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Research Article

# Hilbert Space Decomposition Properties of Complex Functions and Their Applications

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**Abstract:** We analyzed the classical problem of decomposing the Hilbert space of holomorphic functions, especially their splitting into the product or sum of domain-separated components. For the Bergman space of analytical functions, we obtained a special decomposition satisfying the assigned growth degree properties. Concerning a general Hilbert space of analytical functions on a connected domain, we studied its $\alpha$-invariant decomposition and related ergodic consequences. As an interesting consequence, we obtained the decomposition theorem for an ergodic $\alpha$-mapping on the Bergman space of holomorphic functions.

***Keywords*****:** decomposition, Bergman space, Hilbert space decomposition, Wold decomposition, invariant mapping, space invariance, ergodicity

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## 1. Introduction: Additive decomposition of holomorphic functions

Decomposition problems in spaces of holomorphic functions are considered by many researchers in different fields of mathematics and applied sciences. Especially important are the problems of splitting functions into the product or sum of two functions separated by the corresponding domains of the complex plane. For instance, the decomposition problem of holomorphic functions from the Paley-Wiener and Hardy spaces into the sum of two functions, specified by their growth degree, was actively studied in [1-3], solving the important filter identification problem [4-8] subject to electrical, optical, and acoustical signals. Especially, mathematically challenging the factorization problem of holomorphic functions in Bergman space with assigned growth degree properties proved to have many applications [9-12], inspired by recent advances in wavelet, co-orbit, control, and coherent state representation theories.

In our work, we are mainly interested both in the additive decomposition of holomorphic functions of Bergman spaces on connected domains, subject to the related growth degree problem on suitably separated subdomains, inspired by recent studies in [2, 3], and in the Bergman space direct sum invariant decompositions [13-16] with respect to isometry operators, defined by means of some invariant measurable mappings, especially ergodic.


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## 2. Bergman space decomposition and growth degree aspects

### 2.1 *Preliminary statements*

Consider a simply connected domain $D \subset \mathbb{C}$ with a smooth boundary $\partial D$. In the known classical works [17-19], Bers and Nirenberg stated that the continuous $\bar{\partial}$-potential integral operator

$$P(g)(z,\bar{z}) := \frac{1}{2\pi i}\int_D g(\xi)\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) d\bar{\xi}\, d\xi, \tag{1}$$

which is convergent owing to the Calderon-Zygmund theorem [20, 21] for any $g \in L_r(D;\mathbb{C}), r > 2,$ where $L_r(D;\mathbb{C})$ denotes the Hilbert space of $r$ - power integrable measurable functions on the region $D$. In particular, the operator (1) possesses the following crucially important projection properties:

$$\frac{\partial}{\partial \bar{z}} P(g) = g, \quad \frac{\partial}{\partial z} P(g) := T(g), \tag{2}$$

where, by definition, the linear operator

$$T(g) := \frac{1}{2\pi i}\int_D \frac{g(\xi) - g(z)}{(\xi - z)^2} d\bar{\xi}\, d\xi \tag{3}$$

is bounded on $L_r(D;\mathbb{C}), r > 2,$ and realizes an isometry of the Hilbert space $L_2(D;\mathbb{C}) : \| T(g) \|_2 = \| g \|_2$ for any $r \in [1,\infty)$.

Let $r \in [1,\infty)$ define [22, 23] the classical Bergman space $A_r(D) \subset L_r(D)$ of holomorphic functions on a connected domain $D \subset \mathbb{C},$ for which the following norm

$$\| g \|_{A_r} := \left(\int_D |g(\xi)|^r \, d\bar{\xi}\, d\xi\right)^{1/r} \tag{4}$$

is bounded for any $g \in A_r(D)$. If $r = 2,$ the Bergman space $A_2(D)$ is called a "reproducing kernel Hilbert space" [22], whose kernel $1/[\pi(1 - \bar{\lambda}\xi z)], \xi \in D, \lambda \in \bar{D}$ realizes the Bergman projection $P_D : L_2(D) \to A_2(D) \subset L_2(D), P_D^2 = P_D,$ via the integral expression

