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Research Article

On The Solutions of Fractional Boundary Value Problems for a Combined Caputo Fractional Differential Equations

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Abstract: In this paper, a new class of fractional boundary value problem with the combined Caputo derivative is proposed and the physical interpretation of this new derivative has been explained. Under some assumptions, the positive solutions of the fractional differential equation with the help of Leray-Schauder and Krasnoselskii's fixed point theorems in a cone have been investigated. Moreover, the solution of the fractional Maxwell models involving the combined Caputo derivative by using the extended Laplace transform is obtained. Finally, some examples are given to support theoretical findings.

***Keywords*:** positive solutions, fractional differential equation, combined Caputo derivative, fractional Maxwell model

MSC: 26A33, 34A12, 34G20

1. Introduction

Recently, fractional calculus has been used very efficiently in various applications and sciences for modelling physical phenomena, for example, physics [1], bioengineering [2], and engineering [3-6]. Existence results of positive solutions for fractional boundary value problems is particularly one of the fundamental issues of fractional calculus. There have been many works devoted to existence of solutions and positive solutions of fractional boundary value problems by the use of some fixed-point theorems (see, e.g., [7-13] and references therein). The obtained results in these papers were established in the sense of the Riemann-Liouville and Caputo fractional derivative which hold one-sided memory effects. In [7], the author investigated the existence of solution for a nonlinear boundary value problem involving the Caputo fractional derivative subject to the non-homogeneous Dirichlet boundary conditions. The existence of solutions of a coupled system of fractional differential equations with p-Laplacian operator involving the Caputo derivative and infinite-point boundary condition was presented in [8]. The existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem involving the Riemann-Liouville differentiation was investigated in [9]. The existence and multiplicity results of positive solutions for the nonlinear differential equation of fractional order subject to the fractional boundary conditions by using some fixed point theorems were obtained in [13]. Recently, barycentric interpolation collocation algorithm was proposed to solve the fractional differential equations involving the Caputo fractional derivative in [14] and [15].

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Differently from aforementioned approaches, we consider a combined Caputo fractional derivatives ${}^{C}D_{\gamma}^{\mu_1,\mu_2}$ which is a convex combination of the left Caputo fractional derivative of order μ_1 and the right Caputo fractional derivative of order μ_2. This combined Caputo fractional operator holds two-sided non-local memory effects and it was initially introduced in [16]. This property of the combined Caputo derivative is important in the mathematical modelling in physical processes of some nonconservative models which have different physical behaviour over a certain time intervals. Hence, two-sided fractional operator can better describe fractional modelling. In particular, the Riesz-Caputo fractional derivative can be recovered by setting $\gamma = 1/2$ in the combined Caputo fractional derivative operator. Extensive works on numerical solutions of fractional differential equations with the Riesz-Caputo fractional derivative have been discussed in the literature, see. e.g., [17-24] and references therein. However, there is no result about existence of positive solutions for fractional differential equations with the combined Caputo fractional derivative.

Viscoelasticity is the tendecy of the materials which show both viscous and elastic behaviour when a stress is acted in materials and continuum mechanics [25]. In the classical viscoelastic models, the viscosity coefficient is taken as a constant function, however, a time-dependent viscosity coefficient has been proposed in [26-27]. In [27], a integer order modified Maxwell model with a time-dependent viscosity is considered as

$$\frac{ds}{dt} = \frac{1}{E}\frac{dT}{dt} + \frac{T}{\varepsilon(t)}, \tag{1}$$

where s, T are the strain and stress, E is the elastic modulus of the spring and $\varepsilon(t)$ is the time-dependent viscosity coefficient.

A special case when the viscosity coefficient is linear in time is presented in [26-27] by taking $\varepsilon(t) = \varepsilon_0 + \sigma t$, where ε_0 is the initial viscosity and σ is the strain-hardening coefficient. This leads to following modified Maxwell model

$$\frac{ds}{dt} = \frac{1}{E}\frac{dT}{dt} + \frac{T}{\varepsilon_0 + \sigma t}, \tag{2}$$

On the other hand, the fractional differential equations have been commonly proposed and been popular for viscoelastic models. The unsteady natural convection flow and heat transfer of fractional Maxwell viscoelastic nanofluid in magnetic field over a vertical plate is considered in [28]. We refer the interested readers to [29-30] and the references therein.

One of the powerful and frequently used tool for finding exact and numerical solutions of fractional differential equations is the Laplace transformation. We extend the domain of the standard Laplace transformation to apply it more general problems. This extended transformation is called the extended Laplace transformation [31].

In [32], the existence of solutions for fractional boundary value problems (FBVP) of the combined Caputo fractional differential equation has been discussed and presented. In this paper, we will present the positive solution of the following FBVP

$$\begin{gathered} {}^{C}D_{\gamma}^{\mu_1,\mu_2} u(\tau) = F(\tau, u(\tau)),\ \tau \in [0, 1],\ \mu_1 \in (1, 2],\ 0 \le \tau \le 1, \\ u(0) = u_0,\ u(1) = u_1, \end{gathered} \tag{3}$$

where ${}^{C}D_{\gamma}^{\mu_1,\mu_2}$ is the combined Caputo differential operator and $F \in C([0, T] \times \mathbb{R}, \mathbb{R})$, u_0 and u_1 are nonnegative constants. Furthermore, by analogy to the modified Maxwell model (2), we discuss the solutions of the following modified fractional Maxwell equation involving the combined Caputo derivative:

$${}^{C}D_{\gamma}^{\mu_1,\mu_2} s(\tau) = \frac{1}{E}\, {}^{C}D_{\gamma}^{\mu_1,\mu_2} T(\tau) + T(\tau),\ \tau \in [0, 1],\ \mu_1, \mu_2 \in (0, 1],\ 0 \le \tau \le 1, \tag{4}$$

where s, T are strain and stress and E is the elastic modulus of the spring. In fact, we take the strainhardening coefficient σ in (2) as the Heaviside function $H(\tau)$ defined by

$$H(\tau)=\begin{cases}0, & \tau\le 0\\ 1, & \tau>0.\end{cases} \tag{5}$$

Further, we assume that $\varepsilon_0 + t \equiv 1$ in this paper.

