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Research Article

On the Integration of the Higher Order Toda Lattice with a Self-Consistent Integral Type Source

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Abstract: This work presents an algorithm that uses the inverse scattering method to find a solution for the higher-order Toda lattice with a self-consistent source. The higher-order Toda lattice with an integral-type source is also a significant theoretical model belonging to very integrable systems. The problem is solved by applying the direct and inverse scattering methods to the discrete Sturm-Liouville operator, and the time dependence of the scattering data for this operator is attained. The solution to the problem is set up using the inverse scattering transform (IST) approach.

***Keywords*:** higher Toda lattice, self-consistent source, inverse scattering method, discrete Sturm-Liouville operator, one-soliton solution

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1. Introduction

The Toda lattice [1] is a clue in nonlinear one-dimensional crystal which has been used in solid state physics and presented in the following form:

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \quad n \in Z. \end{cases}$$

The Toda lattice is relevant in many fields of science, even as a model for DNA in biology [2]. Soliton solutions, which are related to the integrability of the equations, play a crucial role in the Toda lattice. The presence of such solutions is linked to the integrability of equations. The work [3] shows that all the integrable systems have soliton solutions. Many studies have focused on investigating the Toda lattice and its generalizations, from which we indicate here only [4-13].

Recent interest has been growing in investigating soliton equations and their hierarchies with self-consistent sources. These sources appear in solitary waves with alternating speeds, leading to diverse dynamics in physical models such as plasma physics, hydrodynamics, and solid-state physics [14-26]. The work [17] is dedicated to Korteweg-

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de Vries (KdV) equations containing integral types that are self-consistent. As shown in previous works, [18] have examined soliton equations with self-consistent sources in the context of the KdV equation, capillary-gravity waves, and the nonlinear Schrödinger equation, among others. Furthermore, in [26], it has been considered that other important soliton equations with a self-consistent source are the nonlinear Schrödinger equation, which describes the nonlinear interaction of ions acoustic waves in double-component homogeneous plasma with an electrostatic high-frequency wave. The associated results were collected in [27-34].

The discrete soliton equations with self-consistent sources were first studied by Liu and Zeng [35], who investigated the Darboux transformation for formulating and calculating the Toda lattice with self-consistent sources. An inverse scattering method was also developed to find solutions for the Toda lattice with self-consistent sources [36-38]. Integrability of the periodic Toda lattice and its hierarchy with a source has been shown in previous works [39-44].

In this study, we work on the higher-order Toda lattice with an integral-type source using the standard Zakharov-Shabat algorithm.

We contemplate the isospectral deformation of the L-operator by scalar products of its eigenfunctions, which transforms the nonlinear equation into a prescribed form on the right-hand side. The solution can be constructed using the inverse scattering problem for the L-operator [45]. Analogously in [46, 47], this approach may discover applications in certain models of electric transmission lines.

2. Formulation of the problem

In this part, we will provide a brief review of the statement of the problem. For this purpose, we consider the following system:

$$\begin{cases} \dot{a}_n = a_n(G_{n+1,r+1} - G_{n,r+1}) + a_n \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_{n+1}g_{n+1} - f_n g_n)d\mu, \\ \dot{b}_n = H_{n+1,r+1} - H_{n,r+1} + a_n \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_n g_{n+1} + f_{n+1}g_n)d\mu - a_{n-1} \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_n g_{n-1} + f_{n-1}g_n)d\mu, \\ a_{n-1}f_{n-1} + b_n f_n + a_n f_{n+1} = \frac{\mu+\mu^{-1}}{2} f_n, \\ a_{n-1}g_{n-1} + b_n g_n + a_n g_{n+1} = \frac{\mu+\mu^{-1}}{2} g_n, n \in Z, \end{cases} \tag{1}$$

under initial conditions

$$a_n(0) = a_n^0,\ b_n(0) = b_n^0,\ n \in Z, \tag{2}$$

where

$$G_{n,j}(t) = \sum_{s=0}^{j} c_{j-s} < \delta_n, L(t)^s \delta_n >, 0 \le j \le r+1, r \in Z_+,$$

$$H_{n,j}(t) = \sum_{s=0}^{j} 2a_n(t)c_{j-s} < \delta_{n+1}, L(t)^s \delta_n > +c_j + 10 \le j \le r+1,$$

$$\left(L(t)y\right)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, < \delta_m, \delta_n >= \begin{cases} 0, m \ne n \\ 1, m = n, \end{cases}$$

$c_1, c_2, ..., c_{r+1}$ are given arbitrary real numbers. Here and in the future, dot means the derivative with respect to time. $\{a_n^0\}_{-\infty}^{\infty}, \{b_n^0\}_{-\infty}^{\infty}$ satisfy the following properties:

1. $a_n^0 > 0,\ Imb_n^0 = 0,\ \ n \in Z,$
2. $\sum_{n=-\infty}^{\infty} |n|\left(\left|a_n^0 - \frac{1}{2}\right| + \left|b_n^0\right|\right) < \infty,$
3. The operator

$$\left(L(0)y\right)_n \equiv a_{n-1}(0)y_{n-1} + b_n(0)y_n + a_n(0)y_{n+1} \tag{3}$$

has exactly N eigenvalues

$$\lambda_k(0) = \frac{z_k(0) + z_k^{-1}(0)}{2}, k = 1, 2, ..., N,$$

which are out of the interval [−1; 1].

Varying r, we obtain the hierarchy for the Toda lattice with the integral-type source (1) that is advertised in the title of this paper. In system (1), the functional sequences of the functions $\{a_n(t)\}_{-\infty}^{\infty}, \{b_n(t)\}_{-\infty}^{\infty}, \{f_n(\mu,t)\}_{-\infty}^{\infty}$, and $\{g_n(\mu,t)\}_{-\infty}^{\infty}$ are unknown vector functions. We assume that for all $t \geq 0$ and $|\mu| = 1$ the following asymptotic properties are fulfilled:

$$\begin{aligned} &g_n(\mu,t) \sim p(\mu,t)\mu^n + q(\mu,t)\mu^{-n},\ n \to -\infty \\ &f_n(\mu,t) \sim r(\mu,t)\mu^n + s(\mu,t)\mu^{-n},\ n \to -\infty. \end{aligned} \tag{4}$$

Here $p(\mu,t)$, $q(\mu,t)$, $r(\mu,t)$, and $s(\mu,t)$ are given continuous functions in μ and t with first-order derivatives with respect to μ and satisfy Holder's condition [40] with some degree $\nu \in (0, 1]$ on $|\mu| = 1$ for all nonnegative t. Moreover, let the quantities P and Q of the form

$$\begin{aligned} &P(\mu,t) = p(\mu,t)r(\mu,t) + q(\mu^{-1},t)\ s(\mu^{-1},t), \\ &Q(\mu,t) = p(\mu,t)s(\mu,t) + q(\mu^{-1},t)\ r(\mu^{-1},t), \end{aligned} \tag{5}$$

satisfy the relations

$$P(\mu^{-1},t) = -\overline{P(\mu,t)},\ Q(\mu^{-1},t) = -\overline{Q(\mu,t)},$$

for all $|\mu| = 1$ and $t \geq 0$, where the overbar means complex conjugation.

