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Research Article

Unified Solution of Some Properties Related to λ-Pseudo Starlike Functions

Musthafa Ibrahim[1*], **K. R. Karthikeyan**[2]

[1]College of Engineering, University of Buraimi, Al Buraimi, Oman
[2]Department of Applied Mathematics and Science, National University of Science & Technology, Muscat P.O. Box 620, Oman
Email: musthafa.i@uob.edu.om

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Abstract: To unify and extend the study of various subclasses of starlike and convex functions, here we introduce a new subclass of λ-pseudo starlike symmetric functions. To add more versatility to our study, we have defined a new class of functions subordinate to a conic region impacted by the well-known Janowski functions. This study extends well-known results and unifies the studies of various subclasses of α-convex functions. Coefficient estimates of the inverse function and the Fekete-Szegő result for the function class are the main results. Some interesting special cases of our main results are also presented here.

Keywords: analytic functions, bi-univalent, Fekete-Szegő, coefficient inequalities, starlike functions and convex functions, subordination

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1. Introduction and motivation

We let $\mathcal{A}$ denote the class of analytic functions defined in $\mathcal{U}$, having a series of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, we let $\mathcal{S}$ to denote the class of all functions in $\mathcal{A}$, which are univalent in $\mathcal{U}$. It is well-known that every function $f \in \mathcal{S}$ has a function f^{-1}, defined by

$$f^{-1}[f(z)] = z; \quad (z \in \mathcal{U})$$

and

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$$f[f^{-1}(w)] = w; \qquad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

Lewin in [1] introduced the so-called class of bi-univalent functions, which consists of functions f analytic in unit disc $\mathcal{U} = \{z; |z| < 1\}$ such that both f and f^{-1} are univalent in $\mathcal{U}$ Here we let $\mathcal{BS}$ to denote the class of bi-univalent functions. Examples of functions belonging to the class $\mathcal{BS}$ include

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = -\log(1-z), \quad f_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \ldots.$$

Figure 1 is the mapping of f_1 and f^{-1}, respectively, if the domain is unit disc. On the other hand, the function $\frac{z}{1-z^2}$ belongs to class $\mathcal{S}$ but does not belong to $\mathcal{BS}$. Recently several researchers introduced and studied various subclasses of bi-univalent functions, see [2-16].

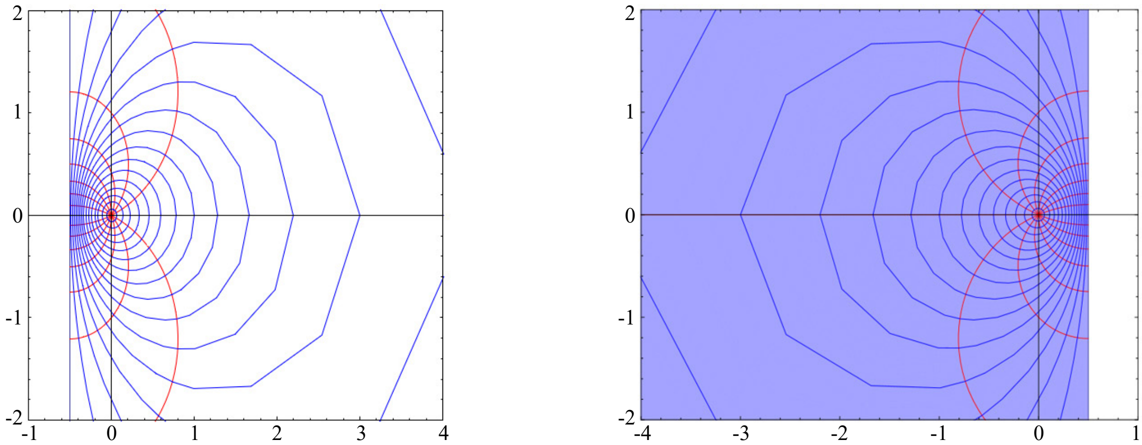

Figure 1. Mapping of $w = \frac{z}{(1-z)}$ and its inverse $z = \frac{w}{(1-w)}$

We let $\mathcal{P}$ to be the class of functions with positive real part which has a power series representation of the form $\ell(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Ma and Minda [17] considered a function $\psi \in \mathcal{P}$ satisfying

(ii) $\psi(0) = 1, \psi'(0) > 0$;

(iii) ψ maps the open unit disc $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Also, they assumed that $\psi(z) = 1 + L_1z + L_2z^2 + \ldots$, with $L_1 > 0$, and introduced the classes:

$$\mathcal{S}^*(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\}$$

and

$$\mathcal{C}(\psi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\}.$$

Well-known special case of $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ which have been dealt with in detail by various researchers, are the so-called *Janowski starlike* functions and *Janowski convex* functions (see [18]) which is defined by

$$\mathcal{S}^*(\Theta_1, \Theta_2) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\Theta_1 z}{1+\Theta_2 z}, -1 \le \Theta_2 < \Theta_1 \le 1 \right\},$$

and

$$\mathcal{C}(\Theta_1, \Theta_2) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+\Theta_1 z}{1+\Theta_2 z}, -1 \le \Theta_2 < \Theta_1 \le 1 \right\}.$$

