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Research Article

Numerical Solutions for Singular Lane-Emden Equations Using Shifted Chebyshev Polynomials of the First Kind

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Abstract: This paper describes an algorithm for obtaining approximate solutions to a variety of well-known Lane-Emden type equations. The algorithm expands the desired solution $y(x) \simeq y_N(x)$, in terms of shifted Chebyshev polynomials of first kind such that $y_N^{(i)}(0) = y^{(i)}(0)$ ($i = 0, 1, ..., N$). The derivative values $y^{(j)}(0)$ for $j = 2, 3, ...,$ are computed by using the given differential equation and its initial conditions. This makes approximate solutions more consistent with the exact solutions of given differential equations. The explicit expressions of the expansion coefficients of $y_N(x)$ are obtained. The suggested method is much simpler compared to any other method for solving this initial value problem. An excellent agreement between the exact and the approximate solutions is found in the given examples. In addition, the error analysis is presented.

Keywords**:** Lane-Emden type equations, isothermal gas spheres, Chebyshev polynomial

MSC: 34A34, 34B16, 34B30

1. Introduction

In recent years, the studies of the Lane-Emden equation model:

$$y''(x) + \frac{k}{x} y'(x) + f(x) g(y(x)) = h(x),\ y(0) = \alpha_0,\ y'(0) = 0,\ k > 0, \tag{1}$$

have attracted the attention of many mathematicians and physicists. The name of this model was inspired by the name of two famous astrophysicists Lane [1] and Emden [2]. In the neighborhood of singular point $x = 0$, the analytical solution of Eq. (1) is always possible [3]. This model is used to describe some physical applications by considering $f(x) = 1$, $h(x) = 0$, $k = 2$ and by various functions $g(y) = y^m(x)$, $e^{y(x)}$, $e^{-y(x)}$, $(y^2(x) - C)^{\frac{3}{2}}$ as follows:

Case 1 ($g(y) = y^m(x)$): This equation is known as the "standard" Lane-Emden equation, and it describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to

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thermodynamics laws [4] with exact solutions are available for $m = 0, 1, 5$, which are $1-\frac{x^2}{6}$, $\frac{\sin x}{x}$, $(1+\frac{x^2}{3})^{-\frac{1}{2}}$, respectively.

Case 2 ($g(y) = e^{y}(x)$): This equation is known as the Poisson-Boltzmann differential equation and it is used to model the isothermal gas spheres [5, 6].

Case 3 ($g(y) = e^{-y(x)}$): This equation arises in the modeling of heat conduction in human head [7, 8],

Case 4 $(g(y) = (y^2(x) - C)^{\frac{3}{2}})$: This equation is known as the "white-dwarf equation" and is used to model the gravitational potential of a degenerate white-dwarf star [9].

The solution of linear and nonlinear singular initial value problems in quantum mechanics and astrophysics are numerically challenging due to the singularity behavior at the origin. The approximate solution to the Lane-Emden equation was given by using many methods, like Adomian decomposition method [6, 10], Homotopy analysis method [11, 12], Homotopy perturbation method [13, 14], series expansion method [15], optimal Homotopy asymptotic method [11], variational iteration method [16], Sinc-Collocation method [17], an implicit series solution [18], spectral methods [19-27], Hermite functions collocation method [28] and an optimized pair of hybrid block techniques [29].

In the present paper, the proposed numerical algorithm based on the computations of derivatives of $y^{(i)}(0) = 2$, 3, ..., N, by using Eq. (1) and the given initial conditions, then the suggested approximated solution $y_N(x)$ is taken as an expansion in shifted Chebyshev polynomials of first kind such that $y_N^{(i)}(0) = y^{(i)}(0)$ $(i = 0, 1, ..., N)$. This process generates a linear algebraic triangular system in the expansion coefficients, which is solved exactly. The obtained maximum pointwise error between the exact and approximate solutions is near $O(10^{-16})$. These procedures and the obtained error enable us to claim that the suggested method gives an acceptable accuracy, reduces computational effort and improves computational accuracy. This partially motivates our interest in developing the proposed algorithm. Another motivation is that the suggested method is much simpler than any other method for solving this initial value problem, in particular, nonlinear problems. The last motivation is that for singular differential equations, the shifted Chebyshev polynomials could be preferable [30].

The current paper is organized as follows: In Section 2, some properties of shifted Chebyshev polynomials of the first kind. In Section 3, the proposed numerical method is provided. The error analysis is presented in Section 4. Numerical examples are given in Section 5. Finally, Section 6 summarises the findings.

2. Some properties of first kind Chebyshev polynomials and their shifted forms

The Chebyshev polynomials $\{T_i(x) : i = 0, 1, 2, ...\}$ can be generated by the recursive formula [31]

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n = 2, 3, ..., \tag{2}$$

with $T_0(x) = 1$, $T_1(x) = x$. They are satisfying the orthogonality relation

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx = \frac{\pi}{2\epsilon_n}\delta_{nm}, \tag{3}$$

where $\epsilon_0 = 1/2$ and $\epsilon_n = 1$, $n \geq 1$. In order to use these polynomials on the interval (0, 1), we define the so-called shifted Chebyshev polynomials $T_i^*(x) = T_i(2x-1)$, $i = 0, 1, 2, \ldots$. These polynomials are satisfying the orthogonality relation [31]

$$\int_{0}^{1} w(x) T_n^*(x) T_m^*(x)\, dx = \frac{\pi}{2\epsilon_n}\delta_{nm}, \tag{4}$$

where $w(x)=(x-x^2)^{-1/2}$. The shifted Chebyshev polynomial $T_n^*(x)$ of degree n has the explicit power form

$$T_n^*(x)=n\sum_{r=0}^{n}\frac{(-1)^{n-r}(n+r-1)!2^{2r}}{(n-r)!(2r)!}x^r,\ n\geq 1. \tag{5}$$