$$P_D g(z) := \frac{1}{\pi}\int_D \frac{g(\xi)}{(1 - \lambda\bar{\xi}\, z)} d\bar{\xi}\, d\xi, \tag{5}$$

whose image $A_2(D)$ is a closed subspace of the Hilbert space $L_2(D)$. In general case, for any $g \in A_2(D)$ owing to the classical Riesz theorem, there exist a mapping $K_D : D^2 \to \mathbb{C}$ called the Bergman reproducing kernel, satisfying the relationship

$$g(z) = \int_D K_D(z,\xi) g(\xi) d\bar{\xi}\, d\xi, \tag{6}$$

where $\overline{K_D(z,\xi)} = K_D(\xi,z), (\xi,z) \in D^2$ and $K_D(z,z) > 0$ for $z \in D.$ The latter makes it possible to define the domain $D \subset \mathbb{C},$ the Hermitian metric $h_D(z) := \frac{\partial^2 \ln K_D(z,z)}{\partial z \partial \bar{z}} dz \otimes d\bar{z},$ subject to which the domain $D$, as a Riemannian space becomes a Kähler manifold: the imaginary part $\mathrm{Im} h_D(z) := \omega^{(2)}(z) \in \Lambda_z^2(D)$ is a nondegenerate 2-form on $D \subset \mathbb{C}.$

*Remark* 2.1. The weighted Bergman space $A_2^{(\mu)}(\mathbb{C})$ of holomorphic functions with the norm $\| g \|_{A_2^{(\mu)}} := \frac{1}{\pi}\left(\int_{\mathbb{C}} |g(\xi)|^2 \exp(-\bar{\xi}\xi) d\bar{\xi}\, d\xi\right)^{1/2}$ for any $g \in A_2^{(\mu)}(\mathbb{C}),$ proves to be a reproducing Hilbert space too [24] with respect to the kernel $\exp(\bar{z}\xi), z, \xi \in \mathbb{C},$ that is for any function $f \in A_2^{(\mu)}(\mathbb{C}),$ there holds the following representation:

$f(z) = \int_{\mathbb{C}} f(\xi)\exp(z\bar{\xi})d\bar{\xi}d\xi, z \in \mathbb{C}$. A related problem of constructing reproducing kernels for the weighted Bergman space $A_2^{(\mu)}(D)$ on a connected domain $D \subset \mathbb{C}$ is, in general, not solved effectively in contrary to the standard Bergman space $A_2(D)$.

Observe the following self-mapping (7) satisfies [17] the important property

$$(\partial e^{s(z,\bar{z})} / \partial\bar{z})^{-1} \frac{\partial}{\partial\bar{z}} P(f\, \partial e^s / \partial\bar{z})(z,\bar{z}) = f(z) \tag{7}$$

for all $z \in D$ and any $f \in A_2(D)$ from the Bergman space $A_2(D)$ of holomorphic functions on $D$, where a mapping $s \in \mathcal{H}^{1+\sigma}(D), \sigma > 0$ is a Hölder continuous on $D$ and such that $f\, \partial e^s / \partial\bar{z} \in L_r(D;\mathbb{C}),\ r > 2.$ The representation (7) also proved very useful [17, 18, 25] for classifying quasiconformal mappings of the complex plane, being reduced to solving some nonuniform elliptic differential equations.

## 2.2 *Main growth degree decomposition result*

Now, consider two functions: an entire function $f \in W_p(\mathbb{C}), p > 2,$ and a Hölder continuous function $\exp s \in \mathcal{H}^{1+\sigma}(\mathbb{C}),$ that the restriction $f \exp s\,|_D \in L_r(D;\mathbb{C})$ $f$ for some $r > 2$. Then, the potential property (7) makes it possible to factorize the function $f \in W_p(\mathbb{C})$ on $D$ the following way:

$$f = \varphi + \psi, \tag{8}$$

where, by definition, the functions

$$\varphi := (\partial e^s / \partial\bar{z})^{-1} \frac{\partial}{\partial\bar{z}} P_+(f\, \partial e^s / \partial\bar{z})(z,\bar{z}) \tag{9}$$