Here, $\prec$ denotes the subordination of analytic function, refer to any standard text for its definition and properties. For arbitrary fixed numbers $\Theta_1, \Theta_2, -1 < \Theta_1 \le 1, -1 \le \Theta_2 < \Theta_1$, we denote $\aleph(\Theta_1, \Theta_2)$ by the family of functions $p(z) \in \mathcal{P}$ satisfying the condition

$$\ell(z) \prec \frac{(1+\Theta_1)\ell(z) + 1 - \Theta_1}{(1+\Theta_2)\ell(z) + 1 - \Theta_2}.$$

This so-called Janowski class [18] was studied by several authors, see [19-21]. Extending the Janowski class of functions [18], Aouf [19] (equation 4) defined the class $\ell(z) \in \mathcal{P}(\Theta_1, \Theta_2, p, \alpha)$ if and only if

$$\ell(z) = \frac{p + \left[p\Theta_2 + (\Theta_1 - \Theta_2)(p-\alpha)\right] w(z)}{\left[1 + \Theta_2 w(z)\right]}, \quad (-1 \le \Theta_2 < \Theta_1 \le 1, 0 \le \alpha < 1), \tag{3}$$

where $w(z)$ is the Schwartz function and $p \in \mathbb{N} = \{1, 2, \ldots\}$. Recently, Breaz et al. [22] (equation 4) used the following expression to study a new class of multivalent function

$$\aleph(p; \Theta_1, \Theta_2; \alpha; \psi; z) = \frac{\left[(1+\Theta_1)p + \alpha(\Theta_2 - \Theta_1)\right]\psi(z) + \left[(1-\Theta_1)p - \alpha(\Theta_2 - \Theta_1)\right]}{\left[(\Theta_2 + 1)\psi(z) + (1 - \Theta_2)\right]}. \tag{4}$$

$\aleph(p; \Theta_1, \Theta_2; \alpha; \psi; z)$ is an extension of the class $\mathcal{P}(\Theta_1, \Theta_2, p, \alpha)$.

The function $p_{k,\sigma}(z)$ plays the role of an extremal functions related to the conic domain and is given by

$$\hat{p}_{k,\sigma}(z) = \begin{cases} \dfrac{1+(1-2\sigma)z}{1-z}, & \text{if } k = 0 \\ 1 + \dfrac{2(1-\sigma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1 \\ 1 + \dfrac{2(1-\sigma)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\operatorname{arctanh}\sqrt{z}\right], & \text{if } 0 < k < 1 \\ 1 + \dfrac{2(1-\sigma)}{1-k^2}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{t}} \dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \dfrac{1}{k^2-1}, & \text{if } k > 1 \end{cases} \tag{5}$$

where $u(z) = \dfrac{z - \sqrt{t}}{1 - \sqrt{tz}}, t \in (0,1)$ and t is chosen such that $k = \cosh\left(\dfrac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{k,\sigma}(z) = 1 + \tau_1 z + \tau_2 z^2 + \cdots, \qquad (\tau_j = p_j(k,\sigma), j = 1,2,3,\ldots), \tag{6}$$

we get

$$\tau_1 = \begin{cases} \dfrac{8(1-\sigma)(\arccos k)^2}{\pi^2(1-k^2)}, & \text{if } 0 \le k < 1, \\ \dfrac{8(1-\sigma)}{\pi^2}, & \text{if } k = 1 \\ \dfrac{\pi^2(1-\sigma)}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & \text{if } k > 1. \end{cases} \tag{7}$$

Noor and Malik [23] studied a class functions involving $\aleph\left(1;\Theta_1,\Theta_2;0;\hat{p}_{k,\sigma};z\right)$. Thereafter several authors studied various subclasses impacted by Janowski functions, refer to [22-28].

Motivated by [29] (also see [30, 31]), here we study a new class of functions omitting the additional stringent criterion of f^{-1} to be one-one.

Definition 1.1. For $0 \le \alpha \le 1, \lambda > 0; \lambda \ne \frac{1}{3}$ and $|t| \le 1, t \ne 1$, a function $f \in \mathcal{A}$ of the form (1) is said to be in the class $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$ if it satisfies:

$$\left(\frac{(1-t)z\left(f'(z)\right)^\lambda}{f(z)-f(tz)}\right)^\alpha \left(\frac{(1-t)\left(zf'(z)\right)^\lambda}{\left[f(z)-f(tz)\right]'}\right)^{1-\alpha} \prec \frac{(\Theta_1+1)\psi(z)-(\Theta_1-1)}{(\Theta_2+1)\psi(z)-(\Theta_2-1)} \tag{8}$$

where $\psi(z) = 1 + L_1 z + L_2 z^2 + \cdots \in \mathcal{P}$.