The paper is outlined as follows. We present the basic preliminaries and definitions which are useful in what follows in Section 2. Section 3 is devoted to positive solutions of the fractional boundary value problems with the Dirichlet boundary conditions. A modified fractional Maxwell model with the combined Caputo derivative is studied in Section 4. Numerical examples are carried out to support the theory for the existence of positive solutions in Section 5.

2. Preliminaries

In this section, we shall give some facts and previous results which will be needed in what follows. These definitions and results with their proofs can be found in the recent works [4-6].

Definition 2.1 [4] Let $\mu_1 > 0$ be a real number. The left-sided Riemann-Liouville fractional integral of order μ_1 of a function $f \in L^1([0, 1])$ is given by

$${}_0I_x^{\mu_1} f(x)=\frac{1}{\Gamma(\mu_1)}\int_0^x (x-s)^{\mu_1-1} f(s)ds,\ x\in[0,1]. \tag{6}$$

The right-sided Riemann-Liouville fractional integral of order μ_1

$${}_xI_1^{\mu_1} f(x)=\frac{1}{\Gamma(\mu_1)}\int_x^1 (s-x)^{\mu_1-1} f(s)ds,\ x\in[0,1]. \tag{7}$$

Definition 2.2 (Combined Riemann-Liouville Fractional Integral) Let μ_1, $\mu_2 \in (0, 1]$, $\gamma \in [0, 1]$. The combined Caputo fractional integral of order μ_1, μ_2 of a function $f \in L^1([0, 1])$ is given by

$${}_{\gamma}^{0}I_1^{\mu_1,\mu_2} f(x)={}_0I_x^{\mu_1} f(x)+{}_xI_1^{\mu_2} f(x),\ x\in[0,1]. \tag{8}$$

Note that the combined Riemann fractional integral operator is reduced to the Riesz fractional integral when $\mu_1 = \mu_2$ and takes the following form

$${}_0I_1^{\mu_1} f(x)=\frac{1}{\Gamma(\mu_1)}\int_0^1 |x-s|^{\mu_1-1}\ f(s)ds,\ x\in[0,1]. \tag{9}$$

Definition 2.3 [4] Let $\mu_1 \in (0, 1]$ be a given real number. The left-sided Caputo fractional derivative of order μ_1 of $f \in AC([0, 1])$ defined as, respectively

$${}_0^CD_x^{\mu_1} f(\tau)=\frac{1}{\Gamma(1-\mu_1)}\int_0^{\tau} (\tau-s)^{-\mu_1} f'(s)ds={}_0I_{\tau}^{1-\mu_1} f'(x). \tag{10}$$

The right-sided Caputo fractional derivative

$$ {}_{x}^{C}D_{1}^{\mu_1} f(\tau) = \frac{-1}{\Gamma(1-\mu_1)} \int_{\tau}^{1} (s-\tau)^{-\mu_1} f'(s)ds = -{}_{\tau}I_{1}^{n+1-\mu_1} f'(x), \tag{11} $$

where $AC([0, 1])$ is the space of absolutely continuous functions on [0, 1].

Definition 2.4 [33] Let $\mu_1, \mu_2 \in (0, 1]$, $\gamma \in [0, 1]$. The combined Caputo fractional differential operator ${}^{C}D_{\gamma}^{\mu_1,\mu_2}$ of order (μ_1, μ_2) of a function $f \in AC[0, 1]$ defined by

$$ {}^{C}D_{\gamma}^{\mu_1,\mu_2} f(x) = \gamma {}_{0}^{C}D_{x}^{\mu_1} f(x) + (1-\gamma) {}_{x}^{C}D_{1}^{\mu_2} f(x) = \gamma {}_{0}I_{x}^{1-\mu_1} f'(x) - (1-\gamma) {}_{x}I_{1}^{1-\mu_2} f'(x). \tag{12} $$

Lemma 2.5 [4-5] The following hold true for $f \in C^n[0, 1]$ with $\mu_1, \mu_2 \in (n, n + 1]$.

$$ {}_{0}I_{x}^{\mu_1} {}_{0}^{C}D_{x}^{\mu_1} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}(x-0)^k, $$

$$ {}_{x}I_{1}^{\mu_2} {}_{x}^{C}D_{1}^{\mu_2} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(1)}{k!}(1-x)^k. $$

