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Research Article

Uniqueness of an Orthogonal 2-Handle Pair on a Surface-Link

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Abstract: The proof of uniqueness of an orthogonal 2-handle pair on a surface-link is given from the viewpoint of a normal form of 2-handle core disks. A version to an immersed orthogonal 2-handle pair on a surface-link is also observed.

***Keywords*:** surface-link, orthogonal 2-handle pair

MSC: 57K45, 57K10

1. Introduction

A surface-link is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When **F** is connected, it is also called a surface-knot. Two surface-links F and F' are equivalent by an equivalence f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f: \mathbf{R}^4 \to \mathbf{R}^4$. A trivial surface-link is a surface-link F which is the boundary of disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial surface-knot is also called an unknotted surface-knot and a trivial disconnected surface-link is also called an unknotted and unlinked surface-link. A trivial surface-link is unique up to equivalences (see [1]). A 2-handle on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a core disk such that $D \times I \cap F = \partial D \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. Let $F(D \times I)$ denote the surface-link obtained from F by surgery along $D \times I$. If D is an immersed disk, then call it an immersed 2-handle. Two (possibly immersed) 2-handles $D \times I$ and $E \times I$ on F are equivalent if there is an equivalence $f: \mathbf{R}^4 \to \mathbf{R}^4$ from F to itself such that the restriction $f|_F : F \to F$ is the identity map and $f(D \times I) = E \times I$. An orthogonal 2-handle pair (or simply, an O2-handle pair) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I$ on F such that

$$D \times I \cap D' \times I = \partial D \times I \cap \partial D' \times I$$

and $\partial D \times I$ and $\partial D' \times I$ meet orthogonally on F, that is, the boundary circles ∂D and $\partial D'$ meet transversely at one point q and the intersection $\partial D \times I \cap \partial D' \times I$ is homeomorphic to the square $Q = q \times I \times I$ (see Figure 1).

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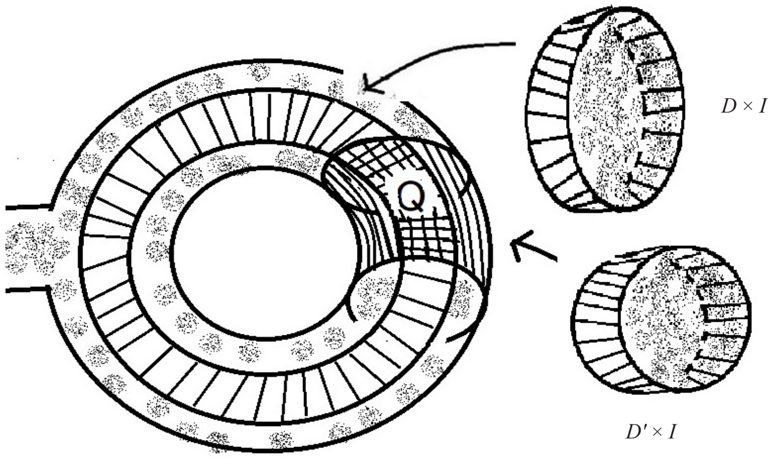

Figure 1. An orthogonal 2-handle pair(=: an O2-handle pair)

Then the three kinds of surface-links $F(D \times I, D' \times I)$, $F(D \times I)$ and $F(D' \times I)$ obtained by surgeries on $(D \times I, D' \times I)$ are all equivalent (see [2, Lemma 2.2]).

An important property of an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F is the following property (see [2, Lemma 2.3, Corollary 2.4] for the proof):

Common 2-handle property. Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. If $D \times I = E \times I$ or $E' \times I = D' \times I$, then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent by an equivalence obtained by 3-cell moves on the unions $D \times I \cup D' \times I$ and $E \times I \cup E' \times I$ which are 3-balls.

In this paper, the following uniqueness theorem of an O2-handle pair on a surface-link is shown by using a normal form of 2-handle core disks discussed in [3] and Common 2-handle property stated above repeatedly which is announced in [2, Section 3] with incomplete proof although the tools of the present proof appear there.