Remark 1.2. Now we will present some special cases of our class.

(i) Let λ = 1 and $\psi(z) = \hat{p}_{k,\sigma}(z)$ (see 5) in Definition 1.1, then the class $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$ reduces to classes $k-\mathcal{US}(\Theta_1,\Theta_2,\sigma,t)$ and $k-\mathcal{UC}(\Theta_1,\Theta_2,\sigma,t)$ for $\alpha = 0$ and $\alpha = 1$, respectively. The classes $k-\mathcal{US}(\Theta_1,\Theta_2,\sigma,t)$ and $k-\mathcal{UC}(\Theta_1,\Theta_2,\sigma,t)$ were defined by Arif et al. [32].

(ii) If we replace λ = 1, $t = 0$ and $\psi(z) = \hat{p}_{k,0}(z)$ in $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$, where $\hat{p}_{k,0}(z)$ is defined as in (5), we can get $k-\mathcal{ST}[\Theta_1,\Theta_2]$ and $k-\mathcal{UC}[\Theta_1,\Theta_2]$ classes defined by Noor and Malik [23] (Definitions 1.3 and 1.4) by choosing $\alpha = 0$ and $\alpha = 1$, respectively. Note that $1-\mathcal{UC}[\Theta_1,\Theta_2] = UP[\Theta_1,\Theta_2](-1 \le \Theta_2 < \Theta_1 \le 1)$ was recently studied by Malik et al. [33].

(iii) If we let $\lambda = 1, \Theta_1 = 1$ and $t = \Theta_2 = -1$ in $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$, we can get $\mathcal{S}_*^s(\psi)$ and $\mathcal{C}_*^s(\psi)$ classes defined by Shanmugam et al. [34] by choosing $\alpha = 0$ and $\alpha = 1$, respectively.

(iv) If we let $\Theta_1 = 1$, $t = \Theta_2 = -1$ and $k(z) = \dfrac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, (\tau = (1-\sqrt{5})/2)$ in (8), then $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$ reduces to

$$\mathcal{R}_\lambda(\alpha,\tau) = \left\{ f \in \mathcal{A}; \left(\frac{2z\left(f'(z)\right)^\lambda}{f(z)-f(-z)}\right)^\alpha \left(\frac{2\left(zf'(z)\right)^\lambda}{\left[f(z)-f(-z)\right]'}\right)^{1-\alpha} \prec k(z) \right\},$$

the class $\mathcal{R}_\lambda(\alpha,\tau)$ was recently introduced by Güney et al. [29], but with an addition criterion for inverse function.

2. Coefficient estimates of inverse in $V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z))$

For all $f \in V_t^\lambda(\alpha;\Theta_1,\Theta_2;\psi(z)$, we have $f'(0) = 1 \ne 0$ for all $z \in \mathcal{U}$ and $f(0) = 0$. Then, there exists an inverse function in some small disk with the center at $w = 0$. Now we will obtain the coefficient estimates of an inverse function belonging to $V_t^{\lambda}(\alpha;\Theta_1,\Theta_2;\psi(z))$.

Now, we will discuss the prerequisite results that are required to obtain our main results.

Lemma 2.1. [17] If $p(z)=1+\sum_{k=1}^{\infty}p_k z^k\in\mathcal{P}$, and v is complex number, then

$$\left|p_2-vp_1^2\right|\le 2\max\{1;|2v-1|\},$$

and the result is sharp for the functions

$$p_1(z)=\frac{1+z}{1-z}\quad\text{and}\quad p_2(z)=\frac{1+z^2}{1-z^2}.$$

Theorem 2.2. Let $f\in V_t^{\lambda}(\alpha;\Theta_1,\Theta_2;\psi(z))$ is given by (1), then the inverse coefficients estimates of $g=f^{-1}$ (provided f^{-1} exists) is given by

$$|b_2|\le\frac{|L_1|(\Theta_1-\Theta_2)}{2|1+t-2\lambda||\alpha-2|}\tag{9}$$

and

$$|b_3|\le\frac{|L_1|(\Theta_1-\Theta_2)}{\left|(4\alpha-6)\left(1+t+t^2\right)+3\lambda\left(6+4\alpha\right)\right|}\max\left[1;|2v-1|\right]\tag{10}$$

with

$$v:=\frac{1}{4}\left[L_1(\Theta_2+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\mathcal{N}\right],\tag{11}$$