and

$$\psi := (\partial e^s / \partial\bar{z})^{-1} \frac{\partial}{\partial\bar{z}} P_-(f\, \partial e^s / \partial\bar{z})(z,\bar{z}), \tag{10}$$

defined by means of the operator (1) as

$$P_{\pm} g(z,\bar{z}) := \frac{1}{2\pi i}\int_{D_{\pm}} g(\xi,\bar{\xi})\left(\frac{1}{\xi - z} - \frac{1}{\xi}\right) d\bar{\xi}d\xi, \tag{11}$$

suitably reduced on functions $g \in L_r(D_{\pm})$ and calculated over the subregions $D_+ := D \cap \mathbb{C}_+$ and $D_- := D \cap \mathbb{C}_-$, being the upper and down intersections of the region $D$ with the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ of the complex plane $\mathbb{C}$, respectively.

The obtained above holomorphic functions $\varphi \in A_2(D_+;\mathbb{C})$ and $\psi \in A_2(D_-;\mathbb{C})$, owing to the representation (5) and (6) on the domains $D_+$ and $D_-$ can be naturally analytically continued from $D_+$ and $D_-$ respectively, on the whole region $D$, giving rise to the functions $\tilde{\varphi} \in A_2(D;\mathbb{C})$ and $\tilde{\psi} \in A_2(D;\mathbb{C})$, respectively, and satisfying, by construction, the expected [2, 3] conditions: the reduced function $\tilde{\varphi}\,|_{D_-}$ is "small" on $D_-$ subject to the wage function $\exp s \in \mathcal{H}^{1+\sigma}(D_-)$ and the reduced function $\tilde{\psi}\,|_{D_+}$ is "small" on $D_+$ subject to the wage function $\exp s \in \mathcal{H}^{1+\sigma}(D_+)$.

Here, one needs to remark that the functions $\tilde{\varphi} : D \to \mathbb{C}$ and $\tilde{\psi} : D \to \mathbb{C}$ are not continuous on the joint boundary part $\partial D_+ \cap \partial D_- := \mathcal{L}$, that is $\tilde{\varphi}_+(t) \neq \tilde{\varphi}_-(t)$ and $\tilde{\psi}_+(t) \neq \tilde{\psi}_-(t)$, respectively, where $\lim_{z \downarrow t} \tilde{\varphi}(z) = \tilde{\varphi}_+(t), \lim_{z \uparrow t} \tilde{\varphi}(z) = \tilde{\varphi}_-(t)$ and $\lim_{z \downarrow t} \tilde{\psi}(z) = \tilde{\psi}_+(t),\ \lim_{z \uparrow t} \tilde{\psi}(z) = \tilde{\psi}_-(t)$ for all $t \in \mathcal{L}$ It is also easy to check that the jump $\bar{f}(t) := (\tilde{\varphi}_+(t) - \tilde{\varphi}_-(t))$ $= -(\tilde{\psi}_+(t) \neq \tilde{\psi}_-(t))$ for all $t \in \mathcal{L}$, since the generating function $f \in W_p(\mathbb{C})$ is, evidently, continuous on the region $D$ giving rise to its zero jump on $\mathcal{L}$, that is $f_+(t) - f_-(t) = 0$ for all $t \in \mathcal{L}$. These properties make it possible to use the well-known Sokhotsky-Plemelj formulas for the possible analytic continuations of the basic functions $\varphi|_{D_+}$ on $D_-$ and $\psi|_{D_-}$ on $D_+$ through the common boundary $\mathcal{L}$ :

$$\tilde{\varphi}(z)|_{z\in D_-} = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\overline{f}(t)dt}{t-z}, \ \tilde{\psi}(z)|_{z\in D_+} = \frac{-1}{2\pi i}\int_{\mathcal{L}} \frac{\overline{f}(t)dt}{t-z}, \tag{12}$$

applied actually to the corresponding reproducing kernels, which should *a priori* give rise to the conditions

$$\tilde{\varphi}(z) + \psi(z) = f(z) \tag{13}$$

for $z \in D_-$ and

$$\varphi(z) + \tilde{\psi}(z) = f(z) \tag{14}$$

for $z \in D_+$. Thus, the following proposition, solving the growth degree problem above, holds.

