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Research Article

Existence Results for Differential Equations of Fourth Order with Non-Homogeneous Boundary Conditions

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Abstract: The objective of this paper is to investigate the existence and uniqueness of solutions to fourth order differential equations

$$v^{(4)}(x) + f(x, v(x)) = 0, x \in [\mathfrak{a}, \mathfrak{b}],$$

satisfying the three-point non-homogeneous conditions

$$v(\mathfrak{a}) = 0, v'(\mathfrak{a}) = 0, v''(\mathfrak{a}) = 0, v'(\mathfrak{b}) - \alpha v'(\zeta) = \mu,$$

where $0 \le \mathfrak{a} < \zeta < \mathfrak{b}$, the constants α, μ are real numbers and $f : [\mathfrak{a}, \mathfrak{b}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The framework for establishing the existence results is based on sharper estimates on the integral of the kernel to connect with fixed point theorems of Banach and Rus.

***Keywords*:** differential equation, three-point non-homogeneous conditions, kernel, existence results, fixed point theorems

MSC: 34B15, 34B10

1. Introduction

The theory of differential equation has emerged as a powerful instrument for comprehending and interpreting problems in a variety of scientific domains. The notion that the laws concerning these situations can be expressed as differential equations of various orders with certain initial or boundary conditions. Especially, the fourth order differential equations appear in the modeling of inelastic flows, viscoelastic, theory of plate deflection, bending of beams and various areas of applied mathematics as well as engineering [1-4]. Due to their enormous importance in theory and

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applications, researchers have studied the existence and uniqueness of solutions to the differential equations of various orders under certain conditions by different methods. We mention a few papers on the existence and uniqueness of solutions to the boundary value problems of third order [5-9] and fourth order [10-14] for readers reference.

In 1988, Gupta [15] demonstrated the existence results for the deformation of stretchy beam with completely supported edges and is characterized by a boundary problem of fourth order of the form

$$\frac{d^4u}{dx^4} - \pi^4 u + g(x, u) = e(x),\ 0 < x < 1,$$

$$u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0.$$

In 2019, Li and Gao [16] discussed the solvability of fourth order boundary value problem

$$u^{(4)}(x) = f(x, u(x), u''(x)),\ x \in [0, 1],$$

$$u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0,$$

which describes the deformation of an elastic beam whose two ends are simply supported in equilibrium state.

In 2021, Khanfer and Bougoffa [17] established the existence and uniqueness theorem for the nonlocal fourth order nonlinear beam differential equations with a parameter

$$u^{(4)} + A(x)u = \lambda f(x, u, u''),\ 0 < x < 1,$$

$$u(0) = u(1) = \int_0^1 p(x)u(x)dx,\ u''(0) = u''(1) = \int_0^1 q(x)u''(x)dx.$$

Inspired by these papers, we consider the fourth order differential equation

$$v^{(4)}(x) + f(x, v(x)) = 0,\ x \in [\mathfrak{a}, \mathfrak{b}], \tag{1}$$

satisfying the three-point non-homogeneous conditions

$$v(\mathfrak{a}) = 0,\ v'(\mathfrak{a}) = 0,\ v''(\mathfrak{a}) = 0,\ v'(\mathfrak{b}) - \alpha v'(\zeta) = \mu, \tag{2}$$

where $0 \le \mathfrak{a} < \zeta < \mathfrak{b}$, the constants α, μ are real numbers, and establish the existence results by employing the fixed point theorems. The main concern of this study is to describe the deformations of an elastic beam by means of a fourth order boundary value problem, where the boundary conditions are given according to the controls at the ends of the beam.

Definition 1.1 A solution of the problem (1)-(2) is a function $v \in C^{(4)}([a, b], \mathbb{R})$ and is satisfying (1)-(2).

The problem must be well-posed in real-world applications and in the interpretation of differential equations with certain constraints. If a problem has only one solution and it has certain ideal conditions, we can validate the "well-posedness" of the problem using a variety of methods. If the problem has two or more solutions or no solution, it is not well posed from a modeling standpoint, and it must be ignored, and must be created a new model [18].

