were proved by Kowalczyk et al. [14-17] and derived the bounds as 4, 1/4, 4/9 and 4/135 respectively. Some more results on sharp bound on third Hankel determinant for different subclass of an analytic functions are obtain by many authors (see [18-26]). Very recently, Rath et al. [27] estimated the sharp bound of the third Hankel determinants for the inverse of starlike functions with respect to symmetric points.

Motivated with the results obtained by the authors specified above, in this paper we are making an attempt to estimate the sharp bound for the third Hankel determinant namely $|H_{3,1}(f^{-1})|$, when f belongs to the class $\Re$.

Let $\mathcal{P}$, be a class of all functions p having a positive real part in $\mathbb{D}$.

$$p(z)=1+\sum_{t=1}^{\infty} c_t z^t. \tag{5}$$

Every such a function is called the Carathéodory function [28]. We apply the technique that has been utilized by Libera and Zlotkiewicz [29] to derive our result. Further, we use the necessary sharp estimates given below in the form of lemmas, which applies equally to functions with a positive real part.

Lemma 1 ([30]) For $p \in \mathcal{P}$, then $|c_t| \le 2$, for $t \in \mathbb{N}$, equality occurs for the function $p_o(z)=\frac{1+z}{1-z}$, $z \in \mathbb{D}$.

Lemma 2 If $p \in \mathcal{P}$, then

$$c_2=\frac{1}{2}\left[c_1^2+t\zeta\right],$$

$$c_3=\frac{1}{4}\left[c_1^3+2c_1t\zeta-c_1t\zeta^2+2t(1-|\zeta|^2)\eta\right]$$

and

$$c_4=\frac{1}{8}\Big\{c_1^4+t\zeta\left[c_1^2\left(\zeta^2-3\zeta+3\right)+4\zeta\right]$$
$$-4t\left(1-|\zeta|^2\right)\left[c_1(\zeta-1)\eta+\bar{\zeta}\eta^2-\left(1-|\eta|^2\right)\xi\right]\Big\}$$

where $t := 4 - c_1^2$, for some ζ, η and ξ with $|\zeta| \le 1$, $|\eta| \le 1$ and $|\zeta| \le 1$. Here for c_2 (see [30], p. 166), c_3 (see [29]) and c_4 can be found in [31].

2. Main result

Theorem 1 If $f \in \mathfrak{R}$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ is the inverse function of f, then

$$|H_{3,1}(f^{-1})| \le \frac{44}{135},$$

the inequality is sharp for $f_1(z) = \log(1 + z)/(1 - z) - z$.

Proof. For $f \in \mathfrak{R}$, there exists a holomorphic function $p \in \mathcal{P}$ such that

$$f'(z) = p(z). \tag{6}$$

Using the series representation for f and p in (6), a simple calculation gives

$$a_n = \frac{c_{n-1}}{n},\ n > 1. \tag{7}$$

Since $f \in \mathfrak{R}$, using the definition of inverse function of f, we have

$$w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n (f^{-1}(w))^n. \tag{8}$$

Further, we have

$$w = f(f^{-1}) = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n (w + \sum_{n=2}^{\infty} t_n w^n)^n. \tag{9}$$

Simplifying, we obtain

$$(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4$$

$$+(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \ldots\ldots = 0. \tag{10}$$

Equating the coefficients of like power in (10), upon simplification, we obtain

$$t_2 = -a_2;\, t_3 = -a_3 + 2a_2^2;\, t_4 = -a_4 + 5a_2a_3 - 5a_2^3;$$

$$t_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{11}$$

Using the values of $a_n (n = 2, 3, 4, 5)$ from (7) in (11), yeilds

$$t_2 = -\frac{c_1}{2},$$

$$t_3 = \frac{-2c_2 + 3c_1^2}{6},$$

$$t_4 = \frac{-6c_3 + 20c_1c_2 - 15c_1^3}{24}$$

$$\text{and } t_5 = \frac{-24c_4 + 90c_1c_3 - 210c_1^2c_2 + 40c_2^2 + 105c_1^4}{120}. \tag{12}$$

