Contemporary Mathematics
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Research Article

Comparative Analysis of Models of Gene and Neural Networks

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Received: 31 January 2023; **Revised:** 29 March 2023; **Accepted:** 3 April 2023

Abstract: In the language of mathematics, the method of cognition of the surrounding world in which the description of the object is carried out the name is mathematical modeling. The study of the model is carried out using certain mathematical methods. The systems of the ordinary differential equations modeling artificial neuronal networks and the systems modeling the gene regulatory networks are considered. The one system consists of a sigmoidal function which depends on linear combinations of the arguments minus the linear part. The other system consists of a sigmoidal function which depends on the hyperbolic tangent function. The linear combinations and hyperbolic tangent functions of the arguments are described by one regulatory matrix. For the three-dimensional cases, two types of matrices are considered and the behavior of the solutions of the system is analyzed. The attracting sets are constructed for several cases. Illustrative examples are provided. The list of references consists of 19 items.

***Keywords*:** gene regulatory network, artificial neural network, chaotic solution, periodic solution, Lyapunov exponents

MSC: 34A34, 34D45, 92B20, 92D15

1. Introduction

Complex regulatory networks are being explored in many areas of science, including biochemistry, biology [1-2], ecology, and engineering. Gene regulatory network (GRN in short) is a complex dynamical system that is present in living organisms and which is constantly changing their states responding to fluctuations in their environment [3]. For a complete description of gene networks, it is necessary to analyze the processes occurring in them at the level of the whole organism. In this case, it is possible to describe gene networks, some parts of which are distributed over various large compartments of their organism, such as organs and tissues. In many cases, it is possible to determine the direction of processes within a specific fragment of the gene network. The main approaches to the description of gene networks and modeling their dynamics are a logical description; a description of the gene network using a system of nonlinear differential equations [3]; stochastic gene network models; graph theory [1], Boolean modeling [1]. Nonlinear ordinary differential equations are the most-widespread formalism for modeling genetic regulatory networks [4-7].

An artificial neural network (ANN in short) is a mathematical model created in the likeness of biological neural networks [8]. Similar to a natural analog, an artificial neural network consists of neurons and synapses [9]. Neural networks are used to solve many problems: recognition and generation of images (face identification in video

DOI: https://doi.org/10.37256/cm.4220232404

surveillance systems); speech and language (language for chat-bots and service robots); weather prediction; medical diagnosis [9-10]; business fields [11-12]; traffic monitoring systems [13].

In our paper, we use nonlinear ordinary differential equations to model the GRN and ANN. Our goal is to describe the behavior of the systems and to compare the results of GRN and ANN. In our previous papers on GRN networks multiple results on attractors and their properties were obtained. By comparison with models of neuronal networks we wish to show that similar results can be presented for neuronal networks. Our consideration is geometrical. The main intent is to use the 2D and 3D projections on different subspaces, to construct the graphs of systems solutions. Visualizations are provided. The dynamics of Lyapunov exponents are shown. Calculations and visualizations are performed using Wolfram Mathematics.

2. Gene regulatory network

Consider the three-dimensional system

$$\begin{cases} x_1' = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_3-\theta_1)}} - v_1x_1, \\ x_2' = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3-\theta_2)}} - v_2x_2, \\ x_3' = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3-\theta_3)}} - v_3x_3. \end{cases} \tag{1}$$

In the context of GRN theory, this system describes the three-element network. The link between any two elements x_i and x_j is associated with the element w_{ij} of the regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}.$$

Positivity of w_{ij} means the activation of x_i by x_j, negativity means inhibition, and zero value is for no relation. System (1) was studied, in particular, in the paper [14].

$$\begin{cases} v_1x_1 = \dfrac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+w_{13}x_3-\theta_1)}}, \\ v_2x_2 = \dfrac{1}{1+e^{-\mu_2(w_{21}x_1+w_{22}x_2+w_{23}x_3-\theta_2)}}, \\ v_3x_3 = \dfrac{1}{1+e^{-\mu_3(w_{31}x_1+w_{32}x_2+w_{33}x_3-\theta_3)}}. \end{cases} \tag{2}$$

All critical points are located in the open parallelepiped

$$\left\{(x_1, x_2, x_3) : 0 < x_1 < \frac{1}{v_1},\ 0 < x_2 < \frac{1}{v_2},\ 0 < x_3 < \frac{1}{v_3}\right\} \tag{3}$$

2.1 *An example of the system (1) with a periodic solution*

Let the coefficient matrix in (1) be

$$W = \begin{pmatrix} 2.5 & -1.5 & 0 \\ 4 & 2.5 & 0 \\ 0 & 0 & 1.2 \end{pmatrix} \tag{4}$$

and $v_1 = v_2 = v_3 = 1$; $\mu_1 = 2.3$; $\mu_2 = 1.9$; $\mu_3 = 1$; $\theta_1 = 0.5$; $\theta_2 = 2.5$; $\theta_3 = 1$.

There is one critical point (0.320154; 0.418536; 0.36235). The characteristic numbers are λ_1 = -0.72 and $\lambda_{2,3}$ = 0.2 ± 1.18i. The type of critical point is an unstable saddle-focus. The nullclines of the system (1) and the stable periodic solution are depicted in Figure 1 and Figure 2. The graphs of $x_i(t)$, i = 1, 2, 3 of the system (1) with the regulatory matrix (4) are depicted in Figure 3.

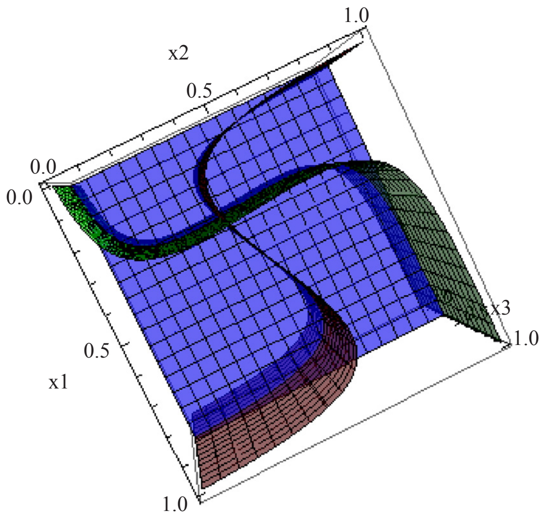

Figure 1. The nullclines of the system (1) with the regulatory matrix (4).