Uniqueness Theorem. Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$. Then the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent.

This theorem for a trivial surface-link is heavily used for confirming the smooth unknotting conjecture of a surface-knot in [2] and the smooth unknotting-unlinking conjecture for a surface-link in [4], whose confirmations are completed by this theo-rem (see [2, Lemma 2.5]). For an immersed O2-handle pair, the following lemma is provided:

Recovery Lemma. If $(D \times I, D' \times I)$ is an immersed O2-pair on a surface-link F in $\mathbf{R}^4$ with $D \times I$ immersed and $D' \times I$ embedded, then there is an embedded 2-handle $D_* \times I$ with $\partial D_* \times I = \partial D \times I$ such that $(D_* \times I, D' \times I)$ is an O2-handle pair on F.

For the proof of Recovery Lemma, Finger move canceling operation is used to cancel a double point of an immersed core disk D of the immersed 2-handle $D \times I$ on F, which is explained in Section 3. By Uniqueness Theorem and Recovery Lemma, we have the following corollary.

Corollary. Let F be a surface-link in $\mathbf{R}^4$, and $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ immersed O2-handle pairs on F in $\mathbf{R}^4$ with $\partial D \times I = \partial E \times I$ and $\partial D' \times I = \partial E' \times I$.

(1) If $D' \times I$ and $E' \times I$ are embedded, then there are embedded 2-handles $D_* \times I$ and $E_* \times I$ on F with $\partial D_* \times I = \partial D \times I$ and $\partial E_* \times I = \partial E \times I$ such that $(D_* \times I, D' \times I)$ and $(E_* \times I, E' \times I)$ are equivalent O2-handle pairs on F, so that the surface-links $F(D' \times I)$ and $F(E' \times I)$ are equivalent.

(2) If $D' \times I$ and $E \times I$ are embedded, then there are embedded 2-handles $D_* \times I$ and $E'_* \times I$ on F with $\partial D_* \times I = \partial D \times I$ and $\partial E'_* \times I = \partial E' \times I$ such that $(D_* \times I, D' \times I)$ and $(E \times I, E'_* \times I)$ are equivalent O2-handle pairs on F, so that the surface-links $F(D' \times I)$ and $F(E \times I)$ are equivalent.

The proof of Uniqueness Theorem is done in Section 2 and and the proof of Recovery Lemma is done in Section 3.

Throughout the paper, the notation

$$XJ = \{(x, t) \in \mathbf{R}^4 | \, x \in X, t \in J\}$$

is used for a subspace X of $\mathbf{R}^3$ and a subinterval J of $\mathbf{R}$.

2. Proof of uniqueness theorem

The proof of Uniqueness Theorem is divided into the proof of the case of a trivial surface-knot F and the proof of the case of a general surface-link F. In the argument, the O2-handle pair $(D \times I, D' \times I)$ is fixed in the 3-space $\mathbf{R}^3[0]$ and consider normal forms of the core disks E, E' of the 2-handles $E \times I$, $E' \times I$ in $\mathbf{R}^4$. To avoid the complexity of handling the intersection point $q = E \cap E'$, a sufficiently small boundary-collar $n(\partial E')$ of E' is fixed in $\mathbf{R}^3[0]$ and consider a normal form of the disk

$$E'_n = \mathrm{cl}(E' \setminus n(\partial E'))$$

in $\mathbf{R}^4$ together with a normal form of E.