Now,

$$H_{3,1}(f^{-1}) = \begin{vmatrix} t_1 = 1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{vmatrix}. \tag{13}$$

Using the values of t_j, ($j = 2, 3, 4, 5$) from (12) in (13), it simplifies to

$$H_{3,1}(f^{-1}) = \frac{1}{8640}\left(135c_1^6 - 540c_1^4c_2 + 720c_1^2c_2^2 + 576c_2c_4 - 432c_1^2c_4 + 720c_1c_2c_3 - 640c_2^3 - 540c_3^2\right). \tag{14}$$

In view of lemma 2 and equation (14), we have

$$-540c_1^4c_2 = -270\left[c_1^6 + c_1^4t\zeta\right];$$

$$-640c_2^3 = -80\left[c_1^6 + 3c_1^4t\zeta + 3c_1^2t^2\zeta^2 + t^3\zeta^3\right];$$

$$720c_1^2c_2^2 = 180\left[c_1^6 + 2c_1^4t\zeta + c_1^2t^2\zeta^2\right];$$

$$720c_1c_2c_3 = 90\Big[c_1^6 + 3c_1^4t\zeta + 2c_1^2t^2\zeta^2 - c_1^4t\zeta^2 - c_1^2t^2\zeta^3$$

$$+2t\left(c_1^3 + c_1t\zeta\right)\left(1 - |\zeta|^2\right)\eta\Big];$$

$$-540c_3^2 = \frac{135}{4}\Big[c_1^6 + 4c_1^4t\zeta + 4c_1^2t^2\zeta^2 - 2c_1^4t\zeta^2 - 4c_1^2t^2\zeta^3$$

$$+c_1^2t^2\zeta^4 + 4t(c_1^3 + 2ct\zeta - c_1t\zeta^2)(1 - |\zeta|^2)\eta$$

$$+4t^2(1 - |\zeta|^2)^2\eta^2\Big];$$

$$576c_2c_4 - 432c_1^2c_4 = 18\Big[-c_1^6 - 3c_1^4t\zeta - c_1^2\left(4-3c_1^2\right) - c_1^4t\zeta^3$$

$$-4c_1^3t(1-\zeta)(1-|\zeta|^2)\eta + 4c_1^2t\left(1-|\zeta|^2\right)\overline{\zeta}\eta^2$$

$$-4c_1^2t\left(1-|\zeta|\right)\left(1-|t|\right)\xi + 2c_1^4t\zeta + 6c_1^2t^2\zeta^2$$

$$+2\left(4-3c_1^2\right)t^2\zeta^3 + 2c_1^2t^2\zeta^4$$

$$+8t^2c_1\zeta\left(1-\zeta\right)\left(1-|\zeta|^2\right)\eta - 8t^2\left(1-|\zeta|^2\right)|\zeta|^2\eta^2$$

$$+8t^2\left(1-|\zeta|^2\right)\left(1-|\eta|^2\right)\zeta\xi\Big]. \tag{15}$$

From the expressions (14) and (15), we have

$$H_{3,1}(f^{-1}) = \frac{1}{8640}\Bigg(\frac{13}{4}c_1^6 + t\Big\{-33c_1^4\zeta + \frac{63}{2}c_1^4\zeta^2 - 72c_1^2\zeta^2 - 18c_1^4\zeta^3 + 93c_1^2t\zeta^2$$

$$+t\left[-63c_1^2\zeta^3 + \frac{9}{4}c_1^2\zeta^4 + 144\zeta^3 - 80t\zeta^3\right]$$

$$+\left[(-27+72\zeta)c_1^3 + c_1t\zeta(54-9\zeta)\right](1-|\zeta|^2)\eta$$

$$+\left[72c_1^2\overline{\zeta} - t(135+9|\zeta|^2)\right](1-|\zeta|^2)\eta^2$$

$$+72\left[2t\zeta - c_1^2\right](1-|\zeta|^2)(1-|\eta|^2)\xi\Big\}\Bigg). \tag{16}$$