The main aim of this work is to obtain the expressions of the solutions $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$, $\{f_n(\mu,t)\}_{-\infty}^{\infty}$, and $\{g_n(\mu,t)\}_{-\infty}^{\infty}$ of the problem (1)-(2) in the framework of inverse scattering method for the operator $L(t)$.

3. The basic facts from scattering problem

In this section, we give some basic information about the scattering theory for the operator $L(t)$. This theory was developed in the work [48].

We consider the second-order difference equation

$$\left(Ly\right)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in Z. \tag{6}$$

Here, $\{y_n\}_{-\infty}^{\infty}$ is an unknown vector and $\lambda = \frac{z + z^{-1}}{2}$ is a spectral parameter. We suppose that the sequences $\{a_n\}_{-\infty}^{\infty}, \{b_n\}_{-\infty}^{\infty}$ satisfy the conditions

$$a_n > 0,\ Imb_n = 0,\ n \in Z,$$
$$\sum_{n=-\infty}^{\infty} |n| \left(\left| a_n - \frac{1}{2} \right| + |b_n| \right) < \infty. \tag{7}$$

If condition (7) is valid, then equation (6) has Jost solutions with the asymptotics:

$$\varphi_n(z) = z^n + o(1) \ \text{ as } \ n \to \infty,\ |z| = 1,$$
$$\psi_n(z) = z^{-n} + o(1) \ \text{ as } \ n \to -\infty,\ |z| = 1. \tag{8}$$

As we know that such solutions exist, and moreover, they are identified by the asymptotic expressions (8) unique and analytically expended into the circle $|z| < 1$.

The function $\varphi_n(z)$ admits the following representation

$$\varphi_n(z) = \sum_{n'=n}^{\infty} K(n, n') z^{n'}, \tag{9}$$

where the coefficients $K(n,n')$ are independent on z, and are related to a_n and b_n by formulas

$$a_n = \frac{1}{2} \frac{K(n+1, n+1)}{K(n,n)},$$
$$b_n = \frac{1}{2} \left(\frac{K(n, n+1)}{K(n,n)} - \frac{K(n-1, n)}{K(n-1, n-1)} \right). \tag{10}$$

For $|z| = 1$, the pairs $\{\varphi_n(z), \varphi_n(z^{-1})\}$ and $\{\psi_n(z), \psi_n(z^{-1})\}$ are the pairs of linearly independent solutions of (6), therefore

$$\psi_n(z) = \alpha(z)\varphi_n(z^{-1}) + \beta(z)\varphi_n(z),$$
$$\varphi_n(z) = \alpha(z)\psi_n(z^{-1}) - \beta(z^{-1})\psi_n(z), \tag{11}$$

with

$$\alpha(z) = \frac{2}{z - z^{-1}} W\{\psi_n(z), \varphi_n(z)\}, \tag{12}$$

and

$$W\{\psi_n(z), \varphi_n(z)\} \equiv a_n(\psi_n(z)\varphi_{n+1}(z) - \psi_{n+1}(z)\varphi_n(z)).$$

The reflection coefficient is given by the formula $R(z) = -\dfrac{\beta(z^{-1})}{\alpha(z)}$ and is regular enough on the circle. The function $\alpha(z)$ is analytically expended into the circle $|z| < 1$, and inside it has a finitely many zeros z_1, z_2, ..., z_N. The points $\lambda_k = \dfrac{z_k + z_k^{-1}}{2},\ k = 1, 2, \ldots, N$ correspond to eigenvalues of the operator L. From (12) we have

$$\varphi_n^k = B_k \psi_n^k, k = 1, 2, \ldots, N, \tag{13}$$

where $\psi_n^k \equiv \psi_n(z_k)$.

The set of the quantities

$$\{R(z), z_1, z_2, \ldots, z_N, B_1, B_2, \ldots, B_N\}$$

is called the scattering data for equations (6).

The coefficients $K(n,n')$ given in representation (9) satisfy the equation of Gelfand-Levitan-Marchenko type

$$\chi(n,m) + F(n+m) + \sum_{n'=n+1}^{\infty} \chi(n,n')F(n'+m) = 0, \quad m > n \tag{14}$$

$$(K(n,n))^{-2} = 1 + F(2n) + \sum_{n'=n+1}^{\infty} \chi(n,n')F(n'+n),$$

where

$$\chi(n,m) = \frac{K(n,m)}{K(n,n)},$$
$$F(n) = \frac{1}{2\pi i}\oint_{|z|=1} R(z)z^{n-1}dz + \sum_{k=1}^{N} C_k^2 z_k^n. \tag{16}$$

Now $\{a_n\}_{-\infty}^{\infty}$ and $\{b_n\}_{-\infty}^{\infty}$ can be expressed via the scattering data by the formulas (10).

It is worthy to remark that the vectors

$$h_n^k = \frac{d}{dz}(\psi_n(z) - \beta_k \varphi_n(z))\bigg|_{z = z_k}$$

are solutions of the equations $Ly = \lambda_k y, k = 1, 2, \ldots, N$. From the equality (12), as $|z| < 1$ we deduce that

$$\varphi_n(z) \to \alpha(z)z^n \text{ as } n \to -\infty,$$

therefore,

$$h_n^k \to -\beta_k \dot{\alpha}(z_k) z_k^n \quad as \quad n \to -\infty, k = 1, 2, \ldots, N, \tag{17}$$

where $\dot{\alpha}(z_k) = \left.\frac{d\alpha(z)}{dz}\right|_{z=z_k}$. From asymptotes (8) and (17), we get $W\{h_n^k, \psi_n^k\} = \frac{\beta_k \dot{\alpha}(z_k)(z_k - z_k^{-1})}{2}$. In the future, we will need the following identity.