From the definitions and lemmas above, we are lead to

$$ {}_{0}^{\gamma}I_{1}^{\mu_1,\mu_2} {}^{C}D_{\gamma}^{\mu_1,\mu_2} f(x) = \gamma \left({}_{0}I_{x}^{\mu_1} {}_{0}^{C}D_{x}^{\mu_1} f(x) + {}_{x}I_{1}^{\mu_2} {}_{0}^{C}D_{x}^{\mu_1} f(x) \right) \tag{13} $$

$$ +(-1)^{n+1}(1-\gamma) \left({}_{0}I_{x}^{\mu_1} {}_{x}^{C}D_{1}^{\mu_2} f(x) + {}_{x}I_{1}^{\mu_2} {}_{x}^{C}D_{1}^{\mu_2} f(x) \right) \tag{14} $$

$$ = \gamma {}_{0}I_{x}^{\mu_1} {}_{0}^{C}D_{x}^{\mu_1} f(x) + (-1)^{n+1}(1-\gamma) {}_{x}I_{1}^{\mu_2} {}_{x}^{C}D_{1}^{\mu_2} f(x). \tag{15} $$

If $\mu_1, \mu_2 \in (0, 1]$ then we have the following simplified form

$$ {}_{0}^{\gamma}I_{1}^{\mu_1,\mu_2} {}^{C}D_{\gamma}^{\mu_1,\mu_2} f(x) = f(x) - \gamma f(0) - (1-\gamma) f(1). \tag{16} $$

The Laplace transform of a function $u(x)$ of a real variable $x \in \mathbb{R}^+$ is defined by

$$ \mathcal{L}(s) := (\mathcal{L}u)(s) = \mathcal{L}[u(x)](s) = \overline{u}(s) := \int_{0}^{\infty} e^{-sx} u(x)dx \quad (s \in \mathbb{C}). \tag{17} $$

The integral in (17) converges absolutely for $s \in \mathbb{C}$ with $Re(s) > Re(s_0)$ provided that the integral converges at some point $s_0 \in \mathbb{C}$.

The inverse Laplace transform is defined for $x \in \mathbb{R}^+$ by

$$ \left(\mathcal{L}^{-1}\overline{u} \right)(x) = \mathcal{L}^{-1}[\overline{u}(s)](x) := \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{sx} \overline{u}(s)ds \quad (\beta = Re(s)). \tag{18} $$

Based on the standard Laplace transformation, we extend the domain of the integral in (17) to derive the extended Laplace transformation as follows.

The extended Laplace transform of a function $u(x)$ of a real variable $x \in [a, \infty)$, $a \in \mathcal{R}$ is defined as [34]

$$(\mathcal{L}_a u)(s) = \mathcal{L}_a[u(x)](s) = \bar{u}_a(s) := \int_a^\infty e^{-sx} u(x)dx \quad (s \in \mathbb{C}). \tag{19}$$

Theorem 2.6 [34] The extended Laplace transformation of the derivatives of a function u is given by

$$\mathcal{L}_a[u^{'}(x)](s) = s(\mathcal{L}_a u)(s) - u(a)e^{-as}. \tag{20}$$

The following two fixed point theorems will be useful in the sequel. The first theorem is Krasnoselskll's fixed point theorem and the second result is Leray-Schauder fixed point theorem in a cone.

Theorem 2.7 [35] Let E be a Banach space and let $\mathcal{K} \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Assume also that $T : \mathcal{K} \cap (\Omega_2 \setminus \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If either

(i) $\|Tu\| \le \|u\|$ for $u \in \mathcal{K} \cap \Omega_1$ and $\|Tu\| \ge \|u\|$ for $u \in \mathcal{K} \cap \Omega_2$, or

(ii) $\|Tu\| \ge \|u\|$ for $u \in \mathcal{K} \cap \Omega_1$ and $\|Tu\| \le \|u\|$ for $u \in \mathcal{K} \cap \Omega_2$,

then T has a fixed point in $\mathcal{K} \cap (\Omega_2 \setminus \Omega_1)$.

Theorem 2.8 [36] Let E be a Banach space and $C \subset E$ be a closed and convex subset of E. Assume that O is a relatively open subset of C with $0 \in O$ and $T : O \to C$ is a continuous and compact map. Then either

(1) T has a fixed point in O or

(2) There is $u \in \partial O$ and $\lambda \in (0, 1)$ such that $u = \lambda Tu$.

Lemma 2.9 [32] The fractional differential equation (3) is equivalent to the following fractional integral equations given as

$$u(\tau) = \gamma u_0 + (1-\gamma)u_1 + \frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} F(s, u(s))ds$$

$$+ \frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s - \tau)^{\mu_2 - 1} F(s, u(s))ds. \tag{21}$$

3. Positive solutions of FBVP

Let $E = C([0, 1])$ be the Banach space with the norm sup norm and $\mathcal{K} = \{u \in E : u(x) \ge 0, x \in [0, 1]\}$ be the cone defined on E. In view of Lemma 2.9 we define the operator $T : \mathcal{K} \to \mathcal{K}$ as follows

$$Tu(\tau) = \gamma u_0 + (1-\gamma)u_1 + \frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} F(s, u(s))ds$$

$$+ \frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s - \tau)^{\mu_2 - 1} F(s, u(s))ds. \tag{22}$$

We shall show that the operator $T : \mathcal{K} \to \mathcal{K}$ is completely continuous operator in the following lemma.

Lemma 3.1 The operator $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.