Now, we assume the following conditions are true in the entire paper:

(C1) $f : [\mathfrak{a}, \mathfrak{b}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function,

(C2) the constants $\mathfrak{a}$ and $\mathfrak{b}$ satisfies $\frac{(\mathfrak{b}-\mathfrak{a})^2}{(\zeta-\mathfrak{a})^2} \neq \alpha$, and

(C3) $|f(x, v) - f(x, w)| \leq \lambda|v - w|$, for all $(x, v), (x, w) \in [\mathfrak{a}, \mathfrak{b}] \times \mathbb{R}$, where $\lambda < \dfrac{9}{(b-a)^4\left[\dfrac{1}{8}+\dfrac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{|(\mathfrak{b}-\mathfrak{a})^2-\alpha(\zeta-\mathfrak{a})^2|}\right]}$

is a Lipschitz constant.

Below is the summary of the remaining part of the study. Section 2 describes the solution to the problem (1)-(2) as the solution to the corresponding integral equation that includes the kernel, after which the sharper estimates of the integral of kernel is calculated. The results for the existence and uniqueness of solutions of (1)-(2) are proven by fixed point theorems using the estimates on the integral of kernel and the conclusions are supported by examples in Section 3.

2. Preliminary findings

We begin by expressing the solution of the problem (1)-(2) into the corresponding integral equation involving kernel. The sharper estimates of the integral of the kernels are then computed. These will be useful in demonstrating our main findings.

Let $\Phi(x) \in C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$, then the unique solution of the problem

$$v^{(4)}(x)+\Phi(x)=0,\ x\in[\mathfrak{a},\ \mathfrak{b}], \tag{3}$$

satisfying the conditions given in (2) is obtained.

Lemma 2.1 Let $\Delta = 3[(\mathfrak{b} - \mathfrak{a})^2 - \alpha(\zeta - \mathfrak{a})^2]$. If the condition (C2) is fulfilled, then the unique solution to the problem (3) with (2) is given by

$$v(x)=\frac{\mu(x-\mathfrak{a})^3}{\Delta}+\int_{\mathfrak{a}}^{\mathfrak{b}}\mathcal{H}(x,\ \Upsilon)\Phi(\Upsilon)d\Upsilon,$$

where

$$\mathcal{H}(x,\ \Upsilon)=\mathcal{M}(x,\ \Upsilon)+\frac{\alpha(x-\mathfrak{a})^3}{\Delta}\mathcal{N}(\zeta,\ \Upsilon), \tag{4}$$

$$\mathcal{M}(x,\ \Upsilon)=\begin{cases}\dfrac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2}-\dfrac{(x-\Upsilon)^3}{3!}, & \mathfrak{a}\le\Upsilon\le x\le\mathfrak{b},\\[2ex] \dfrac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2}, & \mathfrak{a}\le x\le\Upsilon\le\mathfrak{b},\end{cases} \tag{5}$$

and

$$\mathcal{N}(\zeta,\ \Upsilon)=\begin{cases}\dfrac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2}-\dfrac{(\zeta-\Upsilon)^2}{2}, & \mathfrak{a}\le\Upsilon\le\zeta\le\mathfrak{b},\\[2ex] \dfrac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2}, & \mathfrak{a}\le\zeta\le\Upsilon\le\mathfrak{b},\end{cases} \tag{6}$$

Proof. The corresponding integral equation of (3) is

$$v(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 - \frac{1}{3!}\int_a^x (x-\Upsilon)^3 \Phi(\Upsilon)d\Upsilon, \tag{7}$$

where A_0, A_1, A_2 and A_3 are constants. Using the conditions (2) in (7), then we have the following set of equations

$$A_0 + A_1\mathfrak{a} + A_2\mathfrak{a}^2 + A_3\mathfrak{a}^3 = 0,$$

$$A_1 + 2A_2\mathfrak{a} + 3A_3\mathfrak{a}^2 = 0,$$

$$A_2 + 3A_3\mathfrak{a} = 0,$$

$$A_1(1-\alpha) + 2A_2(\mathfrak{b} - \alpha\zeta) + 3A_3(\mathfrak{b}^2 - \alpha\zeta^2) = \Gamma,$$

where

$$\Gamma = \mu + \frac{1}{2}\int_{\mathfrak{a}}^{\mathfrak{b}} (\mathfrak{b}-\Upsilon)^2\Phi(\Upsilon)d\Upsilon - \frac{\alpha}{2}\int_{\mathfrak{a}}^{\zeta}(\zeta-\Upsilon)^2\Phi(\Upsilon)d\Upsilon.$$