For $c_1 := c$ and $t := 4 - c^2$ in (16), it takes the form

$$H_{3,1}(f^{-1}) = \frac{1}{8640}\Bigg(\frac{13}{4}c^6 + (4-c^2)\Big\{-33c^4\zeta + \frac{3}{2}c^2(200-41c^2)\zeta^2 - 18c^4\zeta^3$$

$$+(4-c^2)\left[\frac{9}{4}c^2\zeta^4 - (176-17c^2)\zeta^3\right]$$

$$+\left[(-27+72\zeta)c^3 + c(4-c^2)\zeta(57-9\zeta)\right](1-|\zeta|^2)\eta$$

$$+\left[72c^2\overline{\zeta} - (4-c^2)(135+9|\zeta|^2)\right](1-|\zeta|^2)\eta^2$$

$$+72\left[2(4-c^2)\zeta-c^2\right](1-|\zeta|^2)(1-|\eta|^2)\xi\Big\}\Big). \tag{17}$$

Taking modulus on both sides in expression (17), with $|\zeta| = x \in [0, 1]$, $|\eta| = y \in [0, 1]$, $c_1 = c \in [0, 2]$ and $|\zeta| \le 1$, we obtain

$$\left|H_{3,1}(f^{-1})\right| \le \frac{\Phi(c,x,y)}{8640}, \tag{18}$$

where $\Phi(c, x, y) : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\Phi(c,x,y) = \frac{13}{4}c^6 + (4-c^2)\Big\{33c^4x + \frac{3}{2}c^2(200-41c^2)x^2 + 18c^4x^3$$

$$+(4-c^2)\left[\frac{9}{4}c^2x^4 + (176-17c^2)x^3\right]$$

$$+\left[(27+72x)c^3 + c(4-c^2)x(57+9x)\right](1-x^2)y$$

$$+\left[72c^2x + (4-c^2)(135+9x^2)\right](1-x^2)y^2$$

$$+72\left[2(4-c^2)x+c^2\right](1-x^2)(1-y^2)\Big\}. \tag{19}$$

Now, we will maximize the function $\Phi(c, x, y)$ in the region of the parallelepiped formed by $[0, 2] \times [0, 1] \times [0, 1]$, where $c \in [0, 2]$, $x \in [0, 1]$ and $y \in [0, 1]$.

A. On the vertices of the parallelepiped, we obtain

$$\Phi(0, 0, 0) = 0;\ \Phi(2, 0, 0) = \Phi(2, 1, 0) = \Phi(2, 1, 1) = \Phi(2, 0, 0) = 208,$$

$$\Phi(0, 0, 1) = 2160,\ \Phi(0, 1, 0) = \Phi(0, 1, 1) = 2816.$$

B. Now, considering the eight edges of the parallelepiped.

(i) For $x = 1$ and $y = 0$; $x = 1$ and $y = 1$ in (19) we have

$$\Phi(c,1,y) = 2816 - 444c^2 - 48c^4 - c^6 \le 2816, \text{ for } c \in (0,2).$$

(ii) $c = 0$ and $y = 0$.

$$\Phi(0,x,0) = 2304x + 512x^3 \le 2816, \text{ for } x \in (0,1).$$

(iii) $x = 0$ and $y = 1$.

$$\Phi(c,0,1) = 2160 - 1080c^2 + 108c^3 + 135c^4 - 27c^5 + \frac{13}{4}c^6$$

$$= 2160 + \frac{13}{4}c^6 - 27c^2(40 - 4c - 5c^2 + c^3)$$

$$= 2160 + \frac{13}{4}c^6 - 27c^2[4(5-c) + 5(4-c^2) + c^3]$$

$$\leq 2160 + \frac{13}{4}c^6 \leq 2368.$$

(iv) $c = 0$ and $y = 1$.

$$\Phi(0, x, 1) = 2160 - 2016x^2 + 2816x^3 - 144x^4 \leq 2816, \text{ for } x \in (0, 1).$$