Proof. We first show that $T : \mathcal{K} \to \mathcal{K}$ is continuous operator. Since $F \in C([0, 1] \times [0, \infty), [0, \infty))$, for any $u_1, u_2 \in [0, \infty)$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|u_1(\tau) - u_2(\tau)| < \delta$, we have

$$| F(\tau, u_1(\tau)) - F(\tau, u_2(\tau)) | < \min\{\Gamma(\mu_1 + 1), \Gamma(\mu_2 + 1)\} \frac{\varepsilon}{2}.$$

Hence,

$$| Tu_1(\tau) - Tu_2(\tau) | \le \frac{1}{\Gamma(\mu_1)} \int_0^\tau (\tau - s)^{\mu_1 - 1} | F(s, u_1(s)) - F(s, u_2(s)) | ds \tag{23}$$

$$+ \frac{1}{\Gamma(\mu_2)} \int_\tau^1 (s - \tau)^{\mu_2 - 1} | F(s, u_1(s)) - F(s, u_2(s)) | ds \tag{24}$$

$$\le \min\{\Gamma(\mu_1 + 1), \Gamma(\mu_2 + 1)\} \frac{\varepsilon}{2\Gamma(\mu_1)} \int_0^\tau (\tau - s)^{\mu_1 - 1} ds \tag{25}$$

$$+ \min\{\Gamma(\mu_1 + 1), \Gamma(\mu_2 + 1)\} \frac{\varepsilon}{2\Gamma(\mu_2)} \int_\tau^1 (s - \tau)^{\mu_2 - 1} ds \tag{26}$$

$$\le \frac{\tau^{\mu_1}}{2} \varepsilon + \frac{(1 - \tau)^{\mu_2}}{2} \varepsilon < \varepsilon. \tag{27}$$

This shows that T is continuous. We next show that T is completely continuous. Let $B \subset \mathcal{K}$ be bounded subset and let $M := \max_{s \in [0,1], u \in B} F(s, u(s)) + 1$. Then for $u \in B$ we have

$$| Tu(\tau) | \le \gamma | u_0 | + \frac{1}{\Gamma(\mu_1)} \int_0^\tau (\tau - s)^{\mu_1 - 1} | F(s, u(s)) | ds \tag{28}$$

$$+ (1 - \gamma) | u_1 | + \frac{1}{\Gamma(\mu_2)} \int_\tau^1 (s - \tau)^{\mu_2 - 1} | F(s, u(s)) | ds \tag{29}$$

$$\le \gamma | u_0 | + \frac{M}{\Gamma(\mu_1)} \int_0^\tau (\tau - s)^{\mu_1 - 1} ds + (1 - \gamma) | u_1 | + \frac{M}{\Gamma(\mu_2)} \int_\tau^1 (s - \tau)^{\mu_2 - 1} ds \tag{30}$$

$$\le \gamma | u_0 | + (1 - \gamma) | u_1 | + \frac{M}{\Gamma(\mu_1 + 1)} + \frac{M}{\Gamma(\mu_2 + 1)}, \tag{31}$$

which shows $T(B)$ is bounded, that is, T maps bounded sets to bounded sets. Take $\tau_1, \tau_2 \in [0, 1]$, $\tau_1 < \tau_2$ and $u \in B$. Then we have

$$|Tu(\tau_1)-Tu(\tau_2)|\leq\left|\frac{1}{\Gamma(\mu_1)}\left(\int_0^{\tau_1}(\tau_1-s)^{\mu_1-1}F(s,u(s))ds\int_0^{\tau_2}(\tau_2-s)^{\mu_1-1}F(s,u(s))ds\right)\right|$$

$$+\left|\left(\frac{1}{\Gamma(\mu_2)}\int_{\tau_1}^1(s-\tau_1)^{\mu_2-1}F(s,u(s))ds-\int_{\tau_2}^1(s-\tau_2)^{\mu_2-1}F(s,u(s))ds\right)\right|$$

$$=\left|\frac{1}{\Gamma(\mu_1)}\int_0^{\tau_1}\left((\tau_1-s)^{\mu_1-1}-(\tau_2-s)^{\mu_1-1}\right)F(s,u(s))ds-\frac{1}{\Gamma(\mu_1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\mu_1-1}F(s,u(s))ds\right|$$

$$=\left|\frac{1}{\Gamma(\mu_1)}\int_0^{\tau_1}\left((\tau_1-s)^{\mu_1-1}-(\tau_2-s)^{\mu_1-1}\right)F(s,u(s))ds-\frac{1}{\Gamma(\mu_1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\mu_1-1}F(s,u(s))ds\right|$$

$$+\left|\frac{1}{\Gamma(\mu_2)}\int_{\tau_2}^1\left((s-\tau_1)^{\mu_2-1}-(s-\tau_2)^{\mu_2-1}\right)F(s,u(s))ds+\frac{1}{\Gamma(\mu_2)}\int_{\tau_1}^{\tau_2}(s-\tau_1)^{\mu_2-1}F(s,u(s))ds\right|$$

$$\leq\frac{M}{\Gamma(\mu_1)}\int_0^{\tau_1}\left((\tau_1-s)^{\mu_1-1}-(\tau_2-s)^{\mu_1-1}\right)ds+\frac{M}{\Gamma(\mu_1)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\mu_1-1}ds$$

$$+\frac{M}{\Gamma(\mu_2)}\int_{\tau_2}^1\left((s-\tau_1)^{\mu_2-1}-(s-\tau_2)^{\mu_2-1}\right)ds+\frac{M}{\Gamma(\mu_2)}\int_{\tau_1}^{\tau_2}(s-\tau_1)^{\mu_2-1}ds$$

$$\leq\frac{M}{\Gamma(\mu_1+1)}\left(\tau_1^{\mu_1}-\tau_2^{\mu_1}+2(\tau_2-\tau_1)^{\mu_1}\right)+\frac{M}{\Gamma(\mu_2+1)}\left((1-\tau_1)^{\mu_2}-(1-\tau_2)^{\mu_2}\right)$$