where

$$\mathcal{M}:=\frac{-(\Theta_1-\Theta_2)L_1k_1}{2(\alpha-2)^2(1+t-2\lambda)^2}\quad\text{and}\quad\mathcal{N}:=\frac{L_1(\Theta_1-\Theta_2)k_2}{2(\alpha-2)^2(1+t-2\lambda)^2},\tag{12}$$

with

$$\begin{aligned}k_1&=(8-7\alpha+\alpha^2)(1+2t+t^2)-4\alpha^2(\lambda+\lambda t-\lambda^2)+16\lambda(1-\alpha)(\lambda-t)-4\lambda(8-7\alpha)\\k_2&=(4\alpha-6)(1+t+t^2)+3\lambda(6+4\alpha).\end{aligned}\tag{13}$$

Proof. If $f\in V_t^{\lambda}(\alpha;\Theta_1,\Theta_2;\psi(z))$, then there exists a Schwartz function such that

$$\left(\frac{(1-t)z\left(f'(z)\right)^{\lambda}}{f(z)-f(tz)}\right)^{\alpha}\left(\frac{(1-t)\left(zf'(z)\right)^{\lambda}}{\left[f(z)-f(tz)\right]'}\right)^{1-\alpha}=\frac{(\Theta_1+1)\psi(z)-(\Theta_1-1)}{(\Theta_2+1)\psi(z)-(\Theta_2-1)}.\tag{14}$$

Let $\ell(z)\in\mathcal{P}$ be of the form $\ell(z)=1+\sum_{n=1}^{\infty}p_n z^n$ and it is defined by

$$\ell(z)=\frac{1+w(z)}{1-w(z)},z\in\mathcal{U}.$$

On simple computation gives

$$w(z)=\frac{\ell(z)-1}{\ell(z)+1}=\frac{1}{2}p_1z_1+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{2}\left(p_3-p_1p_2+\frac{1}{4}p_1^3\right)z^3+\cdots,z\in\mathcal{U}$$

and considering

$$\begin{aligned}&\frac{(\Theta_1+1)\psi(z)-(\Theta_1-1)}{(\Theta_2+1)\psi(z)-(\Theta_2-1)}\\&=1+\frac{L_1p_1(\Theta_1-\Theta_2)z}{4}+\frac{(\Theta_1-\Theta_2)L_1}{4}\left[p_2-p_1^2\left(\frac{(\Theta_2+1)L_1+2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right]z^2+\cdots.\end{aligned}\tag{15}$$

Left hand side of (14) is given by

$$\left(\frac{(1-t)z\left(f'(z)\right)^{\lambda}}{f(z)-f(tz)}\right)^{\alpha}\left(\frac{(1-t)\left(zf'(z)\right)^{\lambda}}{\left[f(z)-f(tz)\right]'}\right)^{1-\alpha}=1+(\alpha-2)(1+t-2\lambda)z+\frac{1}{2}\left[a_3k_2-k_1a_2^2\right]z^2+\cdots,\tag{16}$$

where k_1 and k_2 are given by (13). From (2) and (16), we obtain

$$a_2=\frac{L_1p_1(\Theta_1-\Theta_2)}{4(\alpha-2)(1+t-2\lambda)}\tag{17}$$

and

$$a_3=\frac{(\Theta_1-\Theta_2)L_1}{2k_2}\left[p_2-p_1^2\left(\frac{L_1(\Theta_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\Theta_1-\Theta_2)L_1k_1}{8(\alpha-2)^2(1+t-2\lambda)^2}\right)\right]\tag{18}$$

From (2), we see that $b_2=-a_2$ and applying $|p_n|\le 2,(n\ge 1)$ in (17), we obtain the inequality (9). Also, from (2) we have

$$\begin{aligned}b_3&=\frac{(-1)^4}{3!}\begin{pmatrix}3a_2 & 1\\ 6a_2 & 4a_2\end{pmatrix}=2a_2^2-a_3\\&=\frac{L_1^2p_1^2(\Theta_1-\Theta_2)^2}{16(\alpha-2)^2(1+t-2\lambda)^2}-\frac{(\Theta_1-\Theta_2)L_1}{2k_2}\left[p_2-p_1^2\left(\frac{L_1(\Theta_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\Theta_1-\Theta_2)L_1k_1}{8(\alpha-2)^2(1+t-2\lambda)^2}\right)\right]\\&=\frac{-L_1(\Theta_1-\Theta_2)}{2k_2}\left[p_2-\frac{1}{4}p_1^2\left(L_1(\Theta_2+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\mathcal{N}\right)\right],\end{aligned}\tag{19}$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by (12). Applying Lemma 2.1 to (19), we get (10) which is the assertion of the theorem.