(v) $c = 0$ and $x = 0$.

$$\Phi(0, x, 0) = 2160y^2 \leq 2160, \text{ for } y \in (0, 1).$$

(vi) For $c = 0$ and $x = 1$.

$$\Phi(0, 1, y) = 2816.$$

(vii) $c = 2$ and $y = 0$; $c = 2$ and $y = 1$; $c = 2$ and $x = 0$; $c = 2$ and $x = 1$.

$$\Phi(c, x, y) = 208.$$

(viii) $x = 0$ and $y = 0$.

$$\Phi(c, 0, 0) = 288c^2 - 72c^4 + \frac{13}{4}c^6, \text{ for } c \in (0, 2)$$

$$\leq 288c^2 + \frac{13}{4}c^6 \leq 1360.$$

C. Now, we consider the six faces of the parallelepiped.
(i) For $c = 2$, we obtain $\Phi(2, x, y) = 208$ for $x, y \in (0, 1)$.
(ii) If $c = 0$ in (19), then

$$\Phi(0, x, y) = 4\left(704x^3 + 4\left(1 - x^2\right)\left(135 + 9x^2\right)y^2 + 576x\left(1 - x^2\right)\left(1 - y^2\right)\right)$$

$$= 2304x + 512x^3 + 144(15 - x)(1 - x)^2(1 + x)y^2$$

$$\leq 2304x + 512x^3 + 144(15 - x)(1 - x)^2(1 + x) \leq 2816.$$

(iii) If $x = 0$ in (19), then

$$\Phi(c,0,y)=\frac{13c^6}{4}+(4-c^2)\left[27c^3y+135(4-c^2)y^2+72c^2(1-y^2)\right]$$

$$=\frac{13c^6}{4}+(4-c^2)\left[27c^3y+540y^2+c^2(72-207y^2)\right]$$

$$\leq\frac{13c^6}{4}+(4-c^2)\left[27c^3+540+72c^2\right]\leq 2368.$$

(iv) On the face $x = 1$, $c \in (0, 2)$, $y \in (0, 1)$ from (19), we observe that the function $\Phi(c, 1, y)$ is independent of y, discussed in B (i).

$$\Phi(c,1,y)\leq 2816.$$

(v) On the face $y = 0$, $c \in (0, 2)$, $x \in (0, 1)$ from (19), we have

$$\Phi(c,x,0)=\frac{13}{4}c^6+(4-c^2)\left\{33c^4x+\frac{3}{2}c^2(200-41c^2)x^2+18c^4x^3\right.$$

$$+(4-c^2)\left[\frac{9}{4}c^2x^4+(176-17c^2)x^3\right]$$

$$\left.+72\left[2(4-c^2)x+c^2\right](1-x^2)\right\}$$

$$=\frac{13}{4}c^6+(4-c^2)\left\{576x+128x^3+c^4\left(33x-\frac{123}{2}x^3+35x^3-\frac{9}{4}x^4\right)\right.$$

$$\left.+c^2(72-144x+228x^2-100x^3+9x^4)\right\}$$

$$\leq\frac{13}{4}c^6+(4-c^2)\left\{704+72c^2+6c^4\right\}\leq 2816.$$

(vi) On the face $y = 1$, $c \in (0, 2)$, $x \in (0, 1)$ from (19), we have

$$\Phi(c,x,1)=\frac{13}{4}c^6+(4-c^2)\left\{33c^4x+\frac{3}{2}c^2(200-41c^2)x^2+18c^4x^3\right.$$

$$+(4-c^2)\left[\frac{9}{4}c^2x^4+(176-17c^2)x^3\right]$$

$$+\left[(27+72x)c^3+c(4-c^2)x(57+9x)\right](1-x^2)$$

$$+\left[72c^2x+(4-c^2)(135+9x^2)\right](1-x^2)\Big\}$$

$$:= g(c,x),\ \text{with } c\in(0,2) \text{ and } x\in(0,1).$$

Further,

$$\frac{\partial g}{\partial c}=-2160c+324c^2+540c^3-135c^4+\frac{39c^5}{2}+864x+576cx-432c^2x+240c^3x$$

