Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Local Convergence of Parameter Based Derivative Free Continuation Method for the Solution of Non-linear Equations

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Received: 12 January 2023; **Revised:** 27 February 2023; **Accepted:** 3 March 2023

Abstract: This paper's major purpose is to evaluate the local convergence of the parameter-based sixth- and seventh-order continuation iterative approach for solving nonlinear equations in R. This analysis assumes that the Fréchet derivative of the first order satisfies the Lipschitz continuity condition. Under these circumstances, we explore convergence analysis in order to investigate the existence and uniqueness region for the solution of our proposed strategies. Thus, we also offered the theoretical concept of the radii of convergence balls for the proposed approach. By determining the radii of the convergence balls and solving many numerical problems, we can verify the significance of our convergence study.

***Keywords*:** non-linear equations, Lipschitz continuity, Fréchet derivative, local convergence

MSC: 65H05, 65H10

1. Introduction

Our main objective of this research is to consider the problem of approximating a solution ϱ^* of the equation

$$\varphi(\varrho) = 0 \tag{1}$$

where $\varphi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a non-linear operator on an open convex subset Ω. There are numerous difficult issues in physics, numerical analysis, engineering, and applied mathematics that involve finding the roots of an equation (1) for solving such problems, we frequently use iterative methods. In literature, we find many higher order iterative methods. Beginning with an initial approximation of a solution ϱ^* of the equation (1), a sequence $\{\varrho_{\alpha,n}\}$ of approximations is generated such that $\{|\varrho_{\alpha,n} - \varrho^*|\}$ decreases and a better approximation to the solution ϱ^* is obtained at every step. There are numerous iterative approaches to solving (1). The most commonly used approach is the quadratically convergent Newton's method, which is provided by

DOI: https://doi.org/10.37256/cm.4120232338

$$\varrho_{\alpha,n+1} = \varrho_{\alpha,n} - \left(\varphi'(\varrho_{\alpha,n})\right)^{-1} \varphi(\varrho_{\alpha,n}) \quad n = 0,1,2,3,.... \tag{2}$$

where, $\varrho_{\alpha,0}$ is the initial point.

The Newton method and the Newton-like method are well-known iterative techniques for solving nonlinear equations. These techniques quickly converge from any sufficient initial guess. If you can see in the literature, Kantorovich [1] and Rall [2] established the convergence analysis of Newton's technique in a Banach space. Many researchers discussed three types of convergence analysis to prove the convergence of iterative methods those are semi-local convergence, local-convergence, global convergence. Several researchers have generalized and produced local and semilocal convergence analyses of the Newton method Equation (2) under various conditions i.e., lipschitz, Holder, and ω continuity conditions. Various researchers examined the local convergence analysis of several Chebyshev-Halley type schemes, including improved Chebyshev-Halley type methods. In a Banach space, the study of local convergence of iterative methods of higher order can be analyzed under various continuity assumptions. Semilocal convergence of continuation method between chebyshev and Halley method under different continuity conditions discussed in [3-5]. The local convergence of iterative methods of higher order mentioned in [6-12]. In an iterative method the radii of convergence ball plays an important role because choosing initial guess for iterative method is not so easy. The global convergence is of many methods discussed in [13, 14].

Inspired by ongoing research, we design the four-step sixth and seventh order continuation method between Halley and super-Halley for solving non linear equations and discuss the local-convergence under Lipschitz continuity condition. This manuscript divided into four sections.

In section 1, Introduction formed. In Section 2, we present a new continuation approach for solving the nonlinear equation (1) and demonstrate convergence theory of the proposed method. Several numerical examples are examined in Section 3 to validate the theoretical results. lastly, the concluding observations discussed in Section 4.

2. Local convergence

This section describes the local convergence analysis of the proposed iterative method. Consider the proposed family of iterative methods as follows,

$$\begin{cases} \varsigma_{\alpha,n} = \varrho_{\alpha,n} - \varphi(\varrho_{\alpha,n})\left(\varphi'(\varrho_{\alpha,n})\right)^{-1}, \\ \tau_{\alpha,n} = \varrho_{\alpha,n} - \Big[\left(\varphi(\varrho_{\alpha,n}) - \varphi(\varsigma_{\alpha,n})\right)\left(\varphi(\varrho_{\alpha,n}) - 2\varphi(\varsigma_{\alpha,n})\right)^{-1} \\ \qquad -\left(\alpha\varphi(\varsigma_{\alpha,n})^2\left(\varphi(\varrho_{\alpha,n}) - \varphi(\varsigma_{\alpha,n})\right)^{-1}\left(\varphi(\varrho_{\alpha,n}) - 2\varphi(\varsigma_{\alpha,n})\right)^{-1}\right)\Big] \\ \qquad \varphi(\varrho_{\alpha,n})\left(\varphi'(\varrho_{\alpha,n})\right)^{-1}, \\ \varpi_{\alpha,n} = \tau_{\alpha,n} - \varphi(\tau_{\alpha,n})(\varrho_{\alpha,n} - \tau_{\alpha,n})(\varrho_{\alpha,n} - \varsigma_{\alpha,n})^2(\varsigma_{\alpha,n} - \tau_{\alpha,n}) \\ \qquad \left(A(\varphi(\varrho_{\alpha,n}) + B\varphi(\varsigma_{\alpha,n}) + C\varphi(\tau_{\alpha,n}) + D\varphi'(\varrho_{\alpha,n})\right)^{-1}, \\ \varrho_{\alpha,n+1} = \varpi_{\alpha,n} - \varphi(\varpi_{\alpha,n})\left(\varphi'(\varrho_{\alpha,n})\right)^{-1}. \end{cases} \tag{3}$$