An interesting generalization of a class of starlike functions is the so-called class of starlike functions associated with the vertical domain, which is defined as follows:

Definition 2.1. [35] $f\in\mathcal{A}$ is said to be in $\mathcal{S}(\delta,\varepsilon)$ if it satisfies

$$\delta < \text{Re}\frac{zf'(z)}{f(z)} < \varepsilon, z \in \mathcal{U}, \tag{20}$$

where $0 \le \delta < 1 < \varepsilon$.

Letting $t = 0, \alpha = \lambda = 1, \Theta_1 = 1, \Theta_2 = -1$, and

$$\psi(z) = 1 + \frac{\varepsilon - \delta}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\delta)/(\varepsilon-\delta))} z}{1 - z}\right) \tag{21}$$

in Theorem 2.2, we have

Corollary 2.3. [36] Let $f \in \mathcal{S}^*(\delta, \varepsilon)$, then the coefficient estimates of the inverse function are

$$|b_2| \le \frac{2(\varepsilon - \delta)}{\pi} \sin\left(\frac{\pi(1-\delta)}{\varepsilon - \delta}\right)$$

and

$$|b_3| \le \frac{2(\varepsilon - \delta)}{\pi} \sin\left(\frac{\pi(1-\delta)}{\varepsilon - \delta}\right) \max\left\{1; \left|\frac{1}{2} - 3\frac{\varepsilon - \delta}{\pi} i + \left(\frac{1}{2} + 3\frac{\varepsilon - \delta}{\pi} i\right) e^{2\pi i \frac{1-\delta}{\varepsilon - \delta}}\right|\right\}.$$

Letting $\alpha = t = 0$, $\lambda = 1$ and $\psi(z) = \hat{p}_{1,0}(z)$ in Theorem 2.2, we get

Corollary 2.4. [33] Suppose that $f \in UP[\Theta_1, \Theta_2](-1 \le \Theta_2 < \Theta_1 \le 1)$ (see Remark 1.1), then

$$|b_2| \le \frac{2(\Theta_1 - \Theta_2)}{\pi^2}$$

and

$$|b_3| \le \frac{4(\Theta_1 - \Theta_2)}{6\pi^2}.$$

3. Fekete-Szegő inequality for the function of $V_t^\lambda(\alpha; \Theta_1, \Theta_2; \psi(z))$

Here, we will present the Fekete-Szegő inequality for the functions belonging to the class $V_t^\lambda(\alpha; \Theta_1, \Theta_2; \psi(z))$.

Theorem 3.1. Let $f \in V_t^\lambda(\alpha; \Theta_1, \Theta_2; \psi(z))$, then for all $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \le \frac{|L_1|(\Theta_1 - \Theta_2)}{|2(3\lambda - 1)(3 - 2\lambda)|} \max[1; |2\tau - 1|] \tag{22}$$

with

$$\tau := \frac{1}{4}\left[L_1(\Theta_2 + 1) + 2\left(1 - \frac{L_2}{L_1}\right) + \mathcal{M} + \frac{\mu \mathcal{N}}{2}\right], \tag{23}$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by (12). The inequality is sharp for $\mu \in \mathbb{C}$.

Proof. If $f \in V_t^\lambda(\alpha; \Theta_1, \Theta_2; \psi(z))$, in the view of relation (17) and (18), for $\mu \in \mathbb{C}$ we have

$$\left|a_3-\mu a_2^2\right|=\frac{(\Theta_1-\Theta_2)}{2k_2}\left[p_2-p_1^2\left(\frac{L_1(\Theta_2+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}-\frac{(\Theta_1-\Theta_2)L_1k_1}{8(\alpha-2)^2(1+t-2\lambda)^2}\right)\right]$$

$$-\mu\frac{L_1^2p_1^2(\Theta_1-\Theta_2)^2}{16(\alpha-2)^2(1+t-2\lambda)^2}=\frac{(\Theta_1-\Theta_2)L_1}{2k_2}\left[p_2-\frac{1}{4}p_1^2\left(L_1(\Theta_2+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\frac{\mu\mathcal{N}}{2}\right)\right]$$

$$\le\frac{|L_1|(\Theta_1-\Theta_2)}{2|k_2|}\left[2-\frac{1}{4}p_1^2\left(\left|\frac{L_2}{L_1}-L_1(\Theta_2+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|-2\right)\right] \tag{24}$$

now if $\left|\frac{L_2}{L_1}-L_1(\Theta_2+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\le 2$ in the above inequality, we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{|L_1|(\Theta_1-\Theta_2)}{2|k_2|}. \tag{25}$$

Further, if $\left|\frac{L_2}{L_1}-L_1(\Theta_2+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\ge 2$ in the same inequality, we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{|L_1|(\Theta_1-\Theta_2)}{2|k_2|}\left(\left|\frac{L_2}{L_1}-L_1(\Theta_2+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\right). \tag{26}$$