If $\{x_n(\lambda)\}_{-\infty}^{\infty}$ and $\{y_n(\xi)\}_{-\infty}^{\infty}$ are solutions of the equations $Lx = \lambda x$ and $Ly = \xi y$, then the identity holds:

$$(\xi - \lambda)x_n(\lambda)y_n(\xi) = W\{x_n(\lambda), y_n(\xi)\} - W\{x_{n-1}(\lambda), y_{n-1}(\xi)\}, \quad n \in Z. \tag{18}$$

4. Time evolution for $z_k(t)$

In this section, we will show the time independence of the eigenvalues $\lambda_k(t)$, $k = 1, 2, \ldots, N$ of the operator $L(t)$ as well as $z_k(t)$, $k = 1, 2, \ldots, N$.

If $\mathrm{Ker}(L(t) - \lambda)$, $\lambda \in C$ denotes the two-dimensional nullspace of $L(t) - \lambda$, then the system of equations (1) can be rewritten as follows:

$$\begin{cases} \dot{a}_n = a_n[\tilde{H}_{n+1,r+1} + \tilde{H}_{n,r+1} \\ -2(\lambda - b_{n+1})\tilde{G}_{n+1,r}] \\ -a_n \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_{n+1}(\mu,t)g_{n+1}(\mu,t) \\ -f_n(\mu,t)g_n(\mu,t))d\mu \\ \dot{b}_n = 2[a_n^2\tilde{G}_{n+1,r} - a_{n-1}^2\tilde{G}_{n-1,r} \\ +(\lambda - b_n)^2\tilde{G}_{n,r} - (\lambda - b_n)\tilde{H}_{n,r+1}] \\ -a_n \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_n(\mu,t)g_{n+1}(\mu,t) \\ +f_{n+1}(\mu,t)g_n(\mu,t))d\mu \\ +a_{n-1} \oint\limits_{|\mu|=1} \frac{1}{\mu}(f_n(\mu,t)g_{n-1}(\mu,t) \\ +f_{n-1}(\mu,t)g_n(\mu,t))d\mu, \end{cases} \tag{19}$$

where

$$\tilde{G}_{n,r}(z,t) = \sum_{j=0}^{r} \lambda^j G_{n,r-j}(t),$$

$$\tilde{H}_{n,r}(z,t) = \lambda^{r+1} + \sum_{j=0}^{r} \lambda^j H_{n,r-j}(t) - G_{n,r+1}(t)$$

$$r \in N_0, t \in R.$$

Let $\left\{V_n^k(t)\right\}_{-\infty}^{\infty}$ be the normalized eigenvector of the operator $L(t)$, associated with the eigenvalue $\lambda_k(t)$, $k = 1, 2, ..., N$, i.e.,

$$a_{n-1}V_{n-1}^k + b_n V_n^k + a_n V_{n+1}^k = \lambda_k V_n^k, n \in Z. \tag{20}$$

We differentiate identity (20) with respect to t and use (19), then multiply the resulting identity by V_n^k and summing over n from $-\infty$ to ∞, we get

$$\begin{aligned} \dot{\lambda}_k = & \sum_{n=-\infty}^{\infty} (\tilde{H}_{n,r+1} + \tilde{H}_{n-1,r+1}) \\ & \times[(\lambda_k - b_n)V_n^k(z) - a_n V_{n+1}^k(z)]V_n^k(z) \\ & -2\sum_{n=-\infty}^{\infty} (\lambda_k - b_n)[(\lambda_k - b_n)V_n^k(z) - a_n V_{n+1}^k(z)] \\ & \times \tilde{G}_{n,r}V_n^k(z) + 2\sum_{n=-\infty}^{\infty} (a_n^2\tilde{G}_{n+1,r} - a_{n-1}^2\tilde{G}_{n-1,r} \\ & +(\lambda_k - b_n)^2\tilde{G}_{n,r} - (\lambda_k - b_n)\tilde{H}_{n,r+1})(V_n^k(z))^2 \\ & +\sum_{n=-\infty}^{\infty} a_n(\tilde{H}_{n+1,r+1} + \tilde{H}_{n,r+1} - 2(\lambda_k - b_{n+1})\tilde{G}_{n+1,r})V_{n+1}^k(z)V_n^k(z) \\ & +\oint\limits_{|\mu|=1} \frac{1}{\mu}T_k(\mu,t)d\mu, \end{aligned} \tag{21}$$

where

$$
\begin{aligned}
T_k(\mu,t) = \sum_{n=-\infty}^{\infty} & (a_{n-1}(f_n(\mu,t)g_n(\mu,t) \\
& - f_{n-1}(\mu,t)g_{n-1}(\mu,t))V_{n-1}^k(z)V_n^k(z) \\
& + a_n(f_n(\mu,t)g_{n+1}(\mu,t) \\
& + f_{n+1}(\mu,t)g_n(\mu,t))\left(V_n^k(z)\right)^2 \\
& - a_{n-1}(f_n(\mu,t)g_{n-1}(\mu,t) \\
& + f_{n-1}(\mu,t)g_n(\mu,t))\left(V_n^k(z)\right)^2 \\
& + a_n(f_{n+1}(\mu,t)g_{n+1}(\mu,t) \\
& - f_n(\mu,t)g_n(\mu,t))V_{n+1}^k(z)V_n^k(z).
\end{aligned}
$$