Proof of Uniqueness Theorem in the case of a trivial surface-link F. Assume that the trivial surface-knot F is embedded standardly in $\mathbf{R}^3[0]$ with a standard O2-handle pair $(D \times I, D' \times I)$ on F. By [3], the disk union $G = E \cup E'_n$ is deformed into a disk union G_1 in the following form by an isotopy of $\mathbf{R}^4$ keeping the boundary $\partial G = \partial E \cup \partial E'_n$ (which is a trivial link in $\mathbf{R}^3[0]$), $n(\partial E')$ and F fixed:

$$G_1 \cap \mathbf{R}^3[t] = \begin{cases} \varnothing, & \text{for } t > 2, \\ \mathbf{d}'[t], & \text{for } t = 2, \\ o'[t], & \text{for } 1 < t < 2, \\ (\partial G \cup \ell \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial G \cup \ell)[t], & \text{for } 0 \le t < 1, \\ \ell[t], & \text{for } -1 < t < 0, \\ (o \cup \mathbf{b})[t], & \text{for } t = -1, \\ o[t], & \text{for } -2 < t < -1, \\ \mathbf{d}[t], & \text{for } t = -2, \\ \varnothing, & \text{for } t < -2, \end{cases}$$

where the notations o, o' denote trivial links in $\mathbf{R}^3$, the notations $\mathbf{d}$, $\mathbf{d}'$ denote disjoint disk systems in $\mathbf{R}^3$ bounded by o, o', respectively, the notations $\mathbf{b}$, $\mathbf{b}'$ denote disjoint band systems in $\mathbf{R}^3$ spanning o, o', respectively, and the notation ℓ denotes a link in $\mathbf{R}^3$. To obtain this disk union G_1, start the argument of [3] with the assumption that the intersection $G \cap \mathbf{R}^3[0]$ is a link $\ell[0] \cup \partial G$ in $\mathbf{R}^3[0]$ and a boundary-collar $n(\partial G)$ of ∂G in G is in $\mathbf{R}^3[0, c]$ so that

$$n(\partial G) \cap \mathbf{R}^3[t] = \partial G[t], \ t \in [0, c]$$

for a small number $c > 0$, where ∂G is regarded to be in $\mathbf{R}^3$ under the canonical identification $\mathbf{R}^3[0] = \mathbf{R}^3$. Then pull down a minimal point of G in $\mathbf{R}^3(0, \infty)$ to $\mathbf{R}^3(-\infty, 0)$ and pull up a maximal point of G in $\mathbf{R}^3(-\infty, 0)$ to $\mathbf{R}^3(0, \infty)$. In these deformations, trivial components are increased in the intersection link $G \cap \mathbf{R}^3[0]$. After these preparations, do normalizations of $G \cap \mathbf{R}^3[0, \infty)$ and $G \cap \mathbf{R}^3(-\infty, 0]$ keeping $G \cap \mathbf{R}^3[0]$ fixed. The band systems $\mathbf{b}$, $\mathbf{b}'$ are made disjoint by band slide and band thinning and the band system $\mathbf{b}$ is made disjoint from ∂G by a band move keeping the attaching part fixed. By denoting the deformed disks of E and E'_n by E and E'_n again, let $G_1 = E \cup E'_n$. The following notation is used.

Notation. The disk subsystems of the disk system **d** belonging to E or E'_n are denoted by $\mathbf{d}(E)$ or $\mathbf{d}(E'_n)$, respectively. The band subsystems of the band system **b** belonging to E or E'_n are denoted by $\mathbf{b}(E)$ or $\mathbf{b}(E'_n)$, respectively.

A next deformation of G_1 is to change the level of the band system $\mathbf{b}(E)[-1]$ into $\mathbf{b}(E)[1]$ and the level of the disk system $\mathbf{d}(E)[-2]$ into $\mathbf{d}(E)[0.5]$. To do so, it is observed that in $\mathbf{R}^3$, the boundary ∂G and the band system $\mathbf{b}(E'_n)$ meet the disk system $\mathbf{d}(E)$ in finite interior points and in finite interior simple arcs, respectively. For a point $x \in \mathbf{d}(E) \cap \partial G$, find a point $y \in \partial\mathbf{d}(E) \setminus \partial E$ and a simple arc α from x to y in $\mathbf{d}(E)$ which does not meet the band systems $\mathbf{b}$, $\mathbf{b}'$ by band slide and band thinning. Let $n(\alpha)$ be a disk neighborhood of α in $\mathbf{d}(E)$. Deform the disk system $\mathbf{d}'(E)$ so that $n(\alpha) \subset \mathbf{d}'(E)$. Then the intersection $e(\alpha) = n(\alpha)[-2, 2] \cap G_1$ is a disk in the interior of G_1. Let $\tilde{e}(\alpha) = \mathrm{cl}((\partial(n(\alpha)[-2, 2])) \setminus e(\alpha))$ be the complementary disk of the disk $e(\alpha)$ in the 2-sphere $\partial(n(\alpha)[-2, 2])$. The disk union