Following the steps in [25], we can establish that inequality (22) will be sharp if

$$\left(\frac{(1-t)z\left(f'(z)\right)^{\lambda}}{f(z)-f(tz)}\right)^{\alpha}\left(\frac{(1-t)\left(zf'(z)\right)^{\lambda}}{\left[f(z)-f(tz)\right]'}\right)^{1-\alpha}=\frac{(\Theta_1+1)p(z^2)-(\Theta_1-1)}{(\Theta_2+1)p(z^2)-(\Theta_2-1)} \tag{27}$$

and

$$\left(\frac{(1-t)z\left(f'(z)\right)^{\lambda}}{f(z)-f(tz)}\right)^{\alpha}\left(\frac{(1-t)\left(zf'(z)\right)^{\lambda}}{\left[f(z)-f(tz)\right]'}\right)^{1-\alpha}=\frac{(\Theta_1+1)p_1(z)-(\Theta_1-1)}{(\Theta_2+1)p_1(z)-(\Theta_2-1)}, \tag{28}$$

and the proof of the theorem is complete.

Letting $t=0,\alpha=\lambda=1,\Theta_1=1,\Theta_2=-1$ and $\psi(z)$ is of the form (21) in Theorem 3.1, then we have the following result obtained by Sim and Kwon [36].

Corollary 3.2. [36] Let $f\in\mathcal{S}^*(\delta,\varepsilon)$. Then, for any μ,

$$\left|a_3-\mu a_2^2\right|\le\frac{\varepsilon-\delta}{\pi}\sin\left(\frac{\pi(1-\delta)}{\varepsilon-\delta}\right)\max\left\{1;\left|\frac{1}{2}+(1-2\mu)\frac{\varepsilon-\delta}{\pi}i+\left(\frac{1}{2}-(1-2\mu)\frac{\varepsilon-\delta}{\pi}i\right)e^{2\pi i\frac{1-\delta}{\varepsilon-\delta}}\right|\right\}.$$

Letting $t=0$, $\alpha=\lambda=1$, $\Theta_1=1$, and $\Theta_2=-1$ in Theorem 3.1, then we have the following result obtained by Tu and Xiong [37].

Corollary 3.3. [37] Suppose $f(z)\in\mathcal{S}^*(\psi)$ $(z\in\mathcal{U})$, then

$$\left|a_3 - \mu a_2^2\right| \le \frac{L_1}{2} \max\left\{1; \left|L_1 + \frac{L_2}{L_1} - 2\mu L_1\right|\right\} \quad (\mu \in \mathbb{C}).$$

The inequality is sharp for the function given by

$$f(z) = \begin{cases} z \exp \int_0^z \left[\psi(t) - 1\right] \frac{1}{t} dt, & \text{if } \left|L_1 + \frac{L_2}{L_1} - 2\mu L_1\right| \ge 1 \\ z \exp \int_0^z \left[\psi(t^2) - 1\right] \frac{1}{t} dt, & \text{if } \left|L_1 + \frac{L_2}{L_1} - 2\mu L_1\right| \le 1. \end{cases}$$

4. Conclusion

We have defined a new family of pseudo-starlike functions that connect the convex combinations of analytic functions. To make this study more comprehensive, we have defined the class of functions with respect to symmetric points, which amalgamates the study of several classes of well-known analytic functions. Solutions to the Fekete-Szegő problem and coefficient estimates of an inverse function are the foremost results of this paper. Also, we have pointed out appropriate connections and applications of our main results, which are mostly presented in the form of corollaries and remarks. Refer to [5, 23, 25, 38-40], for studies closely related to the results presented here.

This study can be further extended by replacing the superordinate function in (8) with a function that is not Carathéodory (see [41]). Further, this study can be extended by taking a trigonometric hyperbolic function, Gegenbauer polynomial, Laguerre polynomial, Chebyshev polynomial, Fibonacci sequence, or q-Hermite polynomial instead of considering $\psi(z)$ as in (8).

Conflict of interest

Both the authors declare that they have no conflict of interest.

References

[1] Lewin M. On a coefficient problem for bi-univalent functions. *Proceedings of the American Mathematical Society*. 1967; 18(1): 63-68. Available from: https://doi.org/10.2307/2035225.

[2] Ahmad M, Frasin BA, Murugusundaramoorthy G, Al-khazaleh A. An application of Mittag–Leffler-type Poisson distribution on certain subclasses of analytic functions associated with conic domains. *Heliyon*. 2021; 7(10): e08109. Available from: https://doi.org/10.1016/j.heliyon.2021.e08109.

[3] Altinkaya S, Yalcin S. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Acta Universitatis Apulensis*. 2014; 40: 347-354. Available from: https://doi.org/10.17114/j.aua.2014.40.28.