**Proposition 2.2.** Let a function $f \in W_p(\mathbb{C})$ and a wage function $\exp s \in \mathcal{H}^{1+\sigma}(\mathbb{C}), \sigma > 0,$ be such that for a simply connected region $D \subset \mathbb{C}$ the reduction $f \exp\alpha|_D \in L_r(D;\mathbb{C}), r > 2.$ Then, there exists the additive factorizations (13) and (14) on $D_+ = D \cap \mathbb{C}_+$ and $D_+ = D \cap \mathbb{C}_+$ respectively, satisfying the related growth degree properties.

# 3. The invariant function decomposition

## 3.1 *Preliminary invariant decomposition statements*

Now, consider the Hilbert space $L_2(D)$ of complex functions on a simply connected region $D \subset \mathbb{C}$ and the isometry operator $T_\alpha : L_2(D) \to L_2(D)$:

$$T_\alpha f := f^\circ \alpha \tag{15}$$

for any $f \in L_2(\mathbb{C})$ subject to some measurable mapping $\alpha : D \to D,$ for which the Lebesgue measure $\lambda$ is invariant, that is $\lambda(A) = \lambda(\alpha^{-1}A)$ for any Borel measurable set $A \subset D.$ Then, the following proposition holds.

**Proposition 3.1.** The Hilbert space $L_2(D)$ is decomposable as the direct orthogonal sum

$$L_2(D) = H_- \oplus_{n=0}^{+\infty} T_\alpha^n(H_+), \tag{16}$$

where $H_- := \cap_{n=0}^{+\infty} T_\alpha^n(L_2(D))$ and $H_+ = T_\alpha(L_2(D))^\perp$ are the orthogonal complements in $L_2(D)$ to the closed subspace $T_\alpha(L_2(D)) \subset L_2(D).$

To state this decomposition, consider the closed subspace $T_\alpha(L_2(D)) \subset L_2(D)$ and its orthogonal complement $H_+ := T_\alpha(L_2(D))^\perp$ in the Hilbert space $L_2(D),$ one can write down the following splitting:

$$L_2(D) = T_\alpha(L_2(D)) \oplus H_+. \tag{17}$$

Since the closed subspace $T_\alpha(H_+) \perp H_+,$ this splitting can now be applied to the closed subspace $H_+ \subset L_2(D),$ obtaining

$$L_2(D) = T_\alpha(L_2(D)) \cap T_\alpha^2(L_2(D)) \oplus H_+ \oplus T_\alpha(H_+). \tag{18}$$

Having this process iterated, one yields the following Hilbert space direct sum splitting

$$L_2(D) = H_- \oplus_{n=0}^{+\infty} T_\alpha^n(H_+), \tag{19}$$

well-known [16, 26] as the Wold decomposition, where the closed subspace $H_- := \cap_{n=0}^{+\infty} T_\alpha^n (L_2(D))$.

*Remark* 3.2. Here, it is important to observe that the orthogonal direct sum $H_\alpha := \oplus_{n=0}^{+\infty} T_\alpha^n (H_+) \subset L_2(D)$ is, by construction, an invariant subspace of the Hilbert space $L_2(D)$, that is

$$T_\alpha H_\alpha \subset H_\alpha. \tag{20}$$

Let a set $\{\beta_j \in H_+ : j \in \mathbb{N}\}$ denote an orthonormal basis of the closed subspace $H_+ \subset L_2(D)$. Then, the decomposition (16), in particular, means that for any function $g \in H_\alpha$, there exists a set of numbers $\{g_n^{(j)} \in \mathbb{C} : j \in \mathbb{N}, n \in \mathbb{Z}_+\}$, such that the following convergent decomposition