Theorem 1 Let $L_0 > 0$, $L > 0$, $M > 0$, $\alpha \in S$ and $\varphi : \Omega \subset \mathbb{R} \to \mathbb{R}$ be the given parameters and differentiable function respectively. Suppose, there exists $\varrho^* \in \Omega$ such that the following holds true for every ϱ, $\varsigma \in \Omega$:

$$\varphi(\varrho^*) = 0, \; \varphi'\left(\varrho^*\right)^{-1} \in L(S,S), \tag{4}$$

$$|\alpha| ML < 1, \tag{5}$$

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho)-\varphi'(\varrho^*))\right| \le L_0\left|\varrho-\varrho^*\right|, \tag{6}$$

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho)-\varphi'(\varsigma))\right| \le L\left|\varrho-\varsigma\right|, \tag{7}$$

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho))\right| \le M, \tag{8}$$

$$\bar{U}(\varrho^*, r) \subseteq \Omega, \tag{9}$$

where r is given by $r = \min\{r_1, r_4, r_{14}, r_{15}\}$. Then by using the method (3) for $\varrho_{\alpha,0} \in U(\varrho^*, r) \mid \{\varrho^*\}$ the sequence $\{\varrho_{\alpha,n}\}$ is well defined, remains in $\bar{U}(\varrho^*, r)$ for each $n = 0, 1, 2, 3, \ldots$ converges to ϱ^*. In addition, the following estimates hold true.

$$\left|\varsigma_{\alpha,n}-\varrho^*\right| \le g_1\left(\left|\varrho_{\alpha,n}-\varrho^*\right|\right)\left|\varrho_{\alpha,n}-\varrho^*\right| < \left|\varrho_{\alpha,n}-\varrho^*\right| < r, \tag{10}$$

$$\left|\tau_{\alpha,n}-\varrho^*\right| \le g_4\left(\left|\varrho_{\alpha,n}-\varrho^*\right|\right)\left|\varrho_{\alpha,n}-\varrho^*\right| < \left|\varrho_{\alpha,n}-\varrho^*\right| < r, \tag{11}$$

$$\left|\varpi_{\alpha,n}-\varrho^*\right| \le g_{14}\left(\left|\varrho_{\alpha,n}-\varrho^*\right|\right)\left|\varrho_{\alpha,n}-\varrho^*\right| < \left|\varrho_{\alpha,n}-\varrho^*\right| < r, \tag{12}$$

$$\left|\varrho_{\alpha,n+1}-\varrho^*\right| \le g_{15}\left(\left|\varrho_{\alpha,n}-\varrho^*\right|\right)\left|\varrho_{\alpha,n}-\varrho^*\right| < \left|\varrho_{\alpha,n}-\varrho^*\right| < r, \tag{13}$$

where the g functions are defined above. Furthermore, there exist $R \in \left[r, \frac{2}{L_0}\right)$ such that $\bar{U}(\varrho^*, r) \subseteq \Omega$, then the limit point ϱ^* is the only solution of (1) in $\bar{U}(\varrho^*, r)$.

Proof. Let, $\varphi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose ϱ^* is the solution that belongs to the domain Ω. That is $\varrho^* \in \Omega$. $\varrho_{\alpha,0}$ is the initial-point in the domain Ω. From the condition (6)

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho)-\varphi'(\varrho^*))\right| \le L_0\left|\varrho-\varrho^*\right|.$$

and, $\varrho_{\alpha,0} \in \Omega$, we get

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho_{\alpha,0})-\varphi'(\varrho^*))\right| \le L_0\left|\varrho_{\alpha,0}-\varrho^*\right|. \tag{14}$$

For $\left|\varrho_{\alpha,0}-\varrho^*\right| < \frac{1}{L_0}$, gives

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho_{\alpha,0})-\varphi'(\varrho^*))\right| < 1,$$

$$\left|\varphi'\left(\varrho^*\right)^{-1}(\varphi'(\varrho_{\alpha,0})-\varphi'(\varrho^*))\right|<1,$$

$$\left|\varphi'\left(\varrho^*\right)^{-1}\varphi'(\varrho_{\alpha,0})-\varphi'\left(\varrho^*\right)^{-1}\varphi'(\varrho^*)\right|<1,$$

and,

$$\left|I-\varphi'\left(\varrho^*\right)^{-1}\varphi'(\varrho_{\alpha,0})\right|<1. \tag{15}$$

$$\left|I-\varphi'\left(\varrho^*\right)^{-1}\varphi'(\varrho_{\alpha,0})\right|<1. \tag{16}$$