Now, let us simplify the right-hand side of (21)

$$
\begin{aligned}
\dot{\lambda}_k = & \sum_{n=-\infty}^{\infty} \tilde{H}_{n-1,r+1}(\lambda_k - b_n)\left(V_n^k(z)\right)^2 \\
& - \sum_{n=-\infty}^{\infty} (\lambda_k - b_n)\tilde{H}_{n,r+1}(V_n^k(z))^2 \\
& - \sum_{n=-\infty}^{\infty} a_n \tilde{H}_{n-1,r+1} V_n^k(z)V_{n+1}^k(z) \\
& + \sum_{n=-\infty}^{\infty} a_n \tilde{H}_{n+1,r+1} V_n^k(z)V_{n+1}^k(z) \\
& + 2\sum_{n=-\infty}^{\infty} (\lambda_k - b_n)a_n \tilde{G}_{n,r} V_n^k(z)V_{n+1}^k(z) \\
& + 2\sum_{n=-\infty}^{\infty} a_n^2 \tilde{G}_{n+1,r}\left(V_n^k(z)\right)^2 \\
& - 2\sum_{n=-\infty}^{\infty} a_{n-1}^2 \tilde{G}_{n-1,r}\left(V_n^k(z)\right)^2 \\
& - 2\sum_{n=-\infty}^{\infty} a_n(\lambda_k - b_{n+1})\tilde{G}_{n+1,r} V_n^k(z)V_{n+1}^k(z) \\
& + \oint_{|\mu|=1} \frac{1}{\mu} T_k(\mu,t)\,d\mu.
\end{aligned}
$$

Consequently, using (20) we obtain

$$
\begin{aligned}
\dot{\lambda}_k = & 2\sum_{n=-\infty}^{\infty} a_n^2 \tilde{G}_{n+1,r}\left(V_n^k(z)\right)^2 \\
& + 2\sum_{n=-\infty}^{\infty} a_{n-1}a_{n-2}\tilde{G}_{n-1,r} V_{n-2}^k(z)V_n^k(z) \\
& - 2\sum_{n=-\infty}^{\infty} a_n(a_n V_n^k(z) + a_{n+1}V_{n+2}^k(z)) \\
& \times \tilde{G}_{n+1,r} V_n^k(z) + \oint_{|\mu|=1} \frac{1}{\mu} T_k(\mu,t)\,d\mu \\
= & 2\sum_{n=-\infty}^{\infty} a_n^2 \tilde{G}_{n+1,r}\left(V_n^k(z)\right)^2 \\
& + 2\sum_{n=-\infty}^{\infty} a_{n-2}a_{n-1}\tilde{G}_{n-1,r} V_{n-2}^k(z)V_n^k(z) \\
& - 2\sum_{n=-\infty}^{\infty} a_n^2 \tilde{G}_{n+1,r}\left(V_n^k(z)\right)^2 \\
& - 2\sum_{n=-\infty}^{\infty} a_n a_{n+1}\tilde{G}_{n+1,r} V_n^k(z)V_{n+2}^k(z) \\
& + \oint_{|\mu|=1} \frac{1}{\mu} T_k(\mu,t)\,d\mu.
\end{aligned}
$$

It follows that

$$\dot{\lambda}_k = \oint_{|\mu|=1} \frac{1}{\mu} T_k(\mu, t) d\mu. \tag{22}$$

Next, we calculate right-hand side of (22). Note that, by grouping the terms we obtain

$$T_k = \sum_{n=-\infty}^{\infty} \left[f_{n+1} V_{n+1}^k W\left\{V_n^k, g_n\right\} + g_{n+1} V_{n+1}^k W\left\{V_n^k, f_n\right\} \right] + \sum_{n=-\infty}^{\infty} \left[f_n V_{n+1}^k W\left\{V_n^k, f_n\right\} \right].$$

Denote now $\xi = \dfrac{\mu + \mu^{-1}}{2}$, then putting $W_n = W\{V_n^k, g_n\}, D_n = W\{V_n^k, f_n\}$, and using (18), we have

$$T_k = \sum_{n=-\infty}^{\infty} \left[W_n \left(f_{n+1} V_{n+1}^k + f_n V_n^k \right) \right] + D_n \left(g_{n+1} V_{n+1}^k + g_n V_n^k \right) = \frac{2}{\xi - \lambda_k} \sum_{n=-\infty}^{\infty} \left[D_n W_n - D_{n-1} W_{n-1} \right] = 0.$$

Due to (22), we get

$$\frac{d\lambda_k}{dt} = 0, k = 1, 2, \ldots, N.$$

Furthermore, using time independence of the eigenvalues $\lambda_k(t)$, we obtain

$$\frac{dz_k}{dt} = 0, k = 1, 2, \ldots, N. \tag{23}$$

5. Evolution for the scattering function

Let us consider the following system

$$(Ly)_n \equiv a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{24}$$

$$F_{n+1} - F_n = f_{n+1}(\mu) y_{n+1}(z) + f_n(\mu) y_n(z), \lambda = \frac{z + z^{-1}}{2}, n \in Z, \tag{25}$$

for unknown functions $F_n(\mu, z)$, $n \in Z$. By taking an arbitrary solution of this system, we define for all $n \in Z$,

$$S_n^0(z) = \frac{\partial y_n}{\partial t} - 2a_n(t) \tilde{G}_{n,r}(\lambda, t) y_{n+1} - \tilde{H}_{n,r+1}(\lambda, t) y_n + \oint_{|\mu|=1} \frac{1}{\mu} g_n(\mu) F_n(\mu, z) d\mu \tag{26}$$

and

$$S_n(\mu, z) = a_n \left(f_{n+1}(\mu) y_n(z) - f_n(\mu) y_{n+1}(z) \right) + a_{n-1} \left(f_n(\mu) y_{n-1}(z) - f_{n-1}(\mu) y_n(z) \right) + \left(\lambda - \xi \right) F_n(\mu, z),$$

where $\xi = \dfrac{\mu + \mu^{-1}}{2}$. Note that, according to (18), $S_n(\mu, z) \equiv 0$, $n \in Z$.

Now, we determine $(L - \lambda) S_n^0(z)$. For this, we introduce the following notations

$$P_{2r+2} = 2a_n(t)\tilde{G}_{n,r}(\lambda,t)S^+ - \tilde{H}_{n,r+1}(\lambda,t),$$
$$\Omega_n = \oint_{|\mu|=1} \frac{1}{\mu} g_n(\mu) F_n(\mu,z)\, d\mu. \tag{27}$$

Here S^+ is shift operator, i.e., $(S^{\pm} f)(n) = f_{n\pm1}$.

$$\begin{aligned} LS_n^0(z) - \lambda S_n^0(z) &= L(\dot{y}_n - P_{2r+2}y_n + \Omega_n) - \lambda(\dot{y}_n - P_{2r+2}y_n + \Omega_n) \\ &= L\dot{y}_n - \lambda\dot{y}_n - (L-\lambda)P_{2r+2}y_n + (L-\lambda)\Omega_n. \end{aligned}$$