Also, these polynomials are satisfying the recurrence relation

$$T_{n+1}^*(x)=2\left(2x-1\right)T_n^*(x)-T_{n-1}^*(x),\ \ n=1,2,..., \tag{6}$$

with $T_0^*(x)=1$, $T_1^*(x)=2x-1$. Moreover, in view of formula (5), it is easy to see that

$$D^qT_n^*(0)=\frac{n(-1)^{n-q}(n+q-1)!2^{2q}q!}{(n-q)!(2q)!}. \tag{7}$$

Lemma 1 Suppose we are given a polynomial $Q_n(x)$ of degree n which has the expansion

$$Q_n(x)=\sum_{k=0}^{n}a_k(n)T_k^*(x), \tag{8}$$

then the expansion coefficients $a_i(n)$ satisfy the equations

$$\left.\begin{aligned} a_i(n)&=\frac{2^{-2i+1}}{i!}Q_n^{(i)}(0)-\sum_{k=1}^{n-i}\frac{2(i+k)(-1)^k(2i+k-1)!}{(k)!(2i)!}a_{k+i}(n),\ \ i=0,1,...,n-1,\\ a_n(n)&=\frac{2^{-2n+1}}{n!}Q_n^{(n)}(0), \end{aligned}\right\} \tag{9}$$

and they can be computed either using the backward substitution method or using the explicit form

$$a_i(n)=2\epsilon_i\sum_{r=0}^{n-i}\frac{(1/2)_{r+i}}{r!(2i+r)!}Q_n^{(r+i)}(0),\ \ i=0,1,...,n, \tag{10}$$

where $(c)_n=\dfrac{\Gamma(c+n)}{\Gamma(c)}$ is the Pochhammer's symbol.

Proof. By using the expansion (8) and formula (7), one can obtain the following linear algebraic system:

$$Q_n^{(i)}(0)=\sum_{k=0}^{n-i}\frac{(i+k)(-1)^k(2i+k-1)!2^{2i}i!}{k!(2i)!}a_{k+i}(n),\ \ i=0,1,...,n, \tag{11}$$

which is a triangular system of $(n+1)$ equations in the $a_i(n)$ $(0\leq i\leq n)$, which can be written in the form (9). Now, we need to prove that the solution of (11) has the form (10):

For $i=1,...,n$, we have $\epsilon_{i+k}=1$ $(0\leq k\leq n-i)$, so substituting (10) into the right side of (11) yields

$$\sum_{k=0}^{n-i}\frac{(i+k)(-1)^k(2i+k-1)!2^{2i}i!}{k!(2i)!}a_{k+i}(n)=$$

$$\frac{2^{2i+1}i!}{(2i)!}\sum_{k=0}^{n-i}\frac{(i+k)(-1)^k(2i+k-1)!}{k!}\sum_{r=0}^{n-i-k}\frac{(1/2)_{r+i+k}}{r!(2i+2k+r)!}Q_n^{(r+i+k)}(0). \tag{12}$$

By expanding and collecting similar terms at the right-hand side of (12), we obtain

$$\sum_{k=0}^{n-i}\frac{(i+k)(-1)^k(2i+k-1)!2^{2i}i!}{k!(2i)!}a_{k+i}(n)=$$

$$\frac{2^{2i+1}i!}{(2i)!}\sum_{r=0}^{n-i}(1/2)_{i+r}\left[\sum_{k=0}^{r}\frac{(-1)^k(i+k)(2i+k-1)!}{k!(r-k)!(2i+k+r)!}\right]Q_n^{(r+i)}(0). \tag{13}$$

Set

$$S_{r,i}=\sum_{k=1}^{r}\frac{(-1)^k(i+k)(2i+k-1)!}{k!(r-k)!(2i+k+r)!},\ r\geq 1.$$

Using Zeilberger's famous algorithm [32], $S_{r,i}$ fulfils the following recurrence relation:

$$(r+1)(2i+r+1)S_{r+1,i}-S_{r,i}=0,\ S_{0,i}=-\frac{1}{2}.$$

The exact solution to the preceding recurrence relation is

$$S_{r,i}=-\frac{1}{2}\frac{(2i)!}{r!(2i+r)!},$$

then

$$\sum_{k=0}^{r}\frac{(-1)^k(i+k)(2i+k-1)!}{k!(r-k)!(2i+k+r)!}=S_{r,i}-S_{r,i}=0,\ r\geq 1. \tag{14}$$

Using Eq. (14), equation (13) takes the form

$$\sum_{k=0}^{n-i}\frac{(i+k)(-1)^k(2i+k-1)!2^{2i}i!}{k!(2i)!}a_{k+i}(n)=\frac{2^{2i}i!}{(2i)!}(1/2)_i\,Q_n^{(i)}(0)=Q_n^{(i)}(0).$$

Also, for $i=0$, equation (11) takes the form

$$Q_n^{(0)}(0)=\sum_{k=0}^{n}(-1)^k a_k(n),$$

then by following the same procedures, it is easy to show that this equation is satisfied by the given coefficients $a_k(n)$ ($0\leq k\leq n$) in (10). This proves that the solution of system (11) has the form (10).