Then, $(\varphi'(\varrho_{\alpha,0}))^{-1}$ exists, and

$$\left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi'(x)^*\right|=\frac{1}{1-L_0\left|\varrho_{\alpha,0}-\varrho^*\right|} \tag{17}$$

From the first step of the method (3), for $n=0$, we get

$$\begin{aligned}\varsigma_{\alpha,0}-\varrho^* &= \varrho_{\alpha,0}-\varrho^*-(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\\ &=(\varrho_{\alpha,0}-\varrho^*)I-(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\\ &=(\varrho_{\alpha,0}-\varrho^*)\left[(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi'(\varrho_{\alpha,0})\right]-(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0}).\end{aligned}$$

From this, we get

$$\left|\varsigma_{\alpha,0}-\varrho^*\right|=\left|(\varphi'(\varrho_{\alpha,0}))^{-1}\int_0^1\left[\varphi'(\varrho^*+\theta(\varrho_{\alpha,0}-\varrho^*))-\varphi'(\varrho_{\alpha,0})\right]d\theta(\varrho_{\alpha,0}-\varrho^*)\right|$$

$$\leq\left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi'(\varrho^*)\right|\left|(\varrho_{\alpha,0}-\varrho^*))\right| \tag{18}$$

$$\left|\int_0^1(\varphi'(\varrho^*))^{-1}\left[\varphi'(\varrho^*+\theta(\varrho_{\alpha,0}-\varrho^*))-\varphi'(\varrho_{\alpha,0})\ d\theta\right]\right|. \tag{19}$$

Hence,

$$\left|\varsigma_{\alpha,0}-\varrho^*\right|\leq\ \frac{L\left|(\varrho_{\alpha,0}-\varrho^*)\right|^2}{2\left(1-L_0\left|(\varrho_{\alpha,0}-\varrho^*)\right|\right)} \tag{20}$$

This gives,

$$\left|\varsigma_{\alpha,0}-\varrho^{*}\right| \leq g_1\left(\left|(\varrho_{\alpha,0}-\varrho^{*})\right|\right) \leq r \tag{21}$$

For $n = 0$, we conclude that, $\varsigma_{\alpha,0} \in U(\varrho^{*}, r)$, where

$$g_1(t) = \frac{Lt}{2(1-L_0 t)}. \tag{22}$$

then, $h_1(t) = g_1(t) - 1$, with $h_1(0) = -1$, $h_1\left(\frac{1}{L_0}\right) \to \infty$. According to the Intermediate Value Theorem (IVT), h_1 has the smallest zero r_1 in the interval $(0, 1/L_0)$. The result is $0 < r_1 < \frac{1}{L_0}$ and $0 \leq g_1(t) < 1 \forall t \in (0, r_1)$. Now, consider second step of method (3) for $n = 0$,

$$\tau_{\alpha,0} = \varrho_{\alpha,0} - \left[\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}\right) + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-2\varphi(\varsigma_{\alpha,0})}\right)\right]\frac{\varphi(\varrho_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}. \tag{23}$$

We get,

$$\begin{aligned}
|1-PT| &= \left|\left(\varphi'(\varrho^{*})\right)^{-1}(\varrho_{\alpha,0}-\varrho^{*})^{-1}\varphi'(\varrho^{*})(\varrho_{\alpha,0}-\varrho^{*})\right. \\
&\quad \left. -\left(\varphi'(\varrho^{*})\right)^{-1}\left(\varrho_{\alpha,0}-\varrho^{*}\right)^{-1}(\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0}))\right| \\
&\leq \frac{1}{\left|(\varrho_{\alpha,0}-\varrho^{*})\right|} L_0\left|\varrho_{\alpha,0}-\varrho^{*}\right|\int_0^1 \theta d\theta\left|\varrho_{\alpha,0}-\varrho^{*}\right| \\
&\quad + Mg_1\left(\left|\varrho_{\alpha,0}-\varrho\right|^{*}\right)\left|\varrho_{\alpha,0}-\varrho^{*}\right| \\
&\leq L_0\left|\varrho_{\alpha,0}-\varrho^{*}\right|\frac{1}{2} + Mg_1\left(\left|\varrho_{\alpha,0}-\varrho^{*}\right|\right)\left|\varrho_{\alpha,0}-\varrho^{*}\right| \\
&= \frac{L_0\left|\varrho_{\alpha,0}-\varrho^{*}\right|}{2} + Mg_1\left(\left|\varrho_{\alpha,0}-\varrho^{*}\right|\right)\left|\varrho_{\alpha,0}-\varrho^{*}\right| \\
&= \frac{L_0 t}{2} + Mg_1(t)t \\
&= g_2(t) \\
&< g_2(r) \\
&< 1,
\end{aligned}$$