$$\tilde{G}_1 = \mathrm{cl}(G_1 \setminus e(\alpha)) \cup \tilde{e}(\alpha)$$

induces a normal form of the union of a deformed disk $\tilde{E}$ of E and the disk E'_n with $\partial\tilde{G}_1 = \partial G_1$. Note that the disk $\tilde{E}$ may meet with the surface F and the topological position of $\tilde{E}$ in $\tilde{G}_1$ may be changed from G_1, although the disk $E' = E'_n \cup n(\partial E')$ is unchanged and the level configuration of $\tilde{G}_1$ is similar to G_1. Do this deformation for all points of the finite set $\mathbf{d}(E) \cap \partial G$. Further, for an arc β in the finite arc set $\mathbf{d}(E) \cap \mathbf{b}(E'_n)$, find a simple arc α in $\mathbf{d}(E)$ extending this arc β to a point $y \in \partial\mathbf{d}(E) \setminus \partial E$ which does not meet the band systems $\mathbf{b}$, $\mathbf{b}'$ by band slide and band thinning. For a disk neighborhood $n(\alpha)$ in $\mathbf{d}(E)$, do the same deformation as above. Further, do this deformation for all arcs β in the finite arc set $\mathbf{d}(E) \cap \mathbf{b}(E'_n)$. Let $\tilde{G}_1 = \tilde{E} \cup E'_n$ be the disk union obtained from $G_1 = E \cup E'_n$ by all these deformations, which is in a normal form with a level configuration similar to G_1 and we have

$$\mathbf{d}(\tilde{E}) \cap (\partial E \cup n(\partial E')) = \mathbf{d}(\tilde{E}) \cap \mathbf{b}(E'_n) = \emptyset$$

although the disk $\tilde{E}$ may meet F. Now change the level of $\mathbf{b}(\tilde{E})[-1]$ into $\mathbf{b}(\tilde{E})[1]$ and the level of $\mathbf{d}(\tilde{E})[-2]$ into $\mathbf{d}(\tilde{E})[0.5]$. The resulting disk union $G_2 = \tilde{E} \cup E'_n$ is in the following form:

$$G_2 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ \mathbf{d}'[t], & \text{for } t = 2, \\ o'[t], & \text{for } 1 < t < 2, \\ (\partial G \cup o(\tilde{E}) \cup \mathbf{b}(\tilde{E}) \cup \ell(E'_n) \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial G \cup o(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } 0.5 < t < 1, \\ (\partial G \cup \mathbf{d}(\tilde{E}) \cup \ell(E'_n))[t], & \text{for } t = 0.5, \\ (\partial G \cup \ell(E'_n))[t], & \text{for } 0 \le t < 0.5, \\ \ell(E'_n)[t], & \text{for } -1 < t < 0, \\ (o(E'_n) \cup \mathbf{b}(E'_n))[t], & \text{for } t = -1, \\ o(E'_n)[t], & \text{for } -2 < t < -1, \\ \mathbf{d}(E'_n)[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < -2, \end{cases}$$

where $\ell(E'_n)$ denotes the sublink of ℓ belonging to the disk E'_n. In the configuration of the disk union G_2, the pairs $(\tilde{E} \times I, E' \times I)$ and $(\tilde{E} \times I, D' \times I)$ are O2-handle pairs on F and hence are equivalent by Common 2-handle property. Since $(\tilde{E} \times I, E' \times I)$ and $(\tilde{E} \times I, D' \times I)$ are respectively equivalent to the original O2-handle pairs $(E \times I, E' \times I)$ and $(D \times I, D' \times I)$ on F by Common 2-handle property, the original O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent. This completes the proof of Uniqueness Theorem in the case of a trivial surface-link F.