The right-hand side of the above inequality goes to zero when $\tau_2\to\tau_1$. This shows that the set $T(B)$ is equicontinuous set. □

Theorem 3.2 Let $\mu_1,\mu_2\in(0,1]$, $\gamma\in[0,1]$ and $F\in C([0,1]\times[0,\infty),[0,\infty))$. Assume that there are two different positive constants r_1 and r_2 with $r_1<2\gamma u_0+2(1-\gamma)u_1<r_2$ such that

(A1) $F(x,u)\leq\dfrac{\min\{\Gamma(\mu_1+1),\Gamma(\mu_2+1)\}r_2}{2\max\{2^{1-\mu_1},2^{1-\mu_2}\}}$ for $(x,u)\in[0,1]\times[0,r_2]$;

(A2) $F(x,u)\geq\frac{1}{2}\Gamma(\mu_1+1)r_1$ for $(x,u)\in[0,1]\times[0,r_1]$.

Then there exists at least one positive solution for the fractional boundary value problem (3).

Proof. Consider the operator $T:\mathcal{K}\to\mathcal{K}$ defined by (22). From Lemma 3.1, it is clear that T is completely continuous.

Now, set $\Omega_1:=\{u\in\mathcal{K}:\|u\|<r_1\}$. Then for $u\in\mathcal{K}\cap\partial\Omega_1$, $\forall x\in[0,1]$, we have $u(x)\in[0,r_1]$ such that $\|u\|=r_1$. So, for $u\in\mathcal{K}\cap\partial\Omega_1$, by (A2) one can find

$$\|Tu\|=\max_{x\in[0,1]}|Tu(x)|\geq|Tu(1)|$$

$$=\left|\gamma u_0+(1-\gamma)u_1+\frac{1}{\Gamma(\mu_1)}\int_0^1(1-s)^{\mu_1-1}F(s,u(s))ds\right|$$

$$\geq \gamma u_0 + (1-\gamma)u_1 + \frac{\Gamma(\mu_1+1)r_1}{2\Gamma(\mu_1)}\int_0^1 (1-s)^{\mu_1-1}ds$$

$$> r_1. \tag{32}$$

Thus we have shown that $\|Tu\| \geq r_1 = \|u\|$ for $u \in \mathcal{K} \cap \Omega_1$. On the other hand, we set $\Omega_2 := \{u \in \mathcal{K} : \|u\| < r_2\}$. Then for $u \in \mathcal{K} \cap \partial\Omega_2$, $\forall x \in [0, 1]$, we have $u(x) \in [0, r_2]$ such that $\|u\| = r_2$. So, for $u \in \mathcal{K} \cap \partial\Omega_2$, by (A1) one can find

$$|Tu(\tau)| = \left| \gamma u_0 + (1-\gamma)u_1 + \frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau-s)^{\mu_1-1}F(s, u(s))ds \right.$$

$$\left. + \frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s-\tau)^{\mu_2-1}F(s, u(s))ds \right|$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + \frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}r_2}{2\Gamma(\mu_1)\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}}\int_0^\tau (\tau-s)^{\mu_1-1}ds$$

$$+ \frac{1}{\Gamma(\mu_2)}\frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}r_2}{2\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}}\int_\tau^1 (s-\tau)^{\mu_2-1}ds$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + + \frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}r_2}{2\Gamma(\mu_1+1)\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}}2^{1-\mu_1}$$

$$+ \frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}r_2}{2\Gamma(\mu_2+1)\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}}2^{1-\mu_2}$$

$$< r_2. \tag{33}$$

Thus we have shown that $\|Tu\| \leq r_2 = \|u\|$ for $u \in \mathcal{K} \cap \Omega_2$. So, Theorem 2.7 implies that the operator T has a fixed point. This means that the problem (3) has a positive solution, call it by u_p, with $r_1 \leq \|u_p\| \leq r_2$.

This completes the proof. □

Theorem 3.3 Let $\mu_1, \mu_2 \in (0, 1]$, $\gamma \in [0, 1]$ and $F \in C([0, 1] \times [0, \infty), [0, \infty))$. Further assume that

(A3) there exists a continuous and nondecreasing function $h : [0, \infty) \to [0, \infty)$ such that $F(x, u) \leq h(u)$ for $(x, u) \in [0, 1] \times [0, \infty)$;

(A4) there is r with $\frac{r - (\gamma u_0 + (1-\gamma)u_1)}{h(r)} > \max\{\frac{2^{2-\mu_1}}{\Gamma(\mu_1+1)}, \frac{2^{2-\mu_2}}{\Gamma(\mu_2+1)}\}$.

Then the problem (3) has at least one positive solution.