By using the equality $L\dot{y}_n - \lambda\dot{y}_n = -\dot{L}y_n$ and notations (27), we obtain

$$\begin{aligned} &LS_n^0(z) - \lambda S_n^0(z) \\ &= -\dot{L}y_n - 2a_{n-1}^2\tilde{G}_{n-1,r}y_n + a_{n-1}\tilde{H}_{n-1,r+1}y_{n-1} \\ &\quad + a_{n-1}\oint_{|\mu|=1}\frac{1}{\mu}g_{n-1}(\mu)F_{n-1}(\mu,z)d\mu \\ &\quad - 2a_nb_n\tilde{G}_{n,r}y_{n+1} + b_n\tilde{H}_{n,r+1}y_n \\ &\quad + b_n\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n(\mu,z)d\mu \\ &\quad - 2a_na_{n+1}\tilde{G}_{n+1,r}y_{n+2} + a_n\tilde{H}_{n+1,r+1}y_{n+1} \\ &\quad + a_n\oint_{|\mu|=1}\frac{1}{\mu}g_{n+1}(\mu)F_{n+1}(\mu,z)d\mu) \\ &\quad + 2\lambda a_n\tilde{G}_{n,r}y_{n+1} - \lambda\tilde{H}_{n,r+1}y_n \\ &\quad - \lambda\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n(\mu,z)d\mu. \end{aligned}$$

According to (24), (19) and the following equality

$$\begin{aligned} \dot{L}y_n &= \dot{a}_{n-1}y_{n-1} + \dot{b}_ny_n + \dot{a}_ny_{n+1} \\ &= \dot{a}_{n-1}\left(\frac{\lambda - b_n}{a_{n-1}}y_n - \frac{a_n}{a_{n-1}}y_{n+1}\right) + \dot{b}_ny_n + \dot{a}_ny_{n+1}, \end{aligned}$$

we get

$$
\begin{aligned}
LS_n^0(z)-\lambda S_n^0(z) &= \Big[-a_{n-1}(\tilde{H}_{n,r+1}+\tilde{H}_{n-1,r+1}-2(\lambda-b_n)\tilde{G}_{n,r}) \\
&- a_{n-1}\oint_{|\mu|=1}\frac{1}{\mu}(f_n(\mu,t)g_n(\mu,t)-f_{n-1}(\mu,t)g_{n-1}(\mu,t))d\mu\Big] \\
&\times\left(\frac{\lambda-b_n}{a_{n-1}}y_n-\frac{a_n}{a_{n-1}}y_{n+1}\right)-[2(a_n^2\tilde{G}_{n+1,r}-a_{n-1}^2\tilde{G}_{n-1,r}+(\lambda-b_n)^2\tilde{G}_{n,r} \\
&-(\lambda-b_n)\tilde{H}_{n,r+1})+a_n\oint_{|\mu|=1}\frac{1}{\mu}(f_n(\mu,t)g_{n+1}(\mu,t)+f_{n+1}(\mu,t)g_n(\mu,t))d\mu \\
&-[a_n(\tilde{H}_{n+1,r+1}+\tilde{H}_{n,r+1}-2(\lambda-b_{n+1})\tilde{G}_{n+1,r})+a_n\oint_{|\mu|=1}\frac{1}{\mu}(f_{n+1}(\mu,t)g_{n+1}(\mu,t) \\
&-f_n(\mu,t)g_n(\mu,t))d\mu]y_{n+1}-[a_n\tilde{H}_{n-1,r+1}+2a_nb_n\tilde{G}_{n,r}+2a_n\tilde{G}_{n+1,r}(\lambda-b_{n+1}) \\
&-a_n\tilde{H}_{n+1,r+1}-2\lambda a_n\tilde{G}_{n,r}]y_{n+1}-[2a_{n-1}^2\tilde{G}_{n-1,r}-(\lambda-b_n)\tilde{H}_{n-1,r+1}-b_n\tilde{H}_{n,r+1} \\
&-2a_n^2\tilde{G}_{n+1,r}+\lambda\tilde{H}_{n,r+1}]y_n+a_{n-1}\oint_{|\mu|=1}\frac{1}{\mu}g_{n-1}(\mu)F_{n-1}(\mu,z)d\mu \\
&+b_n\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n(\mu,z)d\mu+a_n\oint_{|\mu|=1}\frac{1}{\mu}g_{n+1}(\mu)F_{n+1}(\mu,z)d\mu-\lambda\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n(\mu,z)d\mu.
\end{aligned}
$$

After a simple simplification, on the right-hand side of the last equality, we derive

$$
\begin{aligned}
LS_n^0(z)-\lambda S_n^0(z) = \oint_{|\mu|=1}\frac{1}{\mu}[&-W\{y_{n-1},f_{n-1}(\mu,t)\}g_n(\mu,t)-W\{y_n,f_n(\mu,t)\}g_n(\mu,t) \\
&+a_{n-1}y_{n-1}f_{n-1}(\mu,t)g_{n-1}(\mu,t)+a_{n-1}y_nf_n(\mu,t)g_{n-1}(\mu,t)-a_ny_nf_n(\mu,t)g_{n+1}(\mu,t) \\
&-a_ny_nf_n(\mu,t)g_{n+1}(\mu,t)-a_ny_{n+1}f_{n+1}(\mu,t)g_{n+1}(\mu,t)+a_{n-1}g_{n-1}(\mu)F_{n-1}(\mu,z) \\
&+b_ng_n(\mu)F_n(\mu,z)+a_ng_{n+1}(\mu)F_{n+1}(\mu,z)-\lambda g_n(\mu)F_n(\mu,z)]d\mu.
\end{aligned}
$$

Taking into account (25) and

$$
W\{f_n(\mu),g_n(\mu)\}-W\{f_{n-1}(\mu),g_{n-1}(\mu)\}=0,\quad n\in Z,
$$

we obtain

$$
(L-\lambda)S_n^0(z)=-\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)S_n(\mu,z)d\mu,\ \ n\in Z.
$$