Proof of Uniqueness Theorem in the case of a general surface-link F. For a general surface-link F in $\mathbf{R}^4$ and O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$, let $F(D \times I, D' \times I) = F^c_\delta \cup \delta_{D\times I, D'\times I}$ be the surface-link obtained from F by surgery along $(D \times I, D' \times I)$ where F^c_δ denotes the once-punctured surface of F and the plumbed disk $\delta_{D\times I, D'\times I}$ induced from $(D \times I, D' \times I)$ (see [2]). Let F' be a trivial surface-knot in $\mathbf{R}^4$ obtained from the surface-link $F(D \times I, D' \times I)$ by surgery along 1-handles $h_j (j = 1, 2, \ldots, s)$ embedded in a connected Seifert hypersurface W and attached to F^c_δ avoiding the plumbed disk $\delta_{D\times I, D'\times I}$ and the intersection loops $E \cap W$, $E' \cap W$ (cf. [1]). Then it is seen from the argument of [2, Lemmas 2.2, 2.3] that a trivial torus-knot T and a standard O2-handle pair $(D \times I, D' \times I)$ on T are constructed from the plumbed disk $\delta_{D\times I, D'\times I}$ in $\mathbf{R}^4$ so that the connected sum $F'\#T$ is a trivial surface-knot in $\mathbf{R}^4$ obtained from F by surgery along the 1-handles $h_j (j = 1, 2, \ldots, s)$ without meeting the connected summand T and the O2-handle pair $(D \times I, D' \times I)$. By construction, the pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ are O2-handle pairs on the connected sum $F'\#T$ attached to the connected summand T and disjoint from the "2-handles" $h_j (j = 1, 2, \ldots, s)$ on $F'\#T$ attached to F'. Let $\mathbf{h}$ be the core disk system $D(h_j)$, $(j = 1, 2, \ldots, s)$ of the 2-handle system $h_j (j = 1, 2, \ldots, s)$ on $F'\#T$ attached to F'. By the proof for the case of a trivial surface-link F, the O2-handle pair $(E \times I, E' \times I)$ is equivalent to $(D \times I, D' \times I)$ on $F'\#T$. To obtain such an equivalence without crossing the core disk system $\mathbf{h}$, the proof is revised as follows: A normal form of the disk union $\bar{G} = G \cup \mathbf{h} = E \cup E'_n \cup \mathbf{h}$ can be thought of as the following disk union $\bar{G}_1$:

$$\bar{G}_1 \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ (d'(\mathbf{h}) \cup \mathbf{d}')[t], & \text{for } t = 2, \\ (o'(\mathbf{h}) \cup o')[t], & \text{for } 1 < t < 2, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup b'(\mathbf{h}) \cup \ell \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup \ell)[t], & \text{for } 0 \le t < 1, \\ (\ell(\mathbf{h}) \cup \ell)[t], & \text{for } -1 < t < 0, \\ (o(\mathbf{h}) \cup b(\mathbf{h}) \cup o \cup \mathbf{b})[t], & \text{for } t = -1, \\ (o(\mathbf{h}) \cup o)[t], & \text{for } -2 < t < -1, \\ (d(\mathbf{h}) \cup \mathbf{d})[t], & \text{for } t = -2, \\ \emptyset, & \text{for } t < -2, \end{cases}$$