In the following section, Lemma 1 is utilized to construct the proposed algorithm.

3. Solution of Lane-Emden type equations using $T_n^*(x)$

A function $y(x) \in L_w^2(0, 1)$ may be expanded in terms of $T_n^*(x)$ as (see [31])

$$y(x) = \sum_{i=0}^{\infty} a_i T_i^*(x), \; a_i = \frac{2\epsilon_i}{\pi} \int_0^1 w(x) y(x) T_i^*(x)\, dx. \tag{15}$$

In this section, we aim to discuss an algorithm to obtain approximate solutions to the Lane-Emden equation of the form (1). An approximation to the solution of (1) may be written as

$$y_N(x) = \sum_{i=0}^{N} a_i(N) T_i^*(x), \; N \geq 2, \tag{16}$$

and according to Lemma 1, we have

$$a_i(N) = 2\epsilon_i \sum_{r=0}^{N-i} \frac{(1/2)_{r+i}}{r!(2i+r)!} \alpha_{r+i}, \; i = 0, 1, \ldots, N, \tag{17}$$

where $y_N^{(j)}(0) = y^{(j)}(0) = \alpha_j$, $j = 0, 1, 2, \ldots, N$, $(\alpha_1 = 0)$. The following lemma is needed to compute $y^{(j)}(0)$, $j = 2, 3, \ldots, N$, which enable us to compute the coefficients $a_i(N)$ $i = 0, 1, \ldots, N$, and the proposed algorithm is complete.

Lemma 2 The derivatives of solution $y^{(j)}(x)$, $j \geq 2$, at $x = 0$ can be computed by the formula

$$y^{(j+1)}(0) = \frac{j}{(j+k)} \left[h^{(j-1)}(0) - \sum_{i=0}^{j-1} \binom{j-1}{i} f^{(i)}(0) \left[(g(y(x)))^{(j-i-1)} \right]_{x=0} \right], \; j = 1, 2, \ldots, \tag{18}$$

where $y(0) = \alpha_0$, $y^{(1)}(0) = 0$ and $(g(y(x)))^{(q)} \equiv \frac{d^q}{dx^q} g(y(x))$, $q = 1, 2, \ldots$.

Proof. The proof of this lemma depends on applying Leibniz's rule on Eq. (1).

Now, by the aid of Mathematica, the formula (18) gives the values $y^{(j)}(0) = \alpha_j$, for $j = 2, 3, \ldots, N$.

Note 1 Formula (17) shows that the computations to obtain the coefficients $a_i(N_1)$ $(0 \leq i \leq N_1)$ to get an approximated solution $y_{N_1}(x)$, are still useful to get another one $y_{N_2}(x)$ for every $2 \leq N_2 < N_1$. While that the most other methods such as those in [19, 28, 33, 34] must begin new computations to obtain new approximated solution $y_N(x)$, moreover one must solve a new nonlinear algebraic system at each time. This leads to the conclusion that the suggested method reduces the computational efforts rather than such these methods.

Note 2 The proposed approximation is based on computing $y^{(j)}(0)$ for $j = 0, 1, \ldots, N$. With these values, the Taylor expansion can be used to obtain a polynomial approximation of the same degree up to order N. But one of the preferences of using the truncated Tchebychev series rather than the Taylor one is that the Taylor expansion has slow convergence at points far from the origin of expansion. This means that for a desired level of accuracy, the points that are not near the origin will need more terms than those close to the origin of expansion [35]. Using the expansions of orthogonal functions to obtain numerical solutions for differential equations, such as Chebyshev polynomials, could increase the speed of convergence via the so-called economized power series.

The following section examines the convergence of the approximate solution $y_N(x)$ to $y(x)$, as well as the proposed method's error estimates.

4. Error analysis

In this section, we present a comprehensive study for the convergence analysis of the suggested first kind shifted Chebyshev expansion. Two theorems are given and proved, in the first theorem, we prove that the first kind shifted Chebyshev expansion of a function $y(x)$ with a bounded second derivative, converges uniformly to $y(x)$, and in the second theorem, we give an upper bound for the error (in L_w^2 and L_∞ norms) of the truncated expansion. The following lemma is needed.

Lemma 3 [36, p. 742] Let $y(x)$ be a function such that $y(k) = a_k$. Suppose that the following assumptions are satisfied:

1. $y(x)$ is continuous, positive, decreasing function for $x \le n$.
2. $\sum a_n$ is convergent, and $R_n = \sum\limits_{k=n+1}^{\infty} a_k$,

then

$$R_n \le \int_n^\infty y(x)dx. \tag{19}$$

Theorem 1 (see [37]) A function $y(x) \in L_w^2(0, 1)$, with $|y''(x)| \le M$, can be expanded as an infinite sum of shifted Chebyshev basis as in Eq. (15), and the series converges uniformly to $y(x)$. In addition, the expansion coefficients in (15) satisfy the following inequality

$$|a_i| < \frac{M}{2i^2} \quad \forall i \ge 2. \tag{20}$$

Theorem 2 If $y(x) \in L_w^2(0, 1)$ satisfies the hypotheses of Theorem 1 and we consider the approximate solution $y_N(x)$ as in (16), the following errors estimate are obtained

$$\|y - y_N\|_w < \frac{M}{2} N^{-3/2}, \tag{21}$$

and

$$\|y - y_N\|_\infty < \frac{M}{2} N^{-1}, \tag{22}$$

where $\|y\|_w^2 = \int_0^1 w(x) y^2(x) dx$ and $\|y\|_\infty = \max\limits_{0 \le x \le 1} |y(x)|$.