It follows that

$$
(L-\lambda)S_n^0(z)=0,\ \ n\in Z. \tag{28}
$$

We denote by $\phi_n(z, t)$ and $\psi_n(z, t)$ the Jost solutions of the equation (24) which satisfy condition (8). Setting $y_n^+\equiv\phi_n(z)$ and $y_n^-\equiv\psi_n(z)$ in (25), we define

$$
\begin{aligned}
F_n^-(\mu,z) &= f_n(\mu)\psi_n(z)+2\sum_{j=-\infty}^{n-1}f_j(\mu)\psi_j(z), \\
F_n^+(\mu,z) &= -f_n(\mu)\phi_n(z)-2\sum_{j=n+1}^{\infty}f_j(\mu)\phi_j(z).
\end{aligned} \tag{29}
$$

It is easy to check that these equalities determine the functions $F_n^-(\mu,z)$ and $F_n^+(\mu,z)$ at any $|\mu|=1$ to be analytical functions of the parameter z on the $|z|<1$. Taking into account (18) and (29), the functions $F_n^-(\mu,z)$ and $F_n^+(\mu,z)$ at any value of the parameter z inside the circle $|z| < 1$ can be represented as

$$F_n^-(\mu,z) = -\frac{2\mu}{z-z^{-1}}\left(\frac{1}{\mu-z}-\frac{1}{\mu-z^{-1}}\right)\times\left(W\{f_n(\mu),\psi_n(z)\}+W\{f_{n-1}(\mu),\psi_{n-1}(z)\}\right), \tag{30}$$

and

$$F_n^+(\mu,z) = -\frac{2\mu}{z-z^{-1}}\left(\frac{1}{\mu-z}-\frac{1}{\mu-z^{-1}}\right)\times\left(W\{f_n(\mu),\varphi_n(z)\}+W\{f_{n-1}(\mu),\varphi_{n-1}(z)\}\right). \tag{31}$$

The right-hand side of the expressions (30) and (31) are also meaningful at any $|z| = 1$ of the parameter z satisfying the conditions $z \neq \mu$ and $z \neq \bar{\mu}$.

Next, we introduce $S_n^{0-}(z)$ and $S_n^{0+}(z)$ for $|z| < 1$ as follows:

$$S_n^{0-}(z) = \frac{\partial \psi_n}{\partial t} + 2a_n\tilde{G}_{n,r}\psi_{n+1} - \tilde{H}_{n,r+1}\psi_n + \oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n^-(\mu,z)d\mu, \tag{32}$$

$$S_n^{0+}(z) = \frac{\partial \phi_n}{\partial t} + 2a_n\tilde{G}_{n,r}\phi_{n+1} - \tilde{H}_{n,r+1}\phi_n + \oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n^+(\mu,z)d\mu, \tag{33}$$

In accordance with the aforesaid, the quantities $S_n^{0-}(z) = S_n^{0-}(z)$ and $S_n^{0+}(z) = S_n^{0+}(z)$ thus determined depend analytically on the parameter z in the circle $|z| < 1$. However, since at any $|z| = 1$ and $z \neq \pm 1$ the functions $F_n^-(\mu,z)$ and $F_n^+(\mu,z)$ have singularities at the points $\mu = z$ and $\mu = z - 1$, the limiting values of the functions $S_n^{0-}(z)$ and $S_n^{0+}(z)$ as $|z| \to 1$ must to be determined more accurately. To do it, substituting the right-hand sides of equalities (30) and (31) into expressions (32) and (33), we get that at $|z| = 1$ the following equalities are valid:

$$S_n^{0-}(z) = \frac{\partial \psi_n}{\partial t} + 2a_n\tilde{G}_{n,r}\psi_{n+1} - \tilde{H}_{n,r+1}\psi_n + \ v.p.\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n^-(\mu,z)d\mu + \ \phi_1^-(z)\psi_n(z^{-1}) + \phi_2^-(z)\psi_n(z), \tag{34}$$

$$S_n^{0+}(z) = \frac{\partial \phi_n}{\partial t} + 2a_n\tilde{G}_{n,r}\phi_{n+1} - \tilde{H}_{n,r+1}\phi_n + \ v.p.\oint_{|\mu|=1}\frac{1}{\mu}g_n(\mu)F_n^+(\mu,z)d\mu + \ \phi_1^+(z)\varphi_n(z^{-1}) + \phi_2^+(z)\varphi_n(z), \tag{35}$$

where *v.p.* means that integrals are taken as the principal value, and quantities $\phi_1^-(z), \phi_2^-(z), \phi_1^+(z)$ and $\phi_2^+(z)$ are determined by expressions

$$\begin{aligned}\phi_1^-(z) &= 2\pi\ i(p(z)r(z) + q(z^{-1})s(z^{-1})),\\ \phi_2^-(z) &= 2\pi\ i(q(z)r(z) + p(z^{-1})s(z^{-1}))\end{aligned} \tag{36}$$

and

$$\begin{aligned}\phi_1^+(z) &= -2\pi\ i(a(z)c(z) + b(z^{-1})d(z^{-1})),\\ \phi_2^+(z) &= -2\pi\ i(b(z)c(z) + a(z^{-1})d(z^{-1})).\end{aligned} \tag{37}$$

Due to (5), we have

$$\phi_1^-(z) = 2\pi iP(z), \phi_2^-(z) = 2\pi iQ(z),$$

$$\phi_1^+(z) = -2\pi\ i(P(z)\beta^2(z^{-1}) + Q(z)\alpha(z)\beta(z^{-1}) + Q(z^{-1})\alpha(z)\beta(z^{-1}) + P(z^{-1})\alpha^2(z))$$

and

$$\phi_2^+(z) = -2\pi i(P(z)\alpha(z^{-1})\beta(z^{-1}) + Q(z^{-1})\alpha(z)\alpha(z^{-1}) + Q(z^{-1})\beta(z)\beta(z^{-1}) + P(z^{-1})\alpha(z)\beta(z)).$$

In the next step, we will find the asymptotes of the quantities

$$\phi_-^n = v.p.\oint_{|\mu|=1} \frac{1}{\mu} g_n(\mu) F_n^-(\mu, z) d\mu \text{ as } n \to -\infty$$

and

$$\phi_+^n = v.p.\oint_{|\mu|=1} \frac{1}{\mu} g_n(\mu) F_n^+(\mu, z) d\mu \text{ as } n \to \infty.$$