On solving the above, we get

$$A_0 = -\frac{\mathfrak{a}^3\Gamma}{\Delta}, A_1 = \frac{3\mathfrak{a}^2\Gamma}{\Delta}, A_2 = -\frac{3\mathfrak{a}\Gamma}{\Delta} \text{ and } A_3 = \frac{\Gamma}{\Delta}.$$

Substituting these values in (7), we have

$$v(x) = [-\mathfrak{a}^3 + 3\mathfrak{a}^2 x - 3\mathfrak{a}x^2 + x^3]\frac{\Gamma}{\Delta} - \frac{1}{3!}\int_{\mathfrak{a}}^x (x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{(x-\mathfrak{a})^3}{\Delta}\left[\mu + \frac{1}{2}\int_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{b}-\Upsilon)^2\Phi(\Upsilon)d\Upsilon - \frac{\alpha}{2}\int_{\mathfrak{a}}^{\zeta}(\zeta-\Upsilon)^2\Phi(\Upsilon)d\Upsilon\right]$$

$$-\frac{1}{3!}\int_{\mathfrak{a}}^x (x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \frac{(x-\mathfrak{a})^3[(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2 + \alpha(\zeta-\mathfrak{a})^2]}{3![(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2](\mathfrak{b}-\mathfrak{a})^2}\int_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{b}-\Upsilon)^2\Phi(\Upsilon)d\Upsilon$$

$$-\frac{\alpha(x-\mathfrak{a})^3}{2\Delta}\int_{\mathfrak{a}}^{\zeta}(\zeta-\Upsilon)^2\Phi(\Upsilon)d\Upsilon - \frac{1}{3!}\int_{\mathfrak{a}}^x (x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \frac{(x-\mathfrak{a})^3}{3!(\mathfrak{b}-\mathfrak{a})^2}\int_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{b}-\Upsilon)^2\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathfrak{a})^3(\zeta-\mathfrak{a})^2}{2(\mathfrak{b}-\mathfrak{a})^2\Delta}$$

$$\int_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{b}-\Upsilon)^2\Phi(\Upsilon)d\Upsilon - \frac{\alpha(x-\mathfrak{a})^3}{2\Delta}\int_{\mathfrak{a}}^{\zeta}(\zeta-\Upsilon)^2\Phi(\Upsilon)d\Upsilon - \frac{1}{3!}\int_{\mathfrak{a}}^{x}(x-\Upsilon)^3\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{x}\left[\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2} - \frac{(x-\Upsilon)^3}{3!}\right]\Phi(\Upsilon)d\Upsilon$$

$$+\int_{x}^{\mathfrak{b}}\left[\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2}\right]\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathfrak{a})^3}{\Delta}$$

$$\left\{\int_{\mathfrak{a}}^{\zeta}\left[\frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2} - \frac{(\zeta-\Upsilon)^2}{2}\right]\Phi(\Upsilon)d\Upsilon + \int_{\zeta}^{\mathfrak{b}}\left[\frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2}\right]\Phi(\Upsilon)d\Upsilon\right\}$$

$$= \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{\mathfrak{b}}\mathcal{M}(x,\Upsilon)\Phi(\Upsilon)d\Upsilon + \frac{\alpha(x-\mathfrak{a})^3}{\Delta}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathcal{N}(\zeta,\Upsilon)\Phi(\Upsilon)d\Upsilon$$

$$= \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{\mathfrak{b}}\mathcal{H}(x,\Upsilon)\Phi(\Upsilon)d\Upsilon.$$

To establish the uniqueness, let $u(x)$ be another solution of (3) and (2). Take $w(x) = v(x) - u(x)$. Then

$$w^{(4)}(x) = 0,\ x \in [\mathfrak{a}, \mathfrak{b}], \tag{8}$$

$$w(\mathfrak{a}) = 0,\ w'(\mathfrak{a}) = 0,\ w''(\mathfrak{a}) = 0,\ w'(\mathfrak{b}) - \alpha w'(\zeta) = 0. \tag{9}$$