Hence, we get

$$\left|\left(\varphi'(\varrho^*)\right)^{-1}\varphi(\varsigma_{\alpha,0})\right| \le M\left|\varsigma_{\alpha,0}-\varrho^*\right| \le Mg_1\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\left|\varrho_{\alpha,0}-\varrho^*\right|, \tag{24}$$

where,

$$g_2(t) = \frac{L_0 t}{2} + Mg_1(t)t \tag{25}$$

$$\left|(\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0}))^{-1}\varphi'(\varrho^*)\right| \le \frac{1}{\left|\varrho_{\alpha,0}-\varrho^*\right|\left(1-g_2\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}. \tag{26}$$

$$\left|\left(\varphi'(\varrho^*)\right)^{-1}2\varphi(\varsigma_{\alpha,0})\right| \le 2M\left|\varsigma_{\alpha,0}-\varrho^*\right| \le 2Mg_1\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\left|\varrho_{\alpha,0}-\varrho^*\right|, \tag{27}$$

and

$$g_3 = \frac{L_0 t}{2} + 2Mg_1(t)t. \tag{28}$$

$$\left|(\varphi(\varrho_{\alpha,0})-2\varphi(\varsigma_{\alpha,0}))^{-1}\varphi'(\varrho^*)\right| \le \frac{1}{\left|\varrho_{\alpha,0}-\varrho^*\right|\left(1-g_3\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}. \tag{29}$$

$$\tau_{\alpha,0}-\varrho^* = (\varrho_{\alpha,0}-\varrho^*) - \left[\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}\right) + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-2\varphi(\varsigma_{\alpha,0})}\right)\right]\frac{\varphi(\varrho_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}.$$

$$\left|\tau_{\alpha,0}-\varrho^*\right| = \left|(\varrho_{\alpha,0}-\varrho^*) - \left[\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}\right)\right.\right.$$

$$\left.\left. + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0})-2\varphi(\varsigma_{\alpha,0})}\right)\right]\frac{\varphi(\varrho_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}\right|$$

$$= \left[1 + \left(|\alpha|\, M\frac{1}{1-g_2\left|\varrho_{\alpha,0}-\varrho^*\right|} + \left|1-\alpha\right| M\left(\frac{1-g_1\left|\varrho_{\alpha,0}-\varrho^*\right|}{1-g_3\left|\varrho_{\alpha,0}-\varrho^*\right|}\right)\right)\right.$$

$$\left.\frac{M}{1-L_0\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}\right]\left|\varrho_{\alpha,0}-\varrho^*\right|$$

$$= g_4\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)\left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< \left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< r.$$

Since $|\tau_{\alpha,0} - \varrho^*| < r$, so for $n = 0$, and $\tau_{\alpha,0} \in U(\varrho^*, r)$, where

$$g_4(t) = 1 + \left(\frac{|\alpha| M}{1 - g_2(t)} + |1-\alpha|\ M\left(\frac{1 - g_1(t)}{1 - g_3(t)}\right)\right)\frac{M}{1 - L_0 t}. \tag{30}$$

then, $h_4(t) = g_4(t) - 1$, with $h_4(0) =$, $h_4(r_1) > 0$. According to the Intermediate Value Theorem, h_4 has the smallest zero r_4 in the interval $(0, r_1)$. The result is $0 < r < r_4 < r_1$ and $0 \leq g_4(t) < 1 \forall t \in (0, r_4)$. From step three of the method (3),

$$\varpi_{\alpha,n} = \tau_{\alpha,n} - \varphi(\tau_{\alpha,n})\frac{(\varrho_{\alpha,n} - \tau_{\alpha,n})(\varrho_{\alpha,n} - \varsigma_{\alpha,n})^2(\varsigma_{\alpha,n} - \tau_{\alpha,n})}{A(\varphi(\varrho_{\alpha,n}) + B\varphi(\varsigma_{\alpha,n}) + C\varphi(\tau_{\alpha,n}) + D\varphi'(\varrho_{\alpha,n})}, \tag{31}$$

$$\left|\varrho_{\alpha,0} - \tau_{\alpha,0}\right| = \left|\left[\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}\right) + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - 2\varphi(\varsigma_{\alpha,0})}\right)\right]\frac{\varphi(\varrho_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}\right|$$

$$= \left[|\alpha| M \frac{1}{\left(1 - g_2\left|\varrho_{\alpha,0} - \varrho^*\right|\right)} + \left|1-\alpha\right| M \frac{1 - g_1\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)}{1 - g_3\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)}\right]$$

$$\frac{1}{1 - L_0\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)} M\left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$= g_5\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)$$

$$< g_5(r)$$

$$< 1,$$

where

$$g_5(t) = \left[|\alpha| M \frac{1}{(1 - g_2(t))} + |1-\alpha| M\left(\frac{1 - g_1(t)}{1 - g_3(t)}\right)\right]\frac{Mt}{1 - L_0(t)}. \tag{32}$$

$$\left|\varrho_{\alpha,0} - \varsigma_{\alpha,0}\right| = \left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\right|$$

$$= g_6\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)$$

$$< g_6(r)$$

$$< 1, \tag{33}$$

where,

$$g_6(t) = \frac{Mt}{1 - L_0(t)}. \tag{34}$$

$$\left|\varsigma_{\alpha,0} - \tau_{\alpha,0}\right| = \left[\left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\left[\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}\right)\right.\right.\right.$$