Using expression (29), we find that the following asymptotes is valid:

$$\phi_-^n(z) \underset{n\to-\infty}{\sim} C_1^-(z) z^{-n} + C_2^-(z) z^n,$$
$$\phi_+^n(z) \underset{n\to\infty}{\sim} C_1^+(z) z^n + C_2^+(z) z^{-n},$$

where

$$C_1^-(z) = \frac{1}{z^2-1} v.p. \oint_{|\mu|=1} \left[\frac{q(\mu)r(\mu)(\mu+z)(\mu z-1)}{\mu(\mu-z)} + \frac{p(\mu)s(\mu)(\mu-z)(\mu z+1)}{\mu(\mu-z^{-1})} \right] d\mu,$$
$$C_2^-(z) = -2\pi P(z) \tag{38}$$

and

$$C_1^+(z) = -\frac{1}{z^2-1} v.p.\oint_{|\mu|=1} \left[\frac{b(\mu)c(\mu)(\mu+z)(\mu z-1)}{\mu(\mu-z)} + \frac{a(\mu)d(\mu)(\mu-z)(\mu z+1)}{\mu(\mu-z^{-1})} \right] d\mu$$
$$C_2^+(z) = 2\pi i[a(z)c(z) + b(z^{-1})d(z^{-1})]. \tag{39}$$

Taking into account of (28), (34), and (35), we obtain the equalities

$$S_n^{0-}(z) = \left(\tilde{g}_r(z,0)z^{-1} - \tilde{h}_{r+1}(z,0) + K^-(z)\right) \times \psi_n(z) + K_0^-(z)\psi_n(z^{-1}),$$
$$S_n^{0+}(z) = \left(\tilde{g}_r(z,0)z - \tilde{h}_{r+1}(z,0) + K^+(z)\right) \times \phi_n(z) + K_0^+(z)\phi_n(z^{-1}),$$

where

$$K^-(z) = \phi_2^-(z) + C_1^-(z),$$
$$K_0^-(z) = \phi_1^-(z) + C_2^-(z), \tag{40}$$

$$K^+(z) = \phi_2^+(z) + C_1^+(z),$$
$$K_0^+(z) = \phi_1^+(z) + C_2^+(z), \tag{41}$$

Here, $\tilde{g}_r(z,0)$ and $\tilde{h}_{r+1}(z,0)$ are polynomial in variable z. Note that in accordance with (36), (37), (38), and (39) we deduce

$$K_0^-(z) = K_0^+(z) \equiv 0,$$

in that

$$S_n^{0-}(z) = \left(\tilde{g}_r(z,0)z^{-1} - \tilde{h}_{r+1}(z,0) + K^-(z)\right)\psi_n(z) \tag{42}$$

and

$$S_n^{0+}(z) = \left(\tilde{g}_r(z,0)z - \tilde{h}_{r+1}(z,0) + K^+(z)\right)\phi_n(z), n \in Z. \tag{43}$$

Due to (38)-(41) one can easily be convinced that the quantities $K^-(z)$and $K^+(z)$ admit analytical continuation in z into the circle $|z| < 1$. Moreover, it is easy to show that at $|z| < 1$, the equalities

$$K^-(z) = \frac{1}{z^2-1}\oint_{|\mu|=1}\left[\frac{q(\mu)r(\mu)(\mu+z)(\mu z-1)}{\mu(\mu-z)} + \frac{p(\mu)s(\mu)(\mu-z)(\mu z+1)}{\mu(\mu-z^{-1})}\right]d\mu \tag{44}$$

and

$$K^+(z) = -\frac{1}{z^2-1}\oint_{|\mu|=1}\left[\frac{b(\mu)c(\mu)(\mu+z)(\mu z-1)}{\mu(\mu-z)} + \frac{a(\mu)d(\mu)(\mu-z)(\mu z+1)}{\mu(\mu-z^{-1})}\right]d\mu \tag{45}$$

are fulfilled. Now, we assume

$$G_n(z) = S_n^{0+}(z) - \alpha(z)S_n^{0-}(z^{-1}) + \beta(z^{-1})S_n^{0-}(z).$$

In view of (42) and (43), we deduce

$$\begin{aligned} G_n(z) = &\left(\tilde{g}_r(z,0)z - \tilde{g}_r(z^{-1},0)z - \tilde{h}_{r+1}(z,0) + \tilde{h}_{r+1}(z^{-1},0)\right)\alpha(z)\psi_n(z^{-1}) \\ &+ \left(K^+(z) - K^-(z^{-1})\right)\alpha(z)\psi_n(z^{-1}) - \left(\tilde{g}_r(z,0)z - \tilde{g}_r(z,0)z^{-1} + K^+(z) - K^-(z)\right)\times\ t\beta(z^{-1})\psi_n(z). \end{aligned} \tag{46}$$

On the other hand, from expressions (11), (30), and (31), at any $|z| = 1$ and $|\mu| = 1$ the following equality

$$F_n^+(\mu,z) - \alpha(z)F_n^-(\mu,z^{-1}) + \beta(z^{-1})F_n^-(\mu,z) = 0,$$

is valid. Then, according to (11), (25), (36), and (37) we get

$$\begin{aligned} &\phi_1^+(z)\varphi_n(z^{-1}) + \phi_2^+(z)\varphi_n(z) - \alpha(z)(\phi_1^-(z^{-1})\psi_n(z) + \phi_2^-(z^{-1})\psi_n(z^{-1})) + \beta(z^{-1})(\phi_1^-(z)\psi_n(z^{-1}) + \phi_2^-(z)\psi_n(z)) \\ &= -4\pi i\alpha(z)(P(z^{-1})\psi_n(z) + Q(z)\psi_n(z^{-1})). \end{aligned}$$

By virtue of these equalities and using (11), (34), and (35), we deduce

$$G_n(z,t) = \left[\frac{\partial\alpha(z,t)}{\partial t} - 4\pi i Q(z,t)\alpha(z,t)\right]\psi_n(z^{-1},t) - \left[\frac{\partial\beta(z^{-1},t)}{\partial t} + 4\pi P(z^{-1},t)\alpha(z,t)\right]\psi_n(z,t).$$

Comparing this equality with (46), we obtain

$$\frac{\partial\alpha(z,t)}{\partial t}=[4\pi iQ(z,t)+\tilde{g}_r(z,0)z-\tilde{g}_r(z^{-1},0)z-\tilde{h}_{r+1}(z,0)+\tilde{h}_{r+1}(z^{-1},0)+K^+(z)-K^-(z^{-1})]\alpha(z,t)$$

and

$$\frac{\partial\beta(z^{-1},t)}{\partial t}=[\tilde{g}_r(z,0)(z-z^{-1})+K^+(z)-K^-(z)]\beta(z^{-1},t)-4\pi P(z^{-1},t)\alpha(z,t).$$