Therefore, the solution of (8) is

$$w(x) = D_0 + D_1 x + D_2 x^2 + D_3 x^3,$$

where D_0, D_1, D_2 and D_3 are the arbitrary constants. Apply the conditions (9), it can be written as a matrix form $\mathbf{AD} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & \mathfrak{a} & \mathfrak{a}^2 & \mathfrak{a}^3 \\ 0 & 1 & 2\mathfrak{a} & 3\mathfrak{a}^2 \\ 0 & 0 & 1 & 3\mathfrak{a} \\ 0 & 1-\alpha & 2(\mathfrak{b}-\alpha\zeta) & 3(\mathfrak{b}^2-\alpha\zeta^2) \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 \end{bmatrix}^T$$

and

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

with $|A| = \Delta$, which is not equal to zero from (C2). So, the matrix system $\mathbf{AD} = \mathbf{O}$ has only a trivial solution. As a result, $w(x) \equiv 0$ for all $x \in [\mathfrak{a}, \mathfrak{b}]$. Thus, the uniqueness of solution of the problem (3) and (2) is proved.

□

Remark 2.1 By Lemma 2.1, $v(x)$ is a solution of the boundary value problem (1)-(2) if and only if $v(x)$ satisfies the following integral equation

$$v(x) = \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{H}(x, \Upsilon) f(\Upsilon, v(\Upsilon)) d\Upsilon, \text{ for all } x \in [\mathfrak{a}, \mathfrak{b}]. \tag{10}$$

Lemma 2.2 The kernel $\mathcal{M}(x, \Upsilon)$ given in (5) is non-negative for all $x, \Upsilon \in [\mathfrak{a}, \mathfrak{b}]$.
Proof. The algebraic computations can be used to establish the positivity of $\mathcal{M}(x, \Upsilon)$.

□

Lemma 2.3 The kernel $\mathcal{M}(x, \Upsilon)$ given in (5) satisfies the following integral inequality

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{M}(x, \Upsilon) d\Upsilon \le \frac{1}{72}(\mathfrak{b}-\mathfrak{a})^4, \text{ for all } x \in [\mathfrak{a}, \mathfrak{b}]. \tag{11}$$

Proof. We now consider, then for all $x \in [\mathfrak{a}, \mathfrak{b}]$ to obtain

$$\int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{M}(x, \Upsilon) d\Upsilon = \int_{\mathfrak{a}}^{x} \left[\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2} - \frac{(x-\Upsilon)^3}{3!} \right] d\Upsilon + \int_{x}^{\mathfrak{b}} \frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^2}{3!(\mathfrak{b}-\mathfrak{a})^2} d\Upsilon$$

$$= \left[-\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^3}{18(\mathfrak{b}-\mathfrak{a})^2} + \frac{(x-\Upsilon)^4}{24} \right]_{\mathfrak{a}}^{x} + \left[-\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\Upsilon)^3}{18(\mathfrak{b}-\mathfrak{a})^2} \right]_{x}^{\mathfrak{b}}$$

$$= \frac{(x-\mathfrak{a})^3(\mathfrak{b}-\mathfrak{a})}{18} - \frac{(x-\mathfrak{a})^4}{24}.$$

Let $\Psi(x) = \frac{(x-\mathfrak{a})^3(\mathfrak{b}-\mathfrak{a})}{18} - \frac{(x-\mathfrak{a})^4}{24}$. Then $\Psi'(x) = \frac{(x-\mathfrak{a})^2(\mathfrak{b}-\mathfrak{a})}{6} - \frac{(x-\mathfrak{a})^3}{6}$. For stationary points, we have $\Psi'(x) = 0$. On solving, we get $x = \mathfrak{b}$. Since $\Psi''(\mathfrak{b}) = -\frac{(\mathfrak{b}-\mathfrak{a})^2}{6} < 0$, $\Psi(x)$ has maximum at $x = \mathfrak{b}$ and is given by

$$\max_{x\in[\mathfrak{a},\mathfrak{b}]} \Psi(x) = \max_{x\in[\mathfrak{a},\mathfrak{b}]} \left[\frac{(x-\mathfrak{a})^3(\mathfrak{b}-\mathfrak{a})}{18} - \frac{(x-\mathfrak{a})^4}{24} \right] = \frac{1}{72}(\mathfrak{b}-\mathfrak{a})^4.$$

Hence, the inequality (11).