Proof. Put $V := \{u \in \mathcal{K} : \|u\| < r\}$, so that $V \subset \mathcal{K}$. In view of Lemma 3.1 $T : \overline{V} \to \mathcal{K}$ is completely continuous. Assume that $u \in \partial V$ is a solution of

$$u = \lambda Tu, \tag{34}$$

for each $\lambda \in [0, 1]$. Then by (A3) and (34) and for each $\tau \in [0, 1]$ we have

$$u(\tau) = \lambda Tu(\tau) = \lambda\gamma u_0 + \lambda(1-\gamma)u_1 + \frac{\lambda}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} F(s, u(s))ds \tag{35}$$

$$+\frac{\lambda}{\Gamma(\mu_2)}\int_\tau^1 (s-\tau)^{\mu_2-1} F(s, u(s))ds \tag{36}$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + \frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} F(s, u(s))ds \tag{37}$$

$$+\frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s-\tau)^{\mu_2-1} F(s, u(s))ds \tag{38}$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + \frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} h(u(s))ds \tag{39}$$

$$+\frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s-\tau)^{\mu_2-1} h(u(s))ds \tag{40}$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + h(\| u \|)\Big(\frac{1}{\Gamma(\mu_1)}\int_0^\tau (\tau - s)^{\mu_1 - 1} ds \tag{41}$$

$$+\frac{1}{\Gamma(\mu_2)}\int_\tau^1 (s-\tau)^{\mu_2-1} ds\Big) \tag{42}$$

$$\leq \gamma u_0 + (1-\gamma)u_1 + h(\| u \|)\Big(\frac{2^{1-\mu_1}}{\Gamma(\mu_1+1)} + \frac{2^{1-\mu_2}}{\Gamma(\mu_2+1)}\Big), \tag{43}$$

hence, $\| u \| \leq \gamma u_0 + (1-\gamma)u_1 + h(\| u \|) 2\max\{\frac{2^{1-\mu_1}}{\Gamma(\mu_1+1)}, \frac{2^{1-\mu_2}}{\Gamma(\mu_2+1)}\}$. Consequently, we get

$$\frac{\| u \| - (\gamma u_0 + (1-\gamma)u_1)}{h(\| u \|)} \leq \max\{\frac{2^{2-\mu_1}}{\Gamma(\mu_1+1)}, \frac{2^{2-\mu_2}}{\Gamma(\mu_2+1)}\}. \tag{44}$$

The condition (A4) and (44) lead to $||u|| \neq r$ which contradicts to the fact that $u \in \partial V$. This implies that T has a fixed point $u \in \overline{V}$ by Theorem 2.8. Consequently, there exists at least one positive solution of the problem (3). Thus we complete the proof. □

4. A modified fractional Maxwell model

In this section, we examine the solutions of the modified fractional Maxwell model given by (4). We will prove that we can recover the fractional modified Maxwell model with the left-sided Caputo derivative presented in [37] by letting $\gamma = 1$ and the modified Maxwell model with the ordinary derivative studied in [27]. To this end, we need the following lemmas.

Lemma 4.1 The extended Laplace transformation of the left-sided and right-sided Caputo derivatives are given by

$$(\mathcal{L}\{{}_0^C D_x^{\mu_1} u(x)\})(s) = s^{\mu_1}(\mathcal{L}u)(s) - s^{\mu_1 - 1}u(0), \tag{45}$$

$$(\mathcal{L}_{1x}^{C} D_1^{\mu_1} u(x))(s) = (-1)^{1-\mu_1}\left(s^{\mu_1}(\mathcal{L}_1 u)(s) - s^{\mu_1 - 1}e^{-s}u(1)\right). \tag{46}$$

Proof. The proof of (45) can be found in [4]. We shall prove (46). By the definition of the extended Laplace transformation (19), we have

$$(\mathcal{L}_{1x}^{C} D_1^{\mu_1} u(x))(s) = (\mathcal{L}_1\left\{\frac{-1}{\Gamma(1-\mu_1)}\int_x^1 (t-x)^{-\mu_1} u'(t)dt\right\})(s)$$

$$= (-1)^{1-\mu_1}\mathcal{L}\left\{\frac{x^{-\mu_1}}{\Gamma(1-\mu_1)}\right\}(\mathcal{L}_1\left\{u'(x)\right\})(s)$$

$$= (-1)^{1-\mu_1}\cdot s^{\mu_1 - 1}\left[s(\mathcal{L}_1\{u(x)\})(s) - u(x)\cdot e^{-sx}\right]_{x=1}$$

$$= (-1)^{1-\mu_1}\cdot\left[s^{\mu_1}(\mathcal{L}_1\{u(x)\})(s) - s^{\mu_1 - 1}e^{-s}u(1)\right], \tag{47}$$

where we have used (20) in the second equality. The proof is now completed. □

Lemma 4.2 The extended Laplace transformation of the combined Caputo derivative is given by

$$(\mathcal{L}_a\, {}^C D_\gamma^{\mu_1,\mu_2} u(x))(s) = \mathcal{L}(s)A(s) - u(0)B(s) - u(1)C(s)$$

$$+(-1)^{\mu_2}(1-\gamma)s^{\mu_2}\int_0^1 e^{-st}u(t)dt, \tag{48}$$

where

$$A(s) = \gamma s^{\alpha} + (1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2} \tag{49}$$

$$B(s) = \gamma s^{\alpha - 1} \tag{50}$$

$$C(s) = (1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2 - 1}e^{-s}. \tag{51}$$