$$\left.\left.\left. + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - 2\varphi(\varsigma_{\alpha,0})}\right) - 1\right]\right|\right]$$

$$= \frac{M\left|\varrho_{\alpha,0} - \varrho^*\right|}{1 - L_0\left|\varrho_{\alpha,0} - \varrho^*\right|}\left[\frac{|\alpha|M}{1 - g_2\left|\varrho_{\alpha,0} - \varrho^*\right|} + |1-\alpha|\, M\frac{1 - g_1\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)}{1 - g_3\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)} + 1\right]$$

$$= g_7\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)$$

$$< g_7(r)$$

$$< 1,$$

where

$$g_7(t) = \frac{Mt}{1 - L_0 t}\left[|\alpha|M\frac{1}{1 - g_2(t)} + |1-\alpha|\, M\frac{1 - g_1(t)}{1 - g_3(t)} + 1\right]. \tag{35}$$

$$\left|3\varrho_{\alpha,0} - 2\varsigma_{\alpha,0} - \tau_{\alpha,0}\right| = \left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\alpha\left(\frac{\varphi(\varrho_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}\right)\right.$$

$$\left. + (1-\alpha)\left(\frac{\varphi(\varrho_{\alpha,0}) - \varphi(\varsigma_{\alpha,0})}{\varphi(\varrho_{\alpha,0}) - 2\varphi(\varsigma_{\alpha,0})}\right) + 2\right|$$

$$= \frac{M\left|\varrho_{\alpha,0}-\varrho^*\right|}{1-L_0\left|\varrho_{\alpha,0}-\varrho^*\right|}\left[|\alpha|M\frac{1}{1-g_2\left|m\varrho_{\alpha,0}-\varrho^*\right|}\right.$$

$$\left.+|1-\alpha|\ M\frac{1-g_1\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}{1-g_3\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}+2\right]$$

$$= g_8\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)$$

$$< g_8(r)$$

$$< 1,$$

where

$$g_8(t)=\frac{Mt}{1-L_0t}\left[|\alpha|M\frac{1}{1-g_2(t)}+|1-\alpha|\ M\frac{1-g_1(t)}{1-g_3(t)}+2\right]. \tag{36}$$

$$\left|\varrho_{\alpha,0}+2\varsigma_{\alpha,0}-3\tau_{\alpha,0}\right|\le\left|(\varphi'(\varrho_{\alpha,0}))^{-1}\varphi(\varrho_{\alpha,0})\right|\left[2+3|\alpha|\left|\left(\varphi'(\varrho^*)\right)^{-1}\varphi(\varrho_{\alpha,0})\right|\right.$$

$$\left|\varphi'(\varrho^*)\left(\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0})\right)^{-1}\right|+3\left|(\varphi(\varrho_{\alpha,0})-\varphi(\varsigma_{\alpha,0}))\left(\varphi'(\varrho^*)\right)^{-1}\right|$$

$$\left.|1-\alpha|\left|\left(\varphi(\varrho_{\alpha,0})-2\varphi(\varsigma_{\alpha,0})\right)^{-1}\varphi'(\varrho^*)\right|\right]$$

$$=\frac{1}{1-L_0\left|\varrho_{\alpha,0}-\varrho^*\right|}M\left|\varrho_{\alpha,0}-\varrho^*\right|\left[2+3|\alpha|M\frac{1}{\left(1-g_2\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}+\right.$$

$$\left.3|1-\alpha|\ M\frac{1-g_1\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}{1-g_3\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}\right]$$

$$= g_9\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)$$

$$= g_9(t)$$

$$< g_9(r)$$

$$< 1,$$

where

$$g_9(t)=\frac{Mt}{1-L_0t}\left[3\,|\,\alpha|\;M\,\frac{1}{1-g_2(t)}+3\,|\,1-\alpha|\;M\,\frac{1-g_1(t)}{1-g_3(t)}+2\right]. \tag{37}$$

$$\begin{aligned}|A|&=\left|(\varsigma_{\alpha,0}-\tau_{\alpha,0})^2(3\varrho_{\alpha,0}-2\varsigma_{\alpha,0}-\tau_{\alpha,0})\right|\\&\le(g_7(t))^2(g_8(t))\\&=g_{10}(t),\end{aligned}$$

where

$$g_{10}(t)=(g_7(t))^2(g_8(t)), \tag{38}$$

where

$$g_{11}(t)=(g_6(t))^2g_9(t). \tag{39}$$

$$g_{12}(t)=(g_7(t))^2g_5(t)g_6(t)). \tag{40}$$

$$\varpi_{\alpha,0}-\varrho^*=\tau_{\alpha,0}-\varrho^*-\varphi(\tau_{\alpha,0})(\varrho_{\alpha,0}-\tau_{\alpha,0})(\varrho_{\alpha,0}-\varsigma_{\alpha,0})^2(\varsigma_{\alpha,0}-\tau_{\alpha,0})$$
$$\left(A(\varphi(\varrho_{\alpha,0})+B\varphi(\varsigma_{\alpha,0})+C\varphi(\tau_{\alpha,0})+D\varphi'(\varrho_{\alpha,0})\right)^{-1}.$$