$$g = \sum_{n \in \mathbb{Z}_+} \sum_{j \in \mathbb{N}} g_n^{(j)} \beta_j \circ \alpha^n \tag{21}$$

holds. In particular, if the mapping $\alpha : D \to D$ is ergodic [27, 28], the subspace $H_- \subset L_2(D)$ is trivial, and for any function $f \in L_2(D)$, the decomposition (21) reduces to the orthogonal superposition sum

$$f = \sum_{n \in \mathbb{Z}_+} \sum_{j \in \mathbb{N}} f_n^{(j)} \beta_j \circ \alpha^n \tag{23}$$

for a suitable set of coefficients $\{f_n^{(j)} \in \mathbb{C} : j \in \mathbb{N}, n \in \mathbb{Z}_+\}$.

## 3.2 *Main invariant decomposition result and examples*

Assume that the Bergman space $A_2(D) \subset L_2(D)$ is invariant with respect to the isometry $T_\alpha : A_2(D) \to A_2(D)$, that is $T_\alpha A_2(D) \subset A_2(D)$ for the holomorphic mapping $\alpha : D \to D$, the following direct sum orthogonal decomposition

$$A_2(D) = H_- \oplus_{n=0}^{+\infty} T_\alpha^n (H_+) \tag{23}$$

holds, where $H_- := \cap_{n=1}^{+\infty} T_\alpha^n (A_2(D))$ and $H_+ = (T_\alpha A_2(D))^\perp$ in $A_2(D)$.

In particular, if to take the mapping $\alpha : D \to D$ to be as both conformal and ergodic, then the subspace is trivial and for any holomorphic function $f \in A_2(D)$, one can write down the additive orthogonal decomposition

$$f = \sum_{n \in \mathbb{Z}_+} \sum_{j \in \mathbb{N}} f_n^{(j)} \beta_j \circ \alpha^n \tag{24}$$

for a suitable set of coefficients $\{f_n^{(j)} \in \mathbb{C} : j \in \mathbb{N}, n \in \mathbb{Z}_+\}$, where the basis functions $\{\beta_j \in H_+ : j \in \mathbb{N}\}$ are holomorphic in *D*.

**Example 3.3.** Concerning the general Hilbert space $L_2(D)$ case, as an example one can consider the region $D = \mathbb{C}$ and the isometry mapping $T_\alpha : L_2(\mathbb{C}) \to L_2(\mathbb{C})$, where the mapping $\alpha : \mathbb{C} \to \mathbb{C}$ can be given either by the algebraic expression

$$\alpha(z, \bar{z}) = z - \frac{4\bar{z}i}{z^2 - \bar{z}^2}, \tag{25}$$

or by the expression

$$\alpha(z, \bar{z}) = \bar{z} + \frac{4\bar{z}}{(z^2 - \bar{z}^2)}, \tag{26}$$

for any $z \pm \bar{z} \neq 0 \in \mathbb{C}$.

Then, the following theorem, based on Proposition 3.1 and the ergodicity results of [15, 29, 30], holds.

**Theorem 3.4.** The following identity

$$\int_{\mathbb{C}} h(\alpha(z,\bar{z}),\overline{\alpha(z,\bar{z})})d\lambda(z,\bar{z}) = \int_{\mathbb{C}} h(z,\bar{z})d\lambda(z,\bar{z}) \tag{27}$$

holds for any $h \in L_2(\mathbb{C})$, where $d\lambda(z,\bar{z}) = d\bar{z} \wedge dz/(2i)$ is the Lebesgue measure on $\mathbb{C}$. Moreover, taking into account that the Lebesgue measure $\lambda$ on $\mathbb{C}$ is also ergodic with respect to the mappings (25) and (26), we obtain that any function $g \in L_2(\mathbb{C})$ allows the direct sum orthogonal decomposition (22).