Proof. Using Eq. (15) and the orthogonality relation (4), one can obtain

$$\|y - y_N\|_w^2 = \frac{\pi}{2} \sum_{k=N+1}^{\infty} a_k^2. \tag{23}$$

In view of Theorem 1, formula (23) takes the form

$$\|y - y_N\|_w^2 < \frac{M^2 \pi}{8} \sum_{k=N+1}^{\infty} \frac{1}{i^4},$$

and the application of Lemma 3 leads to

$$\|y-y_N\|_w^2 < \frac{M^2\pi}{8}\int_N^\infty \frac{1}{x^4}\,dx = \frac{M^2\pi}{24N^3} < \frac{M^2}{4N^3},$$

and hence (21) is obtained. Similarly, we can prove (22), which completes the proof of the theorem.

In the following section, we give some numerical results obtained by using the algorithm presented in Section 3.

5. Numerical results

In this section, we present some numerical examples and comparisons between the obtained results by using the present method and those obtained by other methods proposed in [6, 19, 28, 33, 38, 39]. Tables 1, 2, 3 and 4 show the computed errors,

$$E_N = \|y-y_N\|_\infty, \tag{24}$$

for various N values obtained in Examples 1-4, demonstrating that the proposed method has an appropriate convergence rate.

Example 1 Consider the standard Lane-Emden equation of the first kind

$$y''(x)+\frac{2}{x}y'(x)+y^m(x)=0,\ 0\le x\le 1,\ y(0)=1,\ y^{(1)}(0)=0,\ m\in[0,5], \tag{25}$$

whose exact solution is $y(x)=1-\frac{x^2}{6},\ \frac{\sin(x)}{x},\ \left(1+\frac{x^2}{3}\right)^{-1/2}$, in the three cases m = 0, 1, 5, respectively. Applying Lemma 1 (k = 2, $f(x)$ = 1, $h(x)$ = 0, $g(y) = (y(x))^m$), we obtain $y^{(i)}(0)$, i = 2, ..., 10, which enable us, in view of formula (10), to compute the expansion coefficients $a_i(10)$, i = 0, ..., 10, as in Table 5. Now, we can compute $y_{10}(x)=\sum_{i=0}^{10} a_i(10)T_i^*(x)$, which is an analytical approximation to the solution of (25) and has the form

$$y_{10}(x)=1-\frac{1}{6}x^2+\frac{m}{120}x^4-\frac{m(8m-5)}{3\times 7!}x^6+\frac{m(70-183m+122m^2)}{9\times 9!}x^8$$

$$+\frac{m(3150-10805m+12642m^2-5032m^3)}{45\times 11!}x^{10}. \tag{26}$$

This solution is in complete agreement with both the truncated Taylor series expansion $\sum_{k=0}^{10}\frac{y^{(k)}(0)}{k!}x^k$ and the result obtained by Horedt [39, Eq. (2.4.24)].

Remark 1 Wazwaz [6, Eq. (39)] provides an analytical approximation for (25) by using Adomian decomposition method. I believed that this formula need to be reviewed, the coefficient of x^{10} should be corrected. This fact can be shown by comparing the value of this coefficient when m = 1, and Taylor expansion of the solution $y(x)=\frac{\sin(x)}{x}$.

Now, we discuss the numerical solutions of (25) corresponding to the three cases m = 0, 1, 5. In the case of m = 0, we use N = 2, to obtain

$$y_2(x)=\frac{15}{16}-\frac{1}{12}T_1^*(x)-\frac{1}{48}T_2^*(x)=1-\frac{x^2}{6}, \tag{27}$$

which is the exact solution. Furthermore, by substituting $m = 0$, this solution can be obtained directly from (26). Figures 1 and 2 show the absolute errors between the numerical and analytical solutions when $m = 1, 5$, respectively. Moreover, the approximated solutions correspond to $m = 2, 3$ are computed by using $N = 12, 16$, respectively, and these two solutions are shown in Figure 3.

Table 1. Comparison of E_N values between the presented method, [28] and [19] for Example 1 at $m = 1, 5$

	Presented method				[28]	
	E_N				E_N	
N	CT(s)	$m = 1$	CT(s)	$m = 5$	$m = 1$	$m = 5$
5	0.874	1.68E-09	0.875	8.97E-03	7.87765E-02	3.54822E-01
10	0.969	2.22E-16	0.953	1.89E-14	2.30291E-03	6.61006E-03
15	0.985	2.01E-16	0.985	1.58E-16	2.87683E-04	9.08909E-04
20	1.001	3.33E-16	1.015	7.25E-17	4.78916E-05	1.71788E-05
	Presented method				[19]	
3	0.873	8.14E-03	0.844	3.27E-02	1.43E-03	1.21E-02
6	0.921	4.31E-13	0.891	5.43E-10	9.97E-05	5.55E-03
8	0.958	1.11E-16	0.906	3.11E-12	2.79E-08	-

Table 2. Comparison between $y_N(x)$ values between the presented method and the numerical solution given by [39] for Example 1 at $m = 2, 3$

	Presented method		[39]	
x	$m = 2$ ($N = 12$)	$m = 3$ ($N = 16$)	$m = 2$	$m = 3$
0.0	1.000000000	1.00000000	1.00000000	1.00000000
0.1	0.998334999	0.998335829	0.9983350	0.9983358
0.5	0.959352716	0.959839069	0.9593527	0.9598391
1	0.848654158	0.855057607	0.8486541	0.8550576

Table 3. Comparisons of $y_8(x)$ values between the presented method and series solution given by Adomian, HFC and ICSRBF methods for Example 2

x	Presented method	Adomian	HFC	ICSRBF	Residual error
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	0
0.1	0.9985979274	0.9985979274	0.9986051425	0.9985979436	8.99E-15
0.2	0.9943962649	0.9943962649	0.9944062706	0.9943962892	2.75E-12
0.5	0.9651777802	0.9651777802	0.9651881683	0.9651778048	7.19E-09
1.0	0.8636810942	0.8636810942	0.8636881301	0.8636811302	4.47E-06

Table 4. E_N values and CPU times for Example 3

N	CT(s)	E_N
10	1.032	6.86686E-02
20	1.062	4.39055E-05
32	20.297	3.31474E-10
40	73.156	3.77087E-14
42	85.121	3.38275E-15
44	91.321	1.70851E-16

Table 5. The values of $a_i(10)$, $i = 0, ..., 10$, for Example 1

i	$a_i(10)$
0	$\frac{15}{16}+\frac{m(100928894850+m(-5625762595+26(14119683-812668m)m))}{42807066624000}$
1	$-\frac{1}{12}+\frac{m(89037356770+m(-5323332115-962m(-374359+21964m)))}{23543886643200}$
2	$-\frac{1}{48}+\frac{m(179734314790+m(-13469187705+26(39191929-2438004m)m))}{94175546572800}$
3	$\frac{m(2195360810+m(-255826495+2(11811443-812668m)m))}{3923981107200}$
4	$\frac{m(1814791070+m(-485192205+61098898m-4876008m^2))}{23543886643200}$
5	$\frac{m(138660350+m(-249310845+49826098m-4876008m^2))}{58859716608000}$
6	$\frac{m(16083050+m(-32593375+2(6336083-812668m)m))}{62783697715200}$
7	$\frac{m(672350+m(-1904205+2(789737-143412m)m))}{47087773286400}$
8	$\frac{m(121450+m(-366335+347558m-95608m^2))}{94175546572800}$
9	$\frac{m(3150+m(-10805+2(6321-2516m)m))}{47087773286400}$
10	$\frac{m(3150+m(-10805+2(6321-2516m)m))}{941755465728000}$

Note 3 In view of Note 1, the approximate solutions $y_N(x)$ for $N = 2, ..., 9$ can be computed by using the coefficients $a_i(10)$ $(0 \leq i \leq 10)$ given in Table 5, for any value of $m \in [0, 5]$.

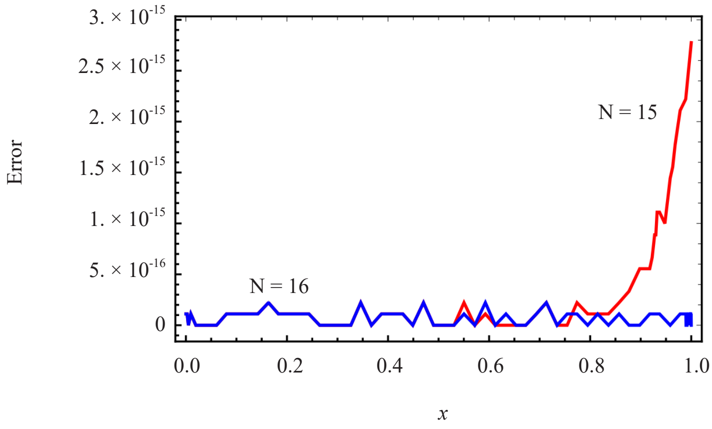

Figure 1. Example 1: The maximum absolute errors E_{15} and E_{16} for $m = 1$

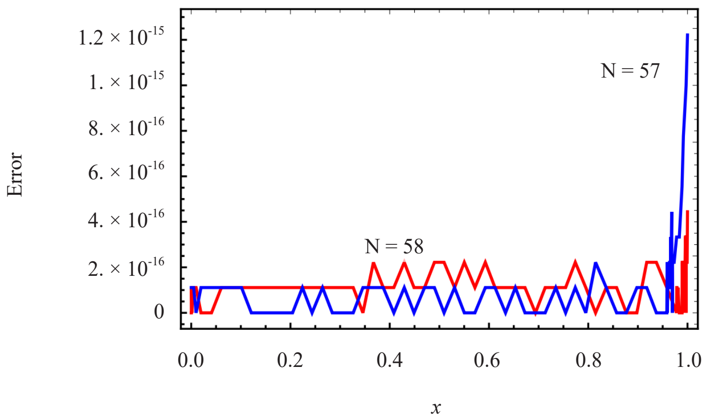

Figure 2. Example 1: The maximum absolute errors E_{57} and E_{58} for $m = 5$

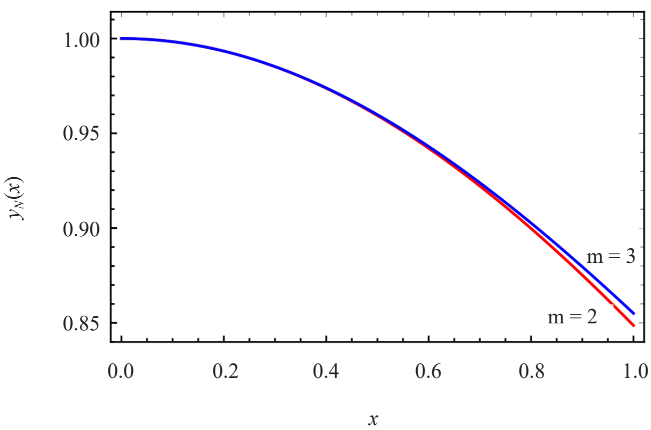

Figure 3. Example 1: The solutions $y_{12}(x)$ and $y_{16}(x)$ for $m = 2$ and 3, respectively

In the cases of $m = 1, 2, 3, 5$, comparisons between numerical results obtained by the presented method and the proposed methods in [19, 28, 39] are given in two Tables 1 and 2. These tables show that our results seem to be either better or close to the results of these methods. Moreover, in the presented method, the expansion coefficients of the approximated solution are obtained explicitly, whereas in the other methods, these coefficients are obtained by solving nonlinear algebraic systems. So, the computational effort in the presented method is much less than in these methods.

Example 2 Consider the Lane-Emden equation [6]

$$y''(x)+\frac{2}{x}y'(x)+\sin(y(x))=0,\ 0\le x\le 1,\ y(0)=1,\ y^{(1)}(0)=0. \tag{28}$$

Applying Lemma 1 ($k = 2$, $f(x) = 1$, $h(x) = 0$, $g(y) = \sin(y(x))$), we get $y^{(2i+1)}(0) = 0$, $i = 1, 2, 3$, and the values of $y^{(2i)}(0) = \alpha_{2i}$, $i = 1, ..., 4$, are given in Table 6. In view of formula (10), we obtain

$$y_8(x)=1-\frac{1}{6}k_1x^2+\frac{1}{120}k_1k_2x^4+k_1\left(\frac{1}{3024}k_1^2-\frac{1}{5040}k_2^2\right)x^6$$

$$+k_1k_2\left(\frac{-113}{3265920}k_1^2+\frac{1}{362880}k_2^2\right)x^8, \tag{29}$$

where $k_1 = \sin(1)$ and $k_2 = \cos(1)$. Also, Wazwaz [6] provides the same analytical approximation by using Adomian decomposition method. Table 3 compares the approximated solution $y_8(x)$ to series solutions obtained by Adomian, ICSRBF and HFC methods described in [6, 28, 38], respectively. Table 7 displays the maximum residual errors obtained by the presented method using various values of N. Additionally, the last column of Table 3 shows that the residual error obtained by the presented method is either in agreement with Adomian method or better than that obtained by HFC and ICSRBF methods (see Table 4 in [38]). The numerical solutions $y_2(x)$ and $y_8(x)$ are shown in Figure 4, as well as the residual error corresponding to $y_{23}(x)$ and $y_{24}(x)$ is given in Figure 5.

Table 6. The values of $y^{(2i)}(0) = \alpha_{2i}$, $i = 1, ..., 4$, for Example 2

i	α_{2i}
1	$-\frac{k_1}{3}$
2	$\frac{k_1k_2}{5}$
3	$-\frac{1}{7}k_1k_2^2+\frac{5}{21}k_1^3$
4	$\frac{-113}{81}k_1^3k_2+\frac{1}{9}k_1k_2^3$

Example 3 [33] The following nonlinear Lane-Emden equation

$$y''(x)+\frac{8}{x}y'(x)+2y(x)(18+4\ln y(x))=0,\ 0\le x\le 1,\ y(0)=1,\ y^{(1)}(0)=0, \tag{30}$$

has the exact solution $y(x)=e^{-2x^2}$.

Table 4 displays the maximum absolute errors obtained by the presented method using various values of N. In Table 8, we list the best obtained maximum absolute errors by our method and two methods CS and ACS in Khuri and Sayfy

[33]. Figure 6 shows the absolute errors between the numerical and analytical solutions for $N = 42, 44$. Additionally, Figure 7 displays the Log-error for different N to demonstrate the stability of the solutions to Eq. (30).

Table 7. Maximum residual error and CPU times for example 2 at $N = 5, 8, 12, 15, 20$

N	CT	Maximum residual error
5	0.735	5.55E-03
8	0.765	4.47E-06
12	0.781	1.69E-07
15	0.797	8.59E-09
20	0.968	4.06E-12

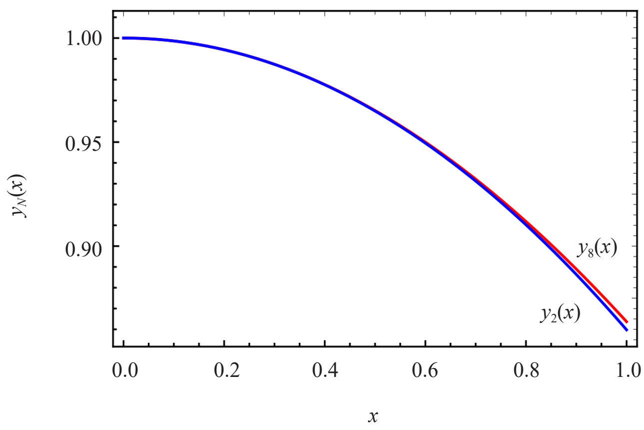

Figure 4. Example 2: The approximated solutions $y_2(x)$ and $y_8(x)$

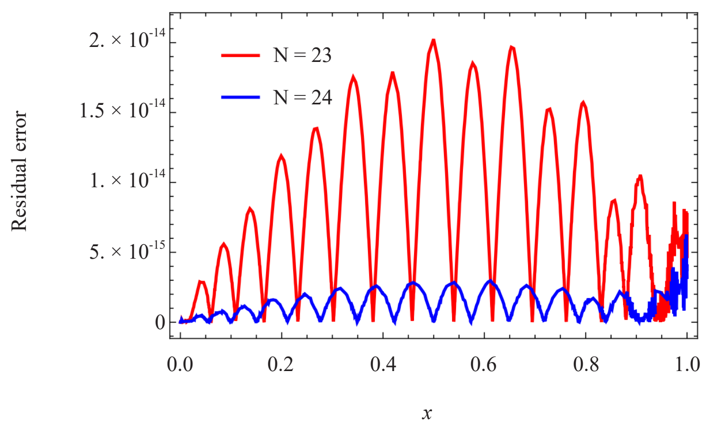

Figure 5. Example 2: The residual errors for $N = 23, 24$

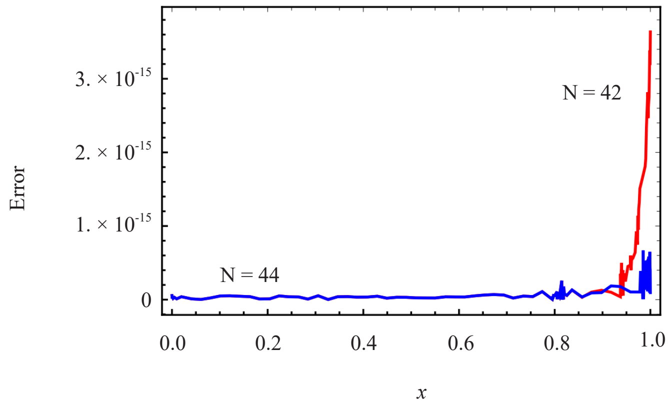

Figure 6. Example 3: The maximum absolute errors E_{42} and E_{44}

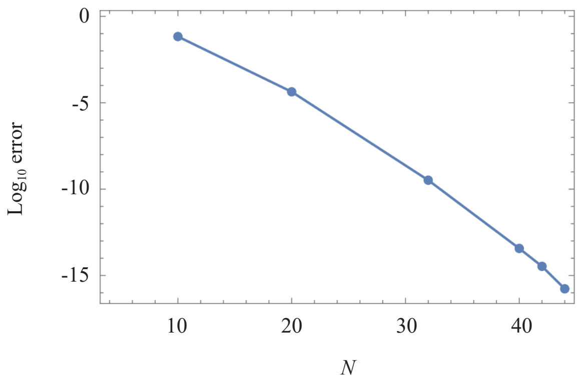

Figure 7. Example 3: Log-error of Example 3 for $N = 10, 20, 32, 40, 42, 44$

Table 8. Comparison of E_N between the presented method and [33] for Example 3

	Presented method	[33]	
	$N = 44$	CS	ACS
E_N	1.70851E-16	2.28651E-06	1.99290E-10

Example 4 Consider Lane-Emden equation in the form

$$y''(x) + \frac{n}{x} y'(x) + g(y) = 0,\ y(0) = \alpha,\ y^{(1)}(0) = 0,\ n \geq 0. \tag{31}$$

Consider $g(y) = (y(x))^m$ and $\alpha = 1$, by applying the proposed algorithm, the series solution for all real values $n \geq 0$ and $m \geq 0$ is expressed in the form:

$$y(x) = 1 - \frac{1}{2(n+1)} x^2 + \frac{m}{8(n+1)(n+3)} x^4 - \frac{2(n+2)m^2 - (n+3)m}{48(n+1)^3 (n+3)(n+5)} x^6$$

$$+\frac{(6n^2+32n+34)m^3-(7n^2+46n+63)m^2+(2n^2+16n+30)m}{348(n+1)^3(n+3)(n+5)(n+7)}x^8$$

$$+\frac{1}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}$$

$$\times\Big[-4(n+4)(6n^3+55n^2+134n+93)m^4$$

$$+(46n^4+28644n^3+259782n^2+631540n+433566)m^3$$

$$-(n+3)(29n^3+383n^2+1515n+1689)m^2+6(n+3)^2(n+5)(n+7)m\Big]x^{10}$$

$$+\cdots. \tag{32}$$

Note 4 In particular, for the special case $n = 2$, Eq. (26) is obtained as a direct consequence of Eq. (32) by using the first ten terms.

For fixed $n = 0$, the following solutions

$$y(x)=1-\frac{1}{2}x^2,$$

$$y(x)=\cos x,$$

$$y(x)=1-\frac{1}{2}x^2+\frac{1}{12}x^4-\frac{1}{72}x^6+\frac{1}{504}x^8-\frac{5}{18144}x^{10}+\cdots, \tag{33}$$

can be obtained for $m = 0, 1, 2$, respectively, where $x = 0$ is an ordinary point. In addition, for fixed $n = \frac{1}{2}$, we find

$$y(x)=1-\frac{1}{3}x^2,$$

$$y(x)=1-\frac{1}{3}x^2+\frac{1}{42}x^4-\frac{1}{1386}x^6+\frac{1}{83160}x^8-\frac{1}{7900200}x^{10}+\cdots,$$

$$y(x)=1-\frac{1}{3}x^2+\frac{1}{21}x^4-\frac{13}{2079}x^6+\frac{23}{31185}x^8-\frac{175}{1885275}x^{10}+\cdots, \tag{34}$$

can be obtained for $m = 0, 1, 2$, respectively. Moreover, the solutions

$$y(x)=1-\frac{1}{4}x^2,$$

$$y(x)=1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+\frac{1}{147456}x^8-\frac{1}{14745600}x^{10}+\cdots,$$

$$y(x)=1-\frac{1}{4}x^2+\frac{1}{32}x^4-\frac{1}{288}x^6+\frac{13}{36864}x^8-\frac{7}{204800}x^{10}+\cdots, \tag{35}$$

are obtained for $n = 1$ and $m = 0, 1, 2$, respectively.

Note 5 All results in this example coincide with the same results obtained by Wazwaz [6, pp. 307-309].

Remark 2 The series solutions of Eq. (31) with specified value of α, can be handled in similar way for other forms for $g(y)$, for example:

(i) For $g(y) = e^y$ and $\alpha = 0$, we have

$$y(x)=-\frac{1}{2(n+1)}x^2+\frac{1}{8(n+1)(n+3)}x^4-\frac{(n+2)}{24(n+1)^2(n+3)(n+5)}x^6$$

$$+\frac{(3n^2+16n+17)}{192(n+1)^3(n+3)(n+5)(n+7)}x^8$$

$$-\frac{(n+4)(6n^3+55n^2+134n+93)}{960(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10}+\cdots, \tag{36}$$

(ii) For $g(y) = \sinh y$ and $\alpha = 1$, we have

$$y(x)=1-\frac{e^2-1}{4e(n+1)}x^2+\frac{e^4-1}{16e^2(n+1)(n+3)}x^4-\frac{(e^2-1)((e^4+1)(n+2)-2e^2)}{192e^3(n+1)^2(n+3)(n+5)}x^6$$

$$+\frac{(e^4-1)((e^4+1)(3n^2+16n+17)-4e^2(n^2+7n+8))}{3072e^4(n+1)^3(n+3)(n+5)(n+7)}x^8+\cdots. \tag{37}$$

Remark 3 The following table demonstrate the steps for the implementation of the presented algorithm. All calculations were carried out on a computer that was running Mathematica 12 (Intel(R) Core(TM) i9-10850 CPU at 3.60 GHz, 3600 MHz, 10 cores, 20 logical processors). The obtained Computational Times (CT) for Examples 1-4 are vitally important measurement. The used programme code is fit for purpose and the aimed numerical solution is obtained within a reasonable time frame, which is considered a useful signpost for the efficiency of the algorithm.

Algorithm

Step 1. Given α_0, $f(x)$, $g(x)$, $h(x)$ and N,

Step 2. Evaluate $y^{(j)}(0) = \alpha_j$, $j = 2, 3, ..., N$ defined in Eq. (18)

Step 3. Evaluate $a_i(N)$ defined in Eq. (17)

Step 4. Evaluate $y_N(x)$ defined in Eq. (16)

Remark 4 It is worth noting that if Eq. (1) is given with $x \in [0, L]$, then by using $x = L\theta$, $\theta \in [0, 1]$ one can obtain:

$$\tilde{y}''(\theta)+\frac{k}{\theta}\tilde{y}'(\theta)+\tilde{f}(\theta)\tilde{g}(y)=\tilde{h}(\theta),\ \tilde{y}(0)=\alpha_0,\ \tilde{y}'(0)=0,\ k>0. \tag{38}$$

Then apply the proposed algorithm to obtain a numerical solution $\tilde{y}_N(\theta)$. The approximate solution $y_N(x) = \tilde{y}_N(x/L)$, may be successful numerical solution for Eq. (1) only for small values of N. For example, if the reader consider Eq. (22) in [29], which is the same Eq. (25) with $x \in [0, 1000]$, then the proposed method's results can be compared to Table 4 in [29] as in Table 9. It is clearly that the method in [29] is better when $x \in [0, 1000]$. And it is obvious that the presented method works flawlessly when $x \in [0, 1]$ is used. Furthermore, when discussing the presented examples, it is worth noting that the computational efforts required by using the presented method are lower.

Table 9. Comparison of E_N values between the presented method and [29]

Presented method			[29]		
N	CT(s)	E_N	NS	CT(s)	E_N
4	0.421	5.9E-06	114	0.26328	1.70751E-07
6	0.876	4.39E-07	120	0.30774	1.72144E-08
8	0.881	1.52E-08	130	0.33337	2.27624E-09
10	0.956	2.25E-09			

6. Results and discussions

We have provided a detailed algorithmic description of how shifted Chebyshev polynomials of the first kind may be used to give highly accurate solutions to the Lane-Emden equation (1). We demonstrate the proposed method returns a valid solution for the given Lane-Emden equation (1) with less computational effort. The main advantage of the presented algorithm is its simplicity and high accurate approximate solutions.

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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