□

Lemma 2.4 The kernel $\mathcal{N}(\zeta, \Upsilon)$ given in (6) satisfies the following integral inequality

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{N}(\zeta, \Upsilon)| d\Upsilon \leq \frac{1}{3}(\mathfrak{b}-\mathfrak{a})^3.$$

Proof. We can consider, then

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{N}(\zeta, \Upsilon)| d\Upsilon = \int_{\mathfrak{a}}^{\zeta} |\mathcal{N}(\zeta, \Upsilon)| d\Upsilon + \int_{\zeta}^{\mathfrak{b}} |\mathcal{N}(\zeta, \Upsilon)| d\Upsilon$$

$$= \int_{\mathfrak{a}}^{\zeta} \left| \frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2} - \frac{(\zeta-\Upsilon)^2}{2} \right| d\Upsilon + \int_{\zeta}^{\mathfrak{b}} \left| \frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2} \right| d\Upsilon$$

$$\leq \int_{\mathfrak{a}}^{\zeta} \left[\frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2} + \frac{(\zeta-\Upsilon)^2}{2} \right] d\Upsilon + \int_{\zeta}^{\mathfrak{b}} \frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^2}{2(\mathfrak{b}-\mathfrak{a})^2} d\Upsilon$$

$$= \left[-\frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^3}{6(\mathfrak{b}-\mathfrak{a})^2} - \frac{(\zeta-\Upsilon)^3}{6} \right]_{\mathfrak{a}}^{\zeta} + \left[-\frac{(\zeta-\mathfrak{a})^2(\mathfrak{b}-\Upsilon)^3}{6(\mathfrak{b}-\mathfrak{a})^2} \right]_{\zeta}^{\mathfrak{b}}$$

$$= \frac{1}{6}(\zeta-\mathfrak{a})^2(\mathfrak{b}-a) + \frac{1}{6}(\zeta-\mathfrak{a})^3$$

$$\leq \frac{1}{3}(\mathfrak{b}-\mathfrak{a})^3.$$

□

Lemma 2.5 The kernel $\mathcal{H}(x, \Upsilon)$ in (4) satisfies the the following integral inequality

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)| d\Upsilon \leq \frac{(\mathfrak{b}-\mathfrak{a})^4}{9} \left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{\left|(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2\right|} \right].$$

Proof. We can consider, then

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)| d\Upsilon = \int_{\mathfrak{a}}^{\mathfrak{b}} \left| \mathcal{M}(x, \Upsilon) + \frac{\alpha(x-\mathfrak{a})^3}{\Delta} \mathcal{N}(\zeta, \Upsilon) \right| d\Upsilon$$

$$\leq \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{M}(x, \Upsilon)| d\Upsilon + \left| \frac{\alpha(x-\mathfrak{a})^3}{\Delta} \right| \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{N}(\zeta, \Upsilon)| d\Upsilon$$

$$\leq \frac{1}{72}(\mathfrak{b}-\mathfrak{a})^4 + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^3}{|\Delta|} \times \frac{1}{3}(\mathfrak{b}-\mathfrak{a})^3$$

$$= \frac{(\mathfrak{b}-\mathfrak{a})^4}{9}\left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{\left|(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2\right|}\right].$$

□

The Banach and Rus fixed point theorems, which are stated below, are important tools for establishing our results.

Theorem 2.6 [19] Let **d** be a metric on a nonempty set $\mathcal{D}$ and the pair ($\mathcal{D}$, **d**) form a complete metric space. If the function $\mathcal{F}: \mathcal{D} \to \mathcal{D}$ satisfies the following inequality for $v, w \in \mathcal{D}$,

$$\mathbf{d}(\mathcal{F}v, \mathcal{F}w) \le \beta \mathbf{d}(v, w), \text{ where } 0 < \beta < 1,$$

then there is a unique point $\vartheta^* \in \mathcal{D}$ with $\mathcal{F}\vartheta^* = \vartheta^*$.

Theorem 2.7 [20] Let **d** and ρ be two metrics on a nonempty set $\mathcal{D}$ and the pair ($\mathcal{D}$, **d**) form a complete metric space. If the function $\mathcal{F}: \mathcal{D} \to \mathcal{D}$ is continuous with respect to the metric **d** on $\mathcal{D}$ and satisfies the following inequalities for $v, w \in \mathcal{D}$,

$$\mathbf{d}(\mathcal{F}v, \mathcal{F}w) \le \theta \rho(v, w), \text{ where } \theta > 0, \tag{12}$$

and

$$\rho(\mathcal{F}v, \mathcal{F}w) \le \kappa \rho(v, w), \text{ where } 0 < \kappa < 1, \tag{13}$$

then there is a unique point $\vartheta^* \in \mathcal{D}$ with $\mathcal{F}\vartheta^* = \vartheta^*$.