*Proof.* Proof easily follows from the ergodicity property of the Lebesgue measure $\lambda$ on $\mathbb{C}$ stated in [29] and the $\alpha$-invariance property of Remark 3.2. Consider now a so-called [31] internal function $\varphi_{(z,\bar{z})} : \mathbb{C}_+ \to \mathbb{C}_+$, analytic in the upper half plane $\mathbb{C}_+$, and take into account that the following Frobenius-Perron type relationship

$$\sum_{\alpha(z_\pm,\bar{z}_\pm)=(z,\bar{z})} \varphi_{(z,\bar{z})}(\omega_\pm)J(\omega_\pm,\bar{\omega}_\pm) = \varphi_{(z,\bar{z})}(\alpha(\omega,\bar{\omega})) \tag{28}$$

holds for all $z \pm \bar{z} \neq 0 \in \mathbb{C}$, where $J(\omega_\pm,\bar{\omega}_\pm) := \det \begin{vmatrix} \frac{\partial \omega_\pm}{\partial \omega} & \frac{\partial \omega_\pm}{\partial \bar{\omega}} \\ \frac{\partial \bar{\omega}_\pm}{\partial \omega} & \frac{\partial \bar{\omega}_\pm}{\partial \bar{\omega}} \end{vmatrix}$ is the inverse to the Jacobian of the $\alpha$-mapping $(\alpha(\omega_\pm,\bar{\omega}_\pm) = \omega,\ \overline{\alpha(\omega_\pm,\bar{\omega}_\pm)} = \bar{\omega})$ at $(\omega_\pm,\bar{\omega}_\pm) \in \mathbb{C}^2$. Taking into account now that the fixed points $\xi_\alpha \in \mathbb{C}_+$, $\alpha(\xi_\alpha,\bar{\xi}_\alpha) = \xi_\alpha$ of the $\alpha$-mapping above produce via the relationship

$$d\mu(z,\bar{z}) := \operatorname{Im}\left(\varphi_{(z,\bar{z})}(\xi_\alpha)\right) d\bar{z} \wedge dz/(2i) \tag{29}$$

an invariant measure $\mu$ on the complex plane $\mathbb{C}$ with respect to this $\alpha$-mapping. Moreover, as for the mappings (25) and (26) above the fixed point $\xi_\alpha = \infty$ proves to be unique, one obtains that the invariant measure (29) is proportional to the usual Lebesgue measure on the complex plane $\mathbb{C}$, thus ensuring that the integral identity (27) is satisfied for any $h \in L_2(\mathbb{C})$. In addition, owing to the ergodicity of the mappings (25) and (26), one derives that the negative invariant subspace $H_- := \cap_{n=0}^{+\infty} T_\alpha^n(L_2(\mathbb{C}))$ is empty, giving rise for any function $g \in L_2(\mathbb{C})$ to the direct sum orthogonal decomposition (22), proving the theorem.

The obtained above result says, in particular, that the Bergman space countable decomposition $A_2(D) = \oplus_{n=0}^{+\infty} T_\alpha^n(H_+)$ holds, where $H_+ = (T_\alpha A_2(D))^\perp$ in $A_2(D)$, being finite dimensional, makes it possible to generate an infinite hierarchy of ergodic matrix valued mappings, widely used in modern quantum computer science [32, 33], neural networks [34, 35], and artificial intelligence [36-38] studies.

## 4. Conclusions

In our paper, we studied a holomorphic function decomposition in Bergman space, specified by means of degree growth on suitably separated sub-domains and having diverse applications in wavelet, co-orbit, control, and coherent state representation theories. A complicated and non-elementary open problem left to be studied consists of describing a class of Hölder continuous functions on special connected domains of the complex plane with prescribed degree growth. There has also been constructed the additive Bergman space direct sum invariant decomposition with respect to multiplicative isometry operators, defined by means of some invariant measurable mappings, especially ergodic. Two examples of additive Bergman space direct sum invariant decompositions, generated by two-dimensional ergodic Boole type mappings, are presented. Subject to this direct sum invariant decomposition problem, one plans a very interesting and important study of classifying dual pairs of domains as invariant mappings, for which the corresponding generating invariant subspaces are finite-dimensional, presenting the main interest for applications in modern quantum computer science, neural networks, and artificial intelligence studies.

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## Conflict of interest

The authors declare no conflict of interest.

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