Proof. Since the Laplace transform is a linear operator, we can write

$$\mathcal{L}_a\left\{{}^C D_\gamma^{\mu_1,\mu_2} u(x))\right\} = \mathcal{L}_a\left\{\gamma\cdot{}_0^C D_x^{\mu_1} u(x) + (1-\gamma)\cdot{}_x^C D_1^{\mu_2} u(x)\right\}$$

$$= \mathcal{L}\left\{\gamma\cdot{}_0^C D_x^{\mu_1} u(x)\right\} + \mathcal{L}_1\left\{(1-\gamma)\cdot{}_x^C D_1^{\mu_2} u(x)\right\}. \tag{52}$$

From (45) and (46), we obtain

$$\mathcal{L}_a\left\{{}^{C}D_{\gamma}^{\mu_1,\mu_2}u(x))\right\}$$

$$=\gamma\cdot\left[s^{\alpha}\mathcal{L}(s)-s^{\mu_1-1}u(0)\right]+(1-\gamma)(-1)^{1-\mu_2}\cdot\left[s^{\mu_2}\mathcal{L}_1(s)-s^{\mu_2-1}e^{-s}u(1)\right]$$

$$=\mathcal{L}(s)\left[\gamma s^{\alpha}+(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2}\right]-\gamma s^{\alpha-1}u(0)$$

$$-(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2-1}e^{-s}u(1)+(-1)^{\mu_2}(1-\gamma)s^{\mu_2}\int_0^1 e^{-st}u(t)dt, \tag{53}$$

where we have used the fact that $\mathcal{L}_1(s)=\mathcal{L}(s)-\int_0^1 e^{-st}u(t)dt$. The result follows. The proof is now finished. □

First, we consider the relaxation test by letting the strain be a constant $s=s_0$ for $\tau>0$ in the equation (4). Thus, we have ${}^{C}D_{\gamma}^{\mu_1,\mu_2}s(\tau)=0$ and we get the following relaxation problem

$${}^{C}D_{\gamma}^{\mu_1,\mu_2}T(\tau)=-ET(\tau). \tag{54}$$

From (48) and noting that u is replaced with T, we have

$$\overline{T}(s)A(s)-T(0)B(s)-T(1)C(s)+(-1)^{\mu_2}(1-\gamma)s^{\mu_2}\int_0^1 e^{-st}T(t)dt=-E\overline{T}(s), \tag{55}$$

where $\overline{T}(s)=(\mathcal{L}T)(s)$. For simplicity, we take $E=1$ from now on. A little algebra reveals that

$$\overline{T}(s)=\frac{1}{A(s)}\left(T(0)B(s)+T(1)C(s)+(-1)^{\mu_2}(1-\gamma)s^{\mu_2}\int_0^1 e^{-st}T(t)dt\right) \tag{56}$$

$$=T(0)\frac{\gamma s^{\mu_1-1}}{\gamma s^{\alpha}+(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2}}+T(1)\frac{(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2-1}e^{-s}}{\gamma s^{\alpha}+(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2}} \tag{57}$$

$$+\frac{(-1)^{\mu_2}(1-\gamma)s^{\mu_2}\int_0^1 e^{-st}T(t)dt}{\gamma s^{\alpha}+(1-\gamma)(-1)^{1-\mu_2}\cdot s^{\mu_2}}. \tag{58}$$

Applying the inverse Laplace transformation (18) to above equation gives that

$$T(\tau)\approx(-1)^{\mu_2}\frac{1-\gamma}{\gamma}\tau^{-(\mu_1-\mu_2)}E_{\mu_1-\mu_2,\mu_1-\mu_2}\left(\frac{1-\gamma}{(-1)^{\mu_2}\gamma}\tau^{\mu_1-\mu_2}\right)+T(0)\cdot E_{\mu_1-\mu_2,1}\left(\frac{1}{(-1)^{\mu_2}\gamma}\tau^{\mu_1-\mu_2}\right)$$

$$+\frac{T(1)(1-\gamma)(-1)^{1-\mu_2}}{\gamma}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\tau^{\mu_1-\mu_2-n}\cdot E_{\mu_1-\mu_2,1+\mu_1-\mu_2-n}\left(\frac{1-\gamma}{(-1)^{\mu_2}\gamma}\tau^{\mu_1-\mu_2}\right), \tag{59}$$

where we have used the fact that $(\mathcal{L}\{x^{\mu_2-1}E_{\mu_1,\mu_2}(\lambda x^{\mu_1})\})(s) = s^{-\mu_2}(1-\lambda s^{1-\mu_1})^{-1}$ and that

$$\left| \int_0^1 e^{-st} T(t)dt \right| < \int_0^1 e^{-st} |T(t)| dt < \varepsilon \tag{60}$$

since the Laplace transformation converges absolutely for large s. Here,

$$E_{\mu_1,\mu_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu_1 + \mu_2)}$$

is a generalized Mittag-Leffler function with two parameters with $z \in \mathbb{C}$, μ_1 and μ_2 are arbitrary positive constants [4]. If $\mu_2 = 1$, then we have the classical Mittag-Leffler function $E_{\mu_1,1}(z) = E_{\mu_1}(z)$ [38]. We plot the stress response $T(\tau)$ with various parameters in Figure 1.