$$g_{13}(t)=\frac{L_0(t)}{2}g_{10}(t)+(g_5(t))^3Mg_1(t)t+g_{11}(t)Mg_4(t)+g_{12}(t)M. \tag{41}$$

$$\left|\left(A\varphi(\varrho_{\alpha,0})+B\varphi(\varsigma_{\alpha,0})+C\varphi(\tau_{\alpha,0})+D\varphi'(\varrho_{\alpha,0})\right)^{-1}\varphi'(\varrho^*)\right|\le\frac{1}{\left|\varrho_{\alpha,0}-\varrho^*\right|\left(1-g_{13}\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}. \tag{42}$$

$$\begin{aligned}\left|\varpi_{\alpha,0}-\varrho^*\right|&=\left|\tau_{\alpha,0}-\varrho^*-\varphi(\tau_{\alpha,0})(\varrho_{\alpha,0}-\tau_{\alpha,0})(\varrho_{\alpha,0}-\varsigma_{\alpha,0})^2(\varsigma_{\alpha,0}-\tau_{\alpha,0})\right|\\&\quad\left|\left(A(\varphi(\varrho_{\alpha,0})+B\varphi(\varsigma_{\alpha,0})+C\varphi(\tau_{\alpha,0})+D\varphi'(\varrho_{\alpha,0})\right)^{-1}\right|\\&\le g_4\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\left|\varrho_{\alpha,0}-\varrho^*\right|\left[1+g_5\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\left(g_6\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\right)^2\right.\\&\quad\left.g_7\left(\left|\varrho_{\alpha,0}-\varrho^*\right|\right)\frac{M}{\left|\varrho_{\alpha,0}-\varrho^*\right|\left(1-g_{13}\left|\varrho_{\alpha,0}-\varrho^*\right|\right)}\right]\end{aligned}$$

$$= g_{14}\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)\left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< \left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< r.$$

$$g_{14}(t) = g_4(t)\left[1 + g_5(t)\left(g_6\right)^2 g_7(t) M \frac{1}{t(1 - g_{13}(t))}\right]. \tag{43}$$

then, $h_{14}(t) = g_{14}(t) - 1$, with $h_{14}(0) = -1$, $h_{14}(r_4) > 0$. Then we say that the function h_{14} have smallest zero r_{14} in the interval $(0, r_4)$ by the intermediate value theorem . Then we get $0 < r < r_{14} < r_4 < r_1$ and $0 \leq g_{14}(t) < 1 \forall t \in (0, r_{14})$

$$\left|x_1 - \varrho^*\right| = \left|\varpi_{\alpha,0} - \varrho^* - \frac{\varphi(\varpi_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}\right|$$

$$\leq \left|\varpi_{\alpha,0} - \varrho^*\right| + \left|\frac{\varphi(\varpi_{\alpha,0})}{\varphi'(\varrho_{\alpha,0})}\right|$$

$$\leq g_{14}\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)\left|\varrho_{\alpha,0} - \varrho^*\right|\left[1 + M + \frac{1}{1 - L_0\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)}\right]$$

$$= g_{15}\left(\left|\varrho_{\alpha,0} - \varrho^*\right|\right)\left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< \left|\varrho_{\alpha,0} - \varrho^*\right|$$

$$< r,$$

where

$$g_{15}(t) = g_{14}(t)\left[1 + M \frac{1}{1 - L_0(t)}\right]. \tag{44}$$

then, $h_{15}(t) = g_{15}(t) - 1$, with $h_{15}(0) = -1$, $h_{15}(r_{14}) > 0$. According to the Intermediate Value Theorem, h_{15} has the smallest zero r_{15} in the interval $(0, 1/L_0)$. The result is $0 < r_{15} < r_{14} < r_4 < r_1$ and $0 \leq g_{15}(t) < 1 \forall t \in (0, r)$. This conclude that for $n = 0$, and $x_1 \in U(\varrho^*, r)$.

Now, we will prove the uniqueness part.

Suppose, there is another solution ς^*. Since, ς^* is a solution, that implies $\varphi(\varsigma^*) = 0$. Let $\varsigma^* \in U(\varrho^*, r)$, that implies $|\varrho^* - \varsigma^*| < r$, and $\varsigma^* \neq \varrho^*$.

Let us consider

$$T = \int_0^1 \varphi'(\varsigma^* + \theta(\varrho^* - \varsigma^*))d\theta$$

$$= \varphi(\varrho^*) - \varphi(\varsigma^*)$$

$$= 0 - 0$$

$$= 0.$$

But T can never be equal to 0, which contradicts our hypothesis. That implies $\varrho^* = \varsigma^*$. □

3. Numerical examples

In this section, numerical examples are provided to illustrate the effectiveness of the method and analysis under consideration. All numerical examples were executed by Mathematica 11.3.

Example 1 Let $\kappa = \mathbb{R}$, $\Omega = [-1, 1]$, $\varrho^* = 0$ and φ be the function defined on Ω by $\varphi(\varrho) = \sin(\varrho)$.