[4] Altınkaya S, Yalçin S. Initial coefficient bounds for a general class of biunivalent functions. *International Journal of Analysis*. 2014; 2014: 867871. Available from: https://doi.org/10.1155/2014/867871.

[5] Altınkaya S, Yalçin S. Faber polynomial coefficient bounds for a subclass of bi-univalent functions. *Comptes Rendus Mathematique*. 2015; 353(12): 1075-1080. Available from: https://doi.org/10.1016/j.crma.2015.09.003.

[6] Altınkaya S, Yalçin S. Coefficient estimates for two new subclasses of biunivalent functions with respect to symmetric points. *Journal of Function Spaces*. 2015; 2015: 145242. Available from: https://doi.org/10.1155/2015/145242.

[7] Altınkaya S, Yalçin S. Coefficient bounds for certain subclasses of m-fold symmetric biunivalent functions. *Journal of Mathematics*. 2015; 2015: 241683. Available from: https://doi.org/10.1155/2015/241683.

[8] Amini E, Al-Omari S, Nonlaopon K, Baleanu D. Estimates for coefficients of bi-univalent functions associated with a fractional q-difference operator. *Symmetry*. 2022; 14(5): 879. Available from: https://doi.org/10.3390/sym14050879.

[9] Amourah A, Frasin BA, Ahmad M, Yousef F. Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions. *Symmetry*. 2022; 14(1): 147. Available from: https://doi.org/10.3390/sym14010147.

[10] Frasin BA, Aouf MK. New subclasses of bi-univalent functions. *Applied Mathematics Letters*. 2011; 24(9): 1569-1573. Available from: https://doi.org/10.1016/j.aml.2011.03.048.

[11] Mahmood S, Jabeen M, Malik SN, Srivastava HM, Manzoor R, Riaz SMJ. Some coefficient inequalities of q-starlike functions associated with conic domain defined by q-derivative. *Journal of Function Spaces*. 2018; 2018: 8492072. Available from: https://doi.org/10.1155/2018/8492072.

[12] Çağlar M, Cotîrlă L-I, Buyankara M. Fekete-Szegő inequalities for a new subclass of biunivalent functions associated with Gegenbauer polynomials. *Symmetry*. 2022; 14(4): 1572. Available from: https://doi.org/10.3390/sym14081572.

[13] Orhan H, Murugusundaramoorthy G, Çağlar M. The Fekete-Szegő problems for subclass of bi-univalent functions associated with Sigmoid function. *Facta Universitatis*. 2022; 37(3): 495-506. Available from: https://doi.org/10.22190/FUMI201022034O.

[14] Wanas AK, Cotîrlă L-I. Applications of (*M,N*)-Lucas polynomials on a certain family of bi-univalent functions. Mathematics. 2022; 10(4): 595. Available from: https://doi.org/10.3390/math10040595.

[15] Wanas AK, Güney HÖ. Coefficient bounds and Fekete-Szegő inequality for a new family of bi-univalent functions defined by Horadam polynomials. *Afrika Matematika*. 2022; 33: 77. Available from: https://doi.org/10.1007/s13370-022-01015-7.

[16] Yousef F, Amourah A, Frasin BA, Bulboača T. An avant-garde construction for subclasses of analytic bi-univalent functions. *Axioms*. 2022; 11(6): 267. Available from: https://doi.org/10.3390/axioms11060267.

[17] Ma WC, Minda D. A unified treatment of some special classes of univalent functions. In: Li Z, Ren F, Yang L, Zhang S. (eds.) *Proceeding of conference on complex analytic*. New York: International Press; 1994. p.157-169.

[18] Janowski W. Some extremal problems for certain families of analytic functions I. *Annales Polonici Mathematici*. 1973; 28(3): 297-326.

[19] Aouf MK. On a class of p-valent starlike functions of order α. *International Journal of Mathematics and Mathematical Sciences*. 1987; 10(4): 733-744. Available from: https://doi.org/10.1155/S0161171287000838.

[20] Noor KI. Applications of certain operators to the classes related with generalized Janowski functions. *Integral Transforms and Special Functions*. 2010; 21(8): 557-567. Available from: https://doi.org/10.1080/10652460903424261.

[21] Sokół J. Classes of multivalent functions associated with a convolution operator. *Computers & Mathematics with Applications*. 2010; 60(5): 1343-1350. Available from: https://doi.org/10.1016/j.camwa.2010.06.015.

[22] Breaz D, Karthikeyan KR, Senguttuvan A. Multivalent prestarlike functions with respect to symmetric points. *Symmetry.* 2022; 14(1): 20. Available from: https://doi.org/10.3390/sym14010020.

[23] Noor KI, Malik SN. On coefficient inequalities of functions associated with conic domains. *Computers & Mathematics with Applications*. 2011; 62(5): 2209-2217. Available from: https://doi.org/10.1016/j.camwa.2011.07.006.