3. Main results based on metrics

The uniqueness of solutions to the problem (1)-(2) are established based on metrics in this section. Let $\mathcal{D}$ be the set of all real-valued functions continuous on $[\mathfrak{a}, \mathfrak{b}]$. For $v(x), w(x) \in \mathcal{D}$, we now define the metrics on $\mathcal{D}$ as follows:

$$\mathbf{d}(v, w) = \max_{x\in[\mathfrak{a},\mathfrak{b}]} |v(x) - w(x)|, \tag{14}$$

and

$$\rho(v, w) = \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |v(x) - w(x)|^p \, dx\right)^{\frac{1}{p}}, p > 1. \tag{15}$$

Here the ordered pair ($\mathcal{D}$, **d**) forms a complete metric space, whereas ($\mathcal{D}$, ρ) is a metric space but not a complete. The following is a useful relation between the two metrics **d** and ρ on $\mathcal{D}$ and is given by

$$\rho(v, w) \le (\mathfrak{b}-\mathfrak{a})^{\frac{1}{p}} \mathbf{d}(v, w), \text{ for all } v, w \in \mathcal{D}. \tag{16}$$

Let us consider the operator $\mathcal{F}: \mathcal{D} \to \mathcal{D}$ as

$$\mathcal{F}v(x) = \frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{H}(x, \Upsilon) f(\Upsilon, v(\Upsilon)) d\Upsilon, \text{ for all } x \in [\mathfrak{a}, \mathfrak{b}], \tag{17}$$

where the kernel $\mathcal{H}(x, \Upsilon)$ is mentioned in (4).

Theorem 3.1 Suppose the conditions (C1)-(C3) are fulfilled. If $\mathfrak{a}$, $\mathfrak{b}$ satisfies the following inequality

$$\frac{(\mathfrak{b}-\mathfrak{a})^4}{9}\left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{\left|(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2\right|}\right] < \frac{1}{\lambda}, \tag{18}$$

then there is a unique solution to the problem (1)-(2).

Proof. We have to prove the problem (1)-(2) has a unique solution. It is enough to prove the operator $\mathcal{F}$ has a unique fixed point $\vartheta^* \in \mathcal{D}$ with $\mathcal{F}\vartheta^* = \vartheta^*$. According to (17), every such fixed point will also lie in $\mathcal{C}^{(4)}([\mathfrak{a}, \mathfrak{b}])$ as can be directly shown by differentiating (10).

Consider, for any $v, w \in \mathcal{D}$ and for $x \in [\mathfrak{a}, \mathfrak{b}]$, we obtain

$$\left|\mathcal{F}v(x) - \mathcal{F}w(x)\right| = \left|\frac{\mu(x-\mathfrak{a})^3}{\Delta} + \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{H}(x, \Upsilon) f(\Upsilon, v(\Upsilon)) d\Upsilon - \frac{\mu(x-\mathfrak{a})^3}{\Delta} - \int_{\mathfrak{a}}^{\mathfrak{b}} \mathcal{H}(x, \Upsilon) f(\Upsilon, w(\Upsilon)) d\Upsilon\right|$$

$$\leq \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)| \left|f(\Upsilon, v(\Upsilon)) - f(\Upsilon, w(\Upsilon))\right| d\Upsilon$$

$$\leq \lambda \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)| \left|v(\Upsilon) - w(\Upsilon)\right| d\Upsilon$$

$$\leq \lambda \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)| \mathbf{d}(v, w) d\Upsilon$$

$$\leq \lambda \frac{(\mathfrak{b}-\mathfrak{a})^4}{9}\left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{\left|(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2\right|}\right] \mathbf{d}(v, w),$$

using (C3). It is evident from the fact that

$$\mathbf{d}(\mathcal{F}v, \mathcal{F}w) \leq \beta \mathbf{d}(v, w),$$

where

$$\beta = \lambda \frac{(\mathfrak{b}-\mathfrak{a})^4}{9}\left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b}-\mathfrak{a})^2}{\left|(\mathfrak{b}-\mathfrak{a})^2 - \alpha(\zeta-\mathfrak{a})^2\right|}\right].$$

Using (18), $\beta < 1$ and hence, the operator $\mathcal{F}$ is fulfilled the condition of Theorem 2.6. This implies that the operator $\mathcal{F}$ has a unique fixed point and is the solution of the problem (1)-(2).