For $\alpha = 0.75$, we get $L_0 = 1$, $L = 1$, $M = 1$. Then, by using the "g" functions, we obtain $r_1 = 0.666667$, $r_4 = 0.569514$, $r_{14} = 0.568376$, $r_{15} = 0.249122$. This gives, $r = \min(r_1, r_4, r_{14}, r_{15}) = 0.249122$.

Table 1. Radius of convergence for $\alpha = 0.75$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.75	0.666667	0.569514	0.568376	0.249122	0.249122

Therefore, from Table 1 we can assure the convergence of the proposed strategy with $\alpha = 0.75$ by using the Theorem 1.

Example 2 Let $\kappa = \mathbb{R}$, $\Omega = [-1, 1]$, $\varrho^* = 0$ and φ be the function defined on Ω by $\varphi(\varrho) = e^{\varrho} - 1$.

Then, for $\alpha = 0.125$, we get $L_0 = e - 1$, $L = e$, $M = e$. Then, by using the "g" functions, we get $r_1 = 0.382692$, $r_4 = 0.366058$, $r_{14} = 0.0883582$, $r_{15} = 0.0874554$. Hence, we get $r = \min(r_1, r_4, r_{14}, r_{15}) = 0.0874554$.

Table 2. Radius of convergence for $\alpha = 0.125$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.125	0.382692	0.3660588	0.0883582	0.0874554	0.0874554

Therefore, from Table 2 we can assure the convergence of the proposed strategy with $\alpha = 0.125$ by using the Theorem 1.

Example 3 Let, $X = Y = R$. Define φ on $\Omega = [1, 3]$ by

$$\varphi(\varrho) = \frac{2}{3}\varrho^{\frac{3}{2}} - \varrho.$$

Then, $\varrho^* = \frac{9}{4}$, $\varphi'(\varrho^*)^{-1} = 2$, $L_0 = L = 1$ and, $\alpha = 0.9870$. Then, by using the "g" functions, we obtain $r_1 = 0.666667$, $r_4 = 0.457978$, $r_{14} = 0.0846284$, $r_{15} = 0.0839528$. hence, we get $r = \min(r_1, r_4, r_{14}, r_{15}) = 0.0839528$.

Table 3. Radius of convergence for $\alpha = 0.0987$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.09870	0.666667	0.457978	0.0846284	0.0839528	0.0839528

Therefore, from Table 3 we can assure the convergence of the proposed strategy with with $\alpha = 0.9870$ by using the Theorem 1.

Example 4 Let φ be a function defined on $X = Y = \mathbb{R}$, $\Omega = \bar{Y}(0, 1)$ by

$$\varphi(\varrho) = \begin{cases} c_1\varrho^3 ln\varrho^2 + c_2\varrho^5 + c_3\varrho^4, & \varrho \neq 0 \\ 0, & \varrho = 0, \end{cases}$$

where $c_1 \neq 0$, c_2, c_3 are the real parameters. Then, we have

$$\varphi'(\varrho) = 3c_1\varrho^3 ln\varrho^2 + 5c_2\varrho^4 + 4c_3\varrho^3 + 2c_1\varrho^2,$$

$$\varphi''(\varrho) = 6c_1\varrho ln\varrho^2 + 20c_2\varrho^3 + 12c_3\varrho^2 + 10c_1\varrho,$$

and

$$\varphi'''(\upsilon) = 6c_1 ln\varrho^2 + 60c_2\varrho^2 + 24c_3\varrho + 22c_1\varrho.$$

Also, we have

$$L = L_0 = 146.6629073, M = 101.5578008.$$

Choose $\alpha = 0.00006$.

Then, by using the "g" function, we obtain $r_1 = 0.00454557$, $r_4 = 0.00454546$, $r_{14} = 0.0000147588$, $r_{15} = 0.00000147587$.

Thus, we get $r = \min(r_1, r_4, r_{14}, r_{15}) = 0.00000147587$.

Table 4. Radius of convergence for $\alpha = 0.00006$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.00006	0.00454557	0.00454546	0.0000147588	0.00000147587	0.00000147587

Therefore, from Table 4 we can assure the convergence of the proposed strategy with $\alpha = 0.00006$ by using the

Theorem 1.

Example 5 (Continuous Stirred Tank Reactor (CSTR))

Let us consider the isothermal CSTR.

A = Component fed to the reactor at the rate Q.

R = Component fed to the reactor at the rate q-Q.

Then we obtain the following reaction schemes in the reactor

$A + R \rightarrow B$

$B + R \rightarrow C$

$C + R \rightarrow D$

$D + R \rightarrow E$

Douglas designed the simple expression for transferring function of the reactor

$$\rho_c \frac{2.98(\varrho + 2.25)}{(\varrho + 1.45)(\varrho + 2.85)^2(\varrho + 4.35)} = -1 \tag{45}$$

Where,

ρ_c = The gain of the proportional controller.

The control system is stable for values of ρ_c that yields roots of the transfer function having negative real part. We will get poles of the open-loop transfer function as the roots of the non-linear equation if we will choose $\rho_c = 0$.