where in addition to the notations on G_1, the following notations are also added. Namely, the notations $o(\mathbf{h})$, $o'(\mathbf{h})$ denote trivial links in $\mathbf{R}^3$ belonging to the disk system h, the notations $d(\mathbf{h})$, $d'(\mathbf{h})$ denote disjoint disk systems in $\mathbf{R}^3$ with $\partial d(\mathbf{h}) = o(\mathbf{h})$, $\partial d'(\mathbf{h}) = o'(\mathbf{h})$ belonging to the disk system $\mathbf{h}$, the notations $b(\mathbf{h})$, $b'(\mathbf{h})$ denote disjoint band systems in $\mathbf{R}^3$ belonging to the disk system $\mathbf{h}$ and spanning $o(\mathbf{h})$, $o'(\mathbf{h})$, respectively, and the notation $\ell(\mathbf{h})$ denotes a link in $\mathbf{R}^3$ belonging to the disk system $\mathbf{h}$. The band systems $\mathbf{b}$, $\mathbf{b}'$, $b(\mathbf{h})$, $b'(\mathbf{h})$ are made disjoint by band slide and band thinning. In this normal form $\bar{G}_1$, the disk system $\mathbf{h}$ can be taken as

$$\mathbf{h} \cap D \times I = \mathbf{h} \cap D' \times I = \emptyset,$$

because the 3-ball $D \times I \cup D' \times I$ is disjoint from the 2-handles $h_j (j = 1, 2, \ldots, s)$ as mentioned before. By a method similar to the process from G_1 to G_2, we have a deformation $\tilde{\bar{G}}_1 = \tilde{E} \cup E'_n \cup \mathbf{h}$ of $\bar{G}_1$ with a level configuration similar to $\bar{G}_1$ such that

$$\mathbf{d}(\tilde{E}) \cap (\partial E \cup n(\partial E')) = \mathbf{d}(\tilde{E}) \cap \mathbf{b}(E'_n) = \mathbf{d}(\tilde{E}) \cap b(\mathbf{h}) = \emptyset,$$

although $\tilde{E}$ may meet $F'\#T$. Now change the level of $\mathbf{b}(\tilde{E})[-1]$ into $\mathbf{b}(\tilde{E})[1]$ and the level of $\mathbf{d}(\tilde{E})[-2]$ into $\mathbf{d}(\tilde{E})[0.5]$. Then the disk union $\bar{G}_2 = \tilde{E} \cup E'_n \cup \mathbf{h}$ obtained from $\tilde{\bar{G}}_1 = \tilde{E} \cup E'_n \cup \mathbf{h}$ is as follows:

$$\bar{G}_2 \cap \mathbf{R}^3[t] = \begin{cases} \varnothing, & \text{for } t > 2, \\ (d'(\mathbf{h}) \cup \mathbf{d}')[t], & \text{for } t = 2, \\ (o'(\mathbf{h}) \cup o')[t], & \text{for } 1 < t < 2, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup b'(\mathbf{h}) \cup o(\bar{E}) \cup \mathbf{b}(\bar{E}) \cup \ell(E'_n) \cup \mathbf{b}')[t], & \text{for } t = 1, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup o(\bar{E}) \cup \ell(E'_n))[t], & \text{for } 0.5 < t < 1, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup \mathbf{d}(\bar{E}) \cup \ell(E'_n))[t], & \text{for } t = 0.5, \\ (\partial\bar{G} \cup \ell(\mathbf{h}) \cup \ell(E'_n))[t], & \text{for } 0 \leq t < 0.5, \\ (\ell(\mathbf{h}) \cup \ell(E'_n))[t], & \text{for } -1 < t < 0, \\ (o(\mathbf{h}) \cup b(\mathbf{h}) \cup o(E'_n) \cup \mathbf{b}(E'_n))[t], & \text{for } t = -1, \\ (o(\mathbf{h}) \cup o(E'_n))[t], & \text{for } -2 < t < -1, \\ (d(\mathbf{h}) \cup \mathbf{d}(E'_n))[t], & \text{for } t = -2, \\ \varnothing, & \text{for } t < -2, \end{cases}$$