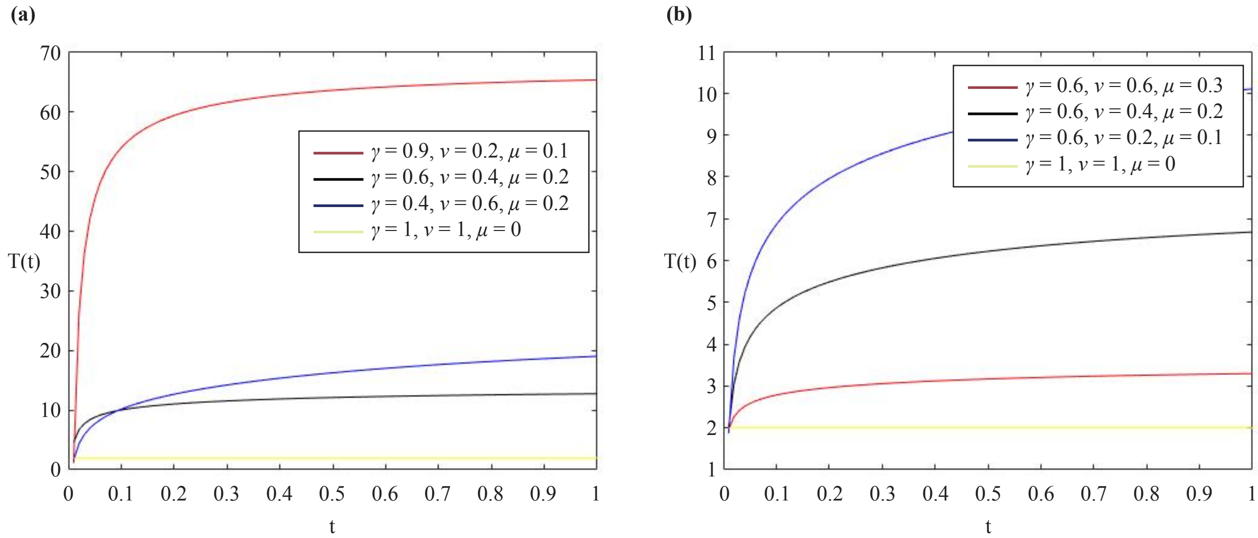

Figure 1. The stress response $T(\tau)$ with various parameters

5. Numerical examples

We give several examples to illustrate applications of the main findings in this paper.

Example 1 Consider the following fractional boundary value problem

$${}^{C}D_{1/2}^{0.5,0.4}u(\tau) = \exp(-u(\tau)) + \tau^2 + 1, x \in [0, 1],$$

$$u(0) = 1, u(1) = 2. \tag{61}$$

Here $\mu_1 = 0.5$, $\mu_2 = 0.6$, $\gamma = 1/2$ and $F(s, u(s)) = \exp(-u(s)) + s^2 + 1$. We take $r_1 = 1$ and $r_2 = 8$. We compute $\frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}}{2\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}} \approx 0.2853$ and $\frac{\Gamma(\mu_1+1)}{2} \approx 0.4431$. We get $r_1 < 2\gamma u_0 + 2(1-\gamma)u_1 < r_2$ and

$$F(s, u(s)) \leq 2 \leq \frac{\min\{\Gamma(\mu_1+1), \Gamma(\mu_2+1)\}}{2\max\{2^{1-\mu_1}, 2^{1-\mu_2}\}} r_2 \text{ for } (s, u) \in [0, 1] \times [0, 8], \tag{62}$$

$$F(s, u(s)) \geq 1 \geq \frac{\Gamma(\mu_1+1)}{2} r_1 \text{ for } (s, u) \in [0, 1] \times [0, 1]. \tag{63}$$

Theorem 3.2 guarantees that there is at least one positive solution u_p of the problem (61) with $1 \leq \|u_p\| \leq 8$.

Example 2 Consider

$$^{C}D_{1/5}^{0.8,0.9} u(\tau) = (\tau - 1/2)^2 u^3(\tau), x \in [0, 1],$$

$$u(0) = 0.2, u(1) = 0.2. \tag{64}$$

Here $\mu_1 = 0.8$, $\mu_2 = 0.9$, $\gamma = 1/5$, $u_0 = u_1 = 0.2$ and $F(s, u(s)) = (s - 1/2)^2 u^3(s)$. We take $h(s) = s^3$ which is a continuous, nondecreasing function and $r = 0.4$. We compute $\max\{\frac{2^{2-\mu_1}}{\Gamma(\mu_1+1)}, \frac{2^{2-\mu_2}}{\Gamma(\mu_2+1)}\} \approx 2.4666$.

$$F(s, u(s)) = (s - 1/2)^2 u^3(s) \leq u^3(s) = h(s) \text{ for } (s, u) \in [0, 1] \times [0, \infty), \tag{65}$$

and $\frac{r - (\gamma u_0 + (1-\gamma)u_1)}{h(r)} = \frac{0.4 - 0.2}{0.4^3} \approx 3.125 > \max\{\frac{2^{2-\mu_1}}{\Gamma(\mu_1+1)}, \frac{2^{2-\mu_2}}{\Gamma(\mu_2+1)}\}$. Theorem 3.3 guarantees that there exists at least one positive solution of the problem (64).

6. Conclusion

In this work, we first presented new results on the existence of positive solutions for the fractional boundary value problems involving the combined Caputo fractional derivative by making use of the Leggett-Williams norm-type theorem for coincidences. We provided some examples to illustrate the main results of the paper.

Secondly, we considered the Maxwell model using the combined Caputo fractional derivative operator. With the help of the extended Laplace transformation and Laplace transform inversion, we observed the behaviour of the stress response with respect to the time $\tau \in (0, 1)$. We will consider viscoelastic models using other type of fractional derivative operator involving non-singular kernel in ongoing study.

Conflict of interest

The author declares no competing financial interest.

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