[24] Karthikeyan KR, Lakshmi S, Varadharajan S, Mohankumar D, Umadevi E. Starlike functions of complex order with respect to symmetric points defined using higher order derivatives. *Fractal and Fractional*. 2022; 6(2): 116. Available from: https://doi.org/10.3390/fractalfract6020116.

[25] Karthikeyan KR, Murugusundaramoorthy G, Bulboača T. Properties of λ-pseudo-starlike functions of complex order defined by subordination. *Axioms*. 2021; 10(2): 86. Available from: https://doi.org/10.3390/axioms10020086.

[26] Srivastava HM, Khan B, Khan N, Ahmad QZ. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Mathematical Journal*. 2019; 48(2): 407-425. Available from: https://doi.org/10.14492/hokmj/1562810517.

[27] Srivastava HM, Khan B, Khan N, Ahmad QZ, Tahir M. A generalized conic domain and its applications to certain subclasses of analytic functions. *The Rocky Mountain Journal of Mathematics*. 2019; 49: 2325-2346. Available

from: https://doi.org/10.1216/RMJ-2019-49-7-2325.
[28] Srivastava HM, Khan N, Darus M, Rahim MT, Ahmad QZ, Zeb Y. Properties of spiral-like close-to-convex functions associated with conic domains. *Mathematics.* 2019; 7(8): 706. Available from: https://doi.org/10.3390/math7080706.
[29] Güney HO, Murugusundaramoorthy G, Vijaya K. Subclasses of˙ λ-bi-pseudo-starlike functions with respect to symmetric points based on shell-like curves. *Cubo*. 2021; 23(2): 299-312. Available from: https://doi.org/10.4067/S0719-06462021000200299.
[30] Malik SN, Raza M, Xin Q, Sokół J, Manzoor R, Zainab R. On convex functions associated with symmetric cardioid domain. *Symmetry*. 2021; 13: 2321. Available from: https://doi.org/10.3390/sym13122321.
[31] Senguttuvan A, Mohankumar A, Ganapathy RR, Karthikeyan KR. Coefficient inequalities of a comprehensive subclass of analytic functions with respect to symmetric points. *Malaysian Journal of Mathematical Sciences*. 2022; 16(3): 437-450. Available from: https://doi.org/10.47836/mjms.16.3.03.
[32] Arif M, Wang Z-G, Khan R, Lee SK. Coefficient inequalities for Janowski-Sakaguchi type functions associated with conic regions. *Hacettepe Journal of Mathematics and Statistics*. 2018; 47(2): 261-271.
[33] Malik SN, Mahmood S, Raza M, Farman S, Zainab S. Coefficient inequalities of functions associated with petal type domains. *Mathematics*. 2018; 6: 298. Available from: https://doi.org/10.3390/math6120298.
[34] Shanmugam TN, Ramachandran C, Ravichandran V. Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points. *Bulletin of the Korean Mathematical Society*. 2006; 43(3): 589-598.
[35] Kuroki K, Owa S, Notes on new class for certain analytic functions. *RIMS Kokyuroku*. 2011; 1772: 21-25.
[36] Sim YJ, Kwon OS. Notes on analytic functions with a bounded positive real part. *Journal of Inequalities and Applications*. 2013; 2013: 370. Available from: https://doi.org/10.1186/1029-242X-2013-370.
[37] Tu Z, Xiong L. Unified solution of Fekete-Szegő problem for subclasses of starlike mappings in several complex variables. *Mathematica Slovaca*. 2019; 69(4): 843-856. Available from: https://doi.org/10.1515/ms-2017-0273.
[38] Karthikeyan KR, Murugusundaramoorthy G, Cho NE. Some inequalities on Bazilevič class of functions involving quasi-subordination. *AIMS Math*. 2021; 6(7): 7111-7124. Available from: https://doi.org/10.3934/math.2021417.
[39] Tang H, Karthikeyan KR, Murugusundaramoorthy G. Certain subclass of analytic functions with respect to symmetric points associated with conic region. *AIMS Mathematics.* 2021; 6(11): 12863-12877. Available from: https://doi.org/10.3934/math.2021742.
[40] Karthikeyan KR, Senguttuvan A. On a characterization of starlike functions with respect to $(2\jmath, \ell)$-symmetric conjugate points. *Asian-European Journal of Mathematics*. 2023. Available from: https://doi.org/10.1142/S1793557123501802.
[41] Karthikeyan KR, Cho NE, Murugusundaramoorthy G. On classes of non-Carathèodory functions associated with a family of functions starlike in the direction of the real axis. *Axioms.* 2023; 12(1): 24. Available from: https://doi.org/10.3390/axioms12010024.