In the configuration of $\bar{G}_2$, the pair $(\bar{E} \times I, E' \times I)$ and $(\bar{E} \times I, D' \times I)$ are O2-handle pairs on $F'\#T$ and hence are equivalent under 3-cell moves disjoint from the 2-handles $h_j (j = 1, 2, \ldots, s)$ by Common 2-handle property. Since $(\bar{E} \times I, E' \times I)$ and $(\bar{E} \times I, D' \times I)$ are respectively equivalent to the original O2-handle pairs $(E \times I, E' \times I)$ and $(D \times I, D' \times I)$ on $F'\#T$ under 3-cell moves disjoint from the 2-handles $h_j (j = 1, 2, \ldots, s)$ by Common 2-handle property, the original O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on $F'\#T$ are equivalent under 3-cell moves disjoint from the 2-handles $h_j (j = 1, 2, \ldots, s)$ by Common 2-handle property. By the back surgery from $F'\#T$ to F on the 2-handles $h_j (j = 1, 2, \ldots, s)$ on $F'\#T$, this means that the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F are equivalent under 3-cell moves disjoint from the 1-handles $h_j (j = 1, 2, \ldots, s)$ on F. This completes the proof of Uniqueness Theorem in the case of a general surface-link F. □

This completes the proof of Uniqueness Theorem.

3. Proof of recovery lemma

The following operation is used for the proof of Recovery Lemma.

Finger Move Canceling. Let D be an immersed disk in $\mathbf{R}^4$ with ∂D embedded, and S a trivial S^2-knot in $\mathbf{R}^4$ meeting the immersed disk D at just one point x different from the double points of D. Let y be a double point of D, and α a simple arc in the disk D joining x and y not meeting the other double points of D. Let d_x be a disk neighborhood of x in the 2-sphere S, and d_y a disk neighborhood of y in D, regarding the disks d_x and d_y as disk fibers of a normal disk bundle over D in $\mathbf{R}^4$. Let V_α be a disk bundle over the arc α in $\mathbf{R}^4$ such that $(D \cup S) \cap V_\alpha = d_y \cup \alpha \cup d_x$. Then an immersed disk D_1 with $\partial D_1 = \partial D$ is constructed from the immersed disk D so that

$$D_1 = \mathrm{cl}(D \setminus d_y) \cup \mathrm{cl}(\partial V_\alpha \setminus (d_y \cup d_x)) \cup \mathrm{cl}(S \setminus d_x).$$

The number of the double points of D_1 is smaller than the number of the double points of D by 1.

The 2-sphere S in Finger Move Canceling is called a canceling sphere. If there is a canceling sphere S, then the immersed disk D is changed into an embedded disk D_* by Finger Move Canceling operations of parallel canceling spheres of S. By using Finger Move Canceling, the proof of Recovery Lemma is done as follows:

Proof of Recovery Lemma. By assumption, the immersed O2-handle pair $(D \times I, D' \times I)$ on a surface-link F in $\mathbf{R}^4$ has $D \times I$ as an immersed 2-handle on F and $D' \times I$ as an embedded 2-handle on F. Let d' be a small disk neighborhood of the intersection point q of ∂D and $\partial D'$ in D' (cf. Figure 1). By shrinking $D' \times I$ as $d' \times I$, one finds a trivial S^2-knot S in $\mathbf{R}^4$ such that S meets the immersed core disk D of $D \times I$ at just one point x different from the double points of D and is disjoint from F and $D' \times I$. In fact, regard $d' \times I$ as a thin thickening of a boundary arc $q \times I$ of D and then take a small trivial S^2-knot S near $q \times I$ meeting D at just one point x which can be constructed from $\partial(d' \times I)$. This 2-sphere S is used for a canceling sphere for the immersed disk D. By Finger Move Canceling, the immersed disk D is changed into an immersed disk with double point number 1 less. By continuing this procedure, the immersed disk D is finally changed into an embedded disk D_*, meaning that the pair $(D_* \times I, D' \times I)$ is an O2-handle pair on F. This completes the proof of Recovery Lemma. □

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Conflict of interest

The author declares no competing financial interest.

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