Finally, from $R(z,t)=-\dfrac{\beta(z^{-1},t)}{\alpha(z,t)}$, (44) and (45) we obtain that at any $|z|=1$, the following equality

$$\begin{aligned}\frac{\partial R(z,t)}{\partial t}&=[\tilde{g}_r(z,0)(z-z^{-1})+\frac{1}{z^2-1}v.p.\oint_{|\mu|=1}D(\mu,t)d\mu]R(z,t)\\&+2\pi\, i(Q(z,t)+Q(z^{-1},t))R(z,t)+4\pi\, iP(z^{-1})\end{aligned}\qquad(47)$$

are fulfilled, where

$$D(\mu,t)=\left(q(\mu,t)r(\mu,t)+p(\mu,t)s(\mu,t)\right)\times\left[\frac{(\mu+z)(\mu z-1)}{\mu(\mu-z)}+\frac{(\mu-z)(\mu z+1)}{\mu(\mu-z^{-1})}\right].$$

6. Time dependence of B_n

We introduce

$$G_n^k=S_n^{0+}(z_k)-B_kS_n^{0-}(z_k),k=1,2,...,N,$$

where the quantity $B_k=B_k(t)$ is determined by (13). Using (42) and (43), we get

$$G_n^k=[\tilde{g}_r(z_k,0)(z_k-z_k^{-1})+K^+(z_k)-K^-(z_k)]B_k\psi_n(z_k),n\in Z.\qquad(48)$$

In (48), the quantities $K^-(zk)$ and $K^+(zk)$ are determined from (44) and (45).

On the other hand, taking into account formulas (30), (31), and the orthogonality relation of $L(t)$, we deduce the following equality

$$F_n^+(\mu,z_k)-B_kF_n^-(\mu,z_k)=-2\sum_{j=-\infty}^{\infty}f_j(\mu)\psi_j(z_k)=0,$$

at any $|\mu|=1$.

In consequence, by virtue of (26), we obtain that

$$G_n^k=\frac{dB_k(t)}{dt}\psi_n(z_k),\quad n\in Z,\quad k=1,2,...,N.$$

Comparing this equality with (48), we get

$$\frac{dB_k(t)}{dt}=[\tilde{g}_r(z_k,0)z_k+K^+(z_k)-\tilde{g}_r(z_k,0)z_k^{-1}-K^-(z_k)]B_k(t),k=1,2,...,N,$$

where quantities $K^{-}(zk, t)$ and $K^{+}(zk, t)$ are determined from (44) and (45). Thus

$$
\begin{aligned}
\frac{dB_k(t)}{dt} &= [(z_k - z_k^{-1})\tilde{g}_r(z_k, 0) - \frac{1}{z_k^2 - 1}\oint_{|\mu|=1} \frac{(\mu + z_k)(\mu z_k - 1)}{\mu(\mu - z_k)} \\
&\times (b(\mu,t)c(\mu,t) + q(\mu,t)r(\mu,t))d\mu - \frac{1}{z_k^2 - 1}\oint_{|\mu|=1} \frac{(\mu - z_k)(\mu z_k + 1)}{\mu(\mu - z_k^{-1})} \\
&k = 1, 2, ..., N,
\end{aligned} \tag{49}
$$

where

$$
\begin{aligned}
a(\mu,t) &= p(\mu,t)\beta(\mu^{-1},t) + q(\mu,t)\alpha(\mu,t), \\
b(\mu,t) &= p(\mu,t)\alpha(\mu^{-1},t) + q(\mu,t)\beta(\mu,t), \\
c(\mu,t) &= r(\mu,t)\beta(\mu^{-1},t) + s(\mu,t)\alpha(\mu,t), \\
d(\mu,t) &= r(\mu,t)\alpha(\mu^{-1},t) + s(\mu,t)\beta(\mu,t)
\end{aligned} \tag{50}
$$

and

$$
\alpha(\mu,t) = \prod_{j=1}^{N} \left| \frac{\mu - z_j}{\mu z_j - 1} \right| \times \exp\left\{ \frac{1}{4\pi\, i} \int_{|\zeta|=1} \ln\left(1 - |R(\zeta,t)|^2\right) \frac{\mu + \zeta}{\mu - \zeta} \frac{d\zeta}{\zeta} \right\}, \beta(\mu^{-1},t) = -R(\mu,t)\alpha(\mu,t).
$$

As consequence of all the previous arguments, we have thereby proved the following assertion.

Theorem 1. If the functions $a_n(t)$, $b_n(t)$, $f_n(\mu,t)$, $g_n(\mu,t)$, $n \in Z$ are solutions of the problem (1)-(3), then the scattering data of the operator

$$
\left(L(t)y\right)_n \equiv a_{n-1}(t)y_{n-1} + b_n(t)y_n + a_n(t)y_{n+1},
$$

by relations (23), (47), and (49).

7. Conclusion

The obtained results completely define the time evolution of the spectral data, which allows us to solve the problem (1-4) by using the following algorithm: Let us give a_n^0 and b_n^0, $n \in Z$.

1. With the given a_n^0 and b_n^0, , $n \in Z$, we find scattering data

$$
\{R(z), z_1, z_2, ..., z_N, B_1, B_2, ..., B_N\} \text{ for}
$$

$$
\left(L(0)y\right)_n;
$$

2. According to the results of Theorem 1, we obtain the time evolution of the scattering data

$$
\{R(z,t), z_1(t), z_2(t), ..., z_N(t), B_1(t), B_2(t), ..., B_N(t)\}
$$

for $\left(L(t)y\right)_n$;

3. With the obtained scattering data, we uniquely define the function $F(n, t)$ from the equality (16);
4. Substituting $F(n,t)$ into the equations (14) and (15), and solving the resulting system we define $\chi(n, m, t)$ then the potentials $a_n(t)$ and $b_n(t)$ can be obtain via the formulas (10);
5. Solving the equation (6), we will construct the eigenfunctions $\{f_n(\mu,t)\}_{-\infty}^{\infty}$ and $\{g_n(\mu,t)\}_{-\infty}^{\infty}, n \in Z$.

The results obtained play an important role in the theory of solitons, and they can be used in some models of a special type of transmission line.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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