$$-90c^4x-198c^5x+144x^2+4416cx^2-540c^2x^2-2688c^3x^2+180c^4x^2$$

$$+369c^5x^2-864x^3-3936cx^3+432c^2x^3+1824c^3x^3+90c^4x^3-210c^5x^3$$

$$-144x^4+216cx^4+216c^2x^4-108c^3x^4-45c^4x^4+\frac{27c^5x^4}{2}$$

and

$$\frac{\partial g}{\partial x}=864c+288c^2-144c^3+60c^4-18c^5-33c^6-4032x+288cx+4416c^2x$$

$$-360c^3x-1344c^4x+72c^5x+123c^6x+8448x^2-2592cx^2-5904c^2x^2$$

$$+432c^3x^2+1368c^4x^2+54c^5x^2-105c^6x^2-576x^3-576cx^3+432c^2x^3$$

$$+288c^3x^3-108c^4x^3-36c^5x^3+9c^6x^3.$$

The solution of the system $\frac{\partial g}{\partial c}=0$ and $\frac{\partial g}{\partial x}=0$ which belongs to the region $(0, 2) \times (0, 1)$, is $(c_1, x_1) \approx (0.248233, 0.445259)$.

But at (c_1, x_1), we observe that

$$\left(\frac{\partial^2 g}{\partial c^2}\right)\left(\frac{\partial^2 g}{\partial x^2}\right)-\left(\frac{\partial^2 g}{\partial c\partial x}\right)^2<0.$$

Therefore, there is no critical point in $y = 1$.

D. Now, considering the interior of the parallelepiped: $(0, 2) \times (0, 1) \times (0, 1)$. We have $\partial\Phi/\partial y = 0$ if, and only if,

$$y_0(c,x)=\frac{4cx(6+x)+c^3(3+2x-x^2)}{2\left(c^2(-23+x)-4(-15+x)\right)(-1+x)},$$

for $(c, x) \in (0, 2)\times(0, 1)$ and $c^2(-23 + x) \neq 4(-15 + x)$. A numerical computation shows that all real solutions of the system of equation $\partial\Phi/\partial c\ (c, x, y_0(c, x)) = 0$ and $\partial\Phi/\partial x\ (c, x, y_0(c, x)) = 0$ are as follows:

$$(c \approx \pm 2, x \approx -0.241686), (c \approx \pm 0.884709, x \approx 16.4392)$$

$$(c \approx \pm 1.54879, x \approx -0.954118), (c \approx \pm 2.10388, x \approx 107.052).$$

Hence, $\Phi(c, x, y)$ has no critical point in the interior of the parallelepiped.
In review of cases A, B, C and D, we obtain

$$\max\left\{\Phi(c, x, y) : c \in [0, 2], x \in [0, 1], y \in [0, 1]\right\} = 2816. \tag{20}$$

From expression (18) and (20), we get

$$\left| H_{3,1}(f^{-1}) \right| \le \frac{44}{135}. \tag{21}$$

From $f_1(z)$, we obtain $a_2 = a_4 = 0$, $a_3 = 2/3$, $a_5 = 2/5$, which inturn gives, $t_1 = 1$, $t_2 = t_4 = 0$, $t_3 = -2/3$, $t_5 = 14/15$ and it follows the result. □

3. Concluding remarks and observations

In this paper, we estimate the sharp bound to the third order Hankel determinant for the inverse function of f, when f is a member of the class of bounded turning functions. In our research, we have used the connection between the coefficients of the functions of the considered class and the Carathéodory class. Further, using the relation between the coefficients of the function f and its inverse f^{-1}, we expressed the coefficient of f^{-1} interms of the coefficient of the functions belongs to the Carathéodory class. Based on the result obtained in this paper researchers may obtain the results of same kind for the other familiar subclasses of univalent functions.

Conflict of interest

All authors have no conflicts of interest.

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