The non-linear equation is

$$\varphi(\varrho) = \varrho^4 + 11.50\varrho^3 + 47.49\varrho^2 + 83.06325\varrho + 51.23266875 \tag{46}$$

The function has 4 approximate roots $\varrho^* = -1.45, -2.85, -2.28, -4.35$. We are choosing -4.35 as the approximate root.

Let us consider, $\Omega = [-4.5, -4]$. Then, we obtain $L = L_0 = 2.760568793$, $M = 2$.

Table 5. Radius of convergence for $\alpha = 0.3$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.3	0.241496	0.228665	0.228298	0.228297	0.228297

Therefore, from Table 5 we can assure the convergence of the proposed strategy with $\alpha = 0.3$ by using the Theorem 1.

Example 6 In the study of the multi-factor effect, the trajectory of an electron in the air gap between two parallel plates is given by:

$$\varrho(s) = \varrho_0 + \left(\lambda_0 + e\frac{E_0}{m\omega}\sin(\omega s_0 + \mu)\right)(s - s_0) + e\frac{E_0}{m(\omega)^2}\left(\cos(\omega s + \mu)) + \sin(\omega s_0 + \mu)\right) \tag{47}$$

Where,

m = Mass of the electron at rest.

e = Charge of the electron at rest.

ϱ_0 = Position of the electron at time s_0.

λ_0 = Velocity of the electron at time s_0.

$E_0 \sin(\omega s + \mu)$ = Radio Frequency (RF) electric field between the plates.

We choose the particulars parameters in the expression (47) in order to get the simpler form which is defined as

$$\varphi(\varrho) = \varrho - \frac{\cos(\varrho)}{2} + \frac{\pi}{4} \tag{48}$$

The required root of the function is $\varrho^* = -0.30909327154179$.
Then we have $L_0 = 1.523542095$,
$L = 1.523542095$,
$M = 1.523542095$.
Choose $\alpha = 0.3$, then by using the "g" function, we obtain

Table 6. Radius of convergence for $\alpha = 0.3$ and 0.5

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.3	0.437577	0.39003	0.388295	0.388294	0.388294
0.5	0.437577	0.381151	0.380218	0.126173	0.126173

Therefore, from Table 6 we can assure the convergence of the proposed strategy with $\alpha = 0.3$, $\alpha = 0.5$ by using the Theorem 1.

Example 7 (Kepler Equation)

Consider the Kepler equation $f: D \subseteq R \rightarrow R$ defined by

$$f(\varrho) = \lambda_1 \varrho - \lambda_2 \sin(\varrho) - \lambda_3, \tag{49}$$

where, $\lambda_1 = 1$, $0 \leq \lambda_2 \leq \pi$, and $0 \leq \lambda_3 \leq \pi$.The parameters involve for evaluating the radius of convergence ball are given by $L = L_0 = \dfrac{\lambda_3}{\left|\lambda_1 - \lambda_2 \cos(\varrho^*)\right|}$. So using theorem 1 we have different radii of convergence.

Table 7. Radius of convergence for $\alpha = 0.5$

λ_1	λ_2	λ_3	ϱ^*	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
1	0.4	0.2	0.329386	2.07168	1.18804	1.17822	0.302895	0.302895
1	0.5	0.3	0.569682	1.28569	0.725293	0.265648	0.261504	0.261504
1	0.6	0.4	0.851271	1.00674	0.752373	0.711516	0.708659	0.708659
1	0.7	0.5	1.134395	0.938831	0.676173	0.675523	0.672947	0.672947
1	0.8	0.6	1.386444	0.948169	0.660695	0.236703	0.233603	0.233603

Therefore, from Table 7 we can assure the convergence of the proposed strategy with $\alpha = 0.5$ by using the Theorem 1.

Example 8 Let us consider a non-linear equation which is defined as, $\varphi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\varrho) = \frac{\varrho^3}{6} + \frac{\varrho^2}{6} - \frac{5\varrho}{6} + \frac{1}{3} \tag{50}$$

The required approximate root of the above equation is $\varrho^* = 0.46259$ and $\varphi'(\varrho^*) = 0.572135$. Then we get, $L_0 = 0.45653$, $L = 0.42385$. So by using theorem 1 we have different radii of convergence

Table 8. Radius of convergence for $\alpha = 0.5$

α	r_1	r_4	r_{14}	r_{15}	$r = \min(r_1, r_4, r_{14}, r_{15})$
0.5	1.49599	0.95156	0.94551	0.272882	0.272882

Therefore, from Table 8 we can assure the convergence of the proposed strategy with $\alpha = 0.5$ by using the Theorem 1.

4. Conclusion

In this research work, we present the local convergence analysis of the of sixth and seventh order derivative free continuation method. Our proposed work will be described only if the first-order fréchet derivative meets the Lipschitz continuity assumption. We have presented the existence and uniqueness for the proposed method given by convergence ball. To check the efficacy of theoretical analysis several numerical examples were carried out.

Conflict of interest

The authors have no conflict of interest.

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