□

Consider an example to demonstrate the above result.

Example 3.1 Let us take $\mathfrak{a} = \frac{1}{2}$, $\mathfrak{b} = 1$, $\zeta = \frac{3}{4}$, $\alpha = \frac{1}{4}$ and $f(x, v) = e^x \sin x + \cos v$. Then consider

$$v^{(4)}(x) + e^x \sin x + \cos v = 0,\ x \in \left[\frac{1}{2}, 1\right], \tag{19}$$

with

$$v\left(\frac{1}{2}\right) = 0,\ v'\left(\frac{1}{2}\right) = 0,\ v''\left(\frac{1}{2}\right) = 0,\ v'(1) - \frac{1}{4} v'\left(\frac{3}{4}\right) = \mu. \tag{20}$$

Then $\Delta = 3[(\mathfrak{b} - \mathfrak{a})^2 - \alpha(\zeta - \mathfrak{a})^2] = \frac{45}{64}$, $\left|\frac{\partial f(x, v)}{\partial v}\right| = |-\sin v| \le 1$ and

$$\frac{(\mathfrak{b} - \mathfrak{a})^4}{9}\left[\frac{1}{8} + \frac{|\alpha|(\mathfrak{b} - \mathfrak{a})^2}{\left|(\mathfrak{b} - \mathfrak{a})^2 - \alpha(\zeta - \mathfrak{a})^2\right|}\right] = \frac{47}{17280} = 0.0027199 < \frac{1}{\lambda}.$$

So, the claims of the Theorem 3.1 are met, and thus the problem (19)-(20) has a unique solution.

We use two metrics in accordance with Rus theorem for establishing the uniqueness of solution to (1)-(2).

Theorem 3.2 Suppose the conditions (C1)-(C3) are fulfilled. If there are two positive numbers $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with the inequality

$$\lambda\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left(\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)|^q d\Upsilon\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < 1, \tag{21}$$

then there is a unique solution to the problem (1)-(2).

Proof. We have to prove the problem (1)-(2) has a unique solution. It is enough to prove the operator $\mathcal{F}$ has a unique fixed point $\vartheta^* \in \mathcal{D}$ with $\mathcal{F}\vartheta^* = \vartheta^*$. According to (17), every such fixed point will also lie in $\mathcal{C}^{(4)}([\mathfrak{a}, \mathfrak{b}])$ as can be directly shown by differentiating (10). We first prove that the inequality (12) of Theorem 2.7 is fulfilled. Consider, for any $v, w \in \mathcal{D}$ and for $x \in [\mathfrak{a}, \mathfrak{b}]$, we obtain

$$|\mathcal{F}v(x) - \mathcal{F}w(x)| \le \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)||f(\Upsilon, v(\Upsilon)) - f(\Upsilon, w(\Upsilon))| d\Upsilon$$

$$\le \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)|\lambda|v(\Upsilon) - w(\Upsilon)| d\Upsilon$$

$$\le \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)|^q d\Upsilon\right)^{\frac{1}{q}} \lambda\left(\int_{\mathfrak{a}}^{\mathfrak{b}} |v(\Upsilon) - w(\Upsilon)|^p d\Upsilon\right)^{\frac{1}{p}}$$

$$\leq \lambda \max_{x\in[\mathfrak{a},\mathfrak{b}]}\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|\mathcal{H}(x,\Upsilon)\right|^{q} d\Upsilon\right)^{\frac{1}{q}} \rho(v,w),$$

using (C3) and Hölder's inequality [21]. Now, we define

$$\theta = \lambda \max_{x\in[\mathfrak{a},\mathfrak{b}]}\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|\mathcal{H}(x,\Upsilon)\right|^{q} d\Upsilon\right)^{\frac{1}{p}}.$$

We conclude that

$$\mathbf{d}(\mathcal{F}v, \mathcal{F}w) \leq \theta\rho(v, w), \text{ for some } \theta > 0 \text{ and for all } v,\ w \in \mathcal{D} \tag{22}$$

and so the inequality (12) of Theorem 2.7 is fulfilled. Using (16), the inequality (22) becomes

$$\mathbf{d}(\mathcal{F}v, \mathcal{F}w) \leq \theta\rho(v, w) \leq \theta(\mathfrak{b}-\mathfrak{a})^{\frac{1}{p}}\, \mathbf{d}(v, w), \text{ for all } v,\ w \in \mathcal{D}.$$

Thus, for any given $\epsilon > 0$, we can take $\delta = \frac{\epsilon}{\theta(\mathfrak{b}-\mathfrak{a})^{\frac{1}{p}}}$ such that $\mathbf{d}(v, w) < \delta$, which implies that $\mathbf{d}(\mathcal{F}v, \mathcal{F}w) < \epsilon$. Hence the operator $\mathcal{F}$ is continuous on $\mathcal{D}$ with respect to the metric $\mathbf{d}$ given in (14).

Furthermore, we prove that the inequality (13) of Theorem 2.7 is fulfilled. Consider for any $v, w \in \mathcal{D}$ and for all $x \in [\mathfrak{a}, \mathfrak{b}]$, we obtain

$$\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|\mathcal{F}v(x)-\mathcal{F}w(x)\right|^{p} dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left[\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|\mathcal{H}(x,\ \Upsilon)\right|^{q} d\Upsilon\right)^{\frac{1}{q}} \lambda\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|v(\Upsilon)-w(\Upsilon)\right|^{p} d\Upsilon\right)^{\frac{1}{p}}\right]^{p} dx\right)^{\frac{1}{p}}$$

$$\leq \lambda\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|\mathcal{H}(x,\ \Upsilon)\right|^{q} d\Upsilon\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \rho(v,\ w)$$

and so we conclude

$$\rho(\mathcal{F}v, \mathcal{F}w) \le \lambda \left(\int_{\mathfrak{a}}^{\mathfrak{b}} \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)|^q \, d\Upsilon \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \rho(v, w)$$

$$= \kappa\rho(v, w),$$

where

$$\kappa = \lambda \left(\int_{\mathfrak{a}}^{\mathfrak{b}} \left(\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{H}(x, \Upsilon)|^q \, d\Upsilon \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Using (21), $\kappa < 1$ and hence, the operator $\mathcal{F}$ satisfies all the conditions of the Theorem 2.7. This implies that the operator $\mathcal{F}$ has a unique fixed point and is the solution of (1)-(2).

□

Now, we consider the example to demonstrate the above result.

Example 3.2 Let us take $\mathfrak{a} = 0$, $\mathfrak{b} = 1$, $\zeta = \frac{1}{2}$, $\alpha = 1$ and $f(x, v) = 1 + x + \sin v$. Then consider

$$v^{(4)} + 1 + x + \sin v = 0, \; x \in [0, 1], \tag{23}$$

with

$$v(0) = 0, \; v'(0) = 0, \; v''(0) = 0, \; v'(1) - v'\left(\frac{1}{2}\right) = \mu. \tag{24}$$

Then $\Delta = 3[(\mathfrak{b} - \mathfrak{a})^2 - \alpha(\zeta - \mathfrak{a})^2] = \frac{9}{4}$ and $\left|\frac{\partial f(x, v)}{\partial v}\right| = |\cos v| \le 1$. For simplicity, we take $p = 2$ and $q = 2$ then by algebraic computation, we get

$$\int_0^1 |\mathcal{H}(x, \Upsilon)|^2 d\Upsilon = \frac{4}{243}x^9 - \frac{1}{30}x^8 + \frac{1}{56}x^7 + \frac{4}{405}x^6(1-x)^5,$$

$$\left(\int_0^1 \left(\int_0^1 |\mathcal{H}(x, \Upsilon)|^2 \, d\Upsilon \right) dx \right) = \frac{3167}{17962560} = 0.00017631$$

and so

$$\left(\int_0^1 \left(\int_0^1 |\mathcal{H}(x, \Upsilon)|^2 \, d\Upsilon \right) dx \right)^{\frac{1}{2}} = 0.0132782 < \frac{1}{\lambda}.$$

So, the claims of the Theorem 3.2 are met, and thus the problem (23)-(24) has a unique solution.

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Conflict of interest

The authors declare no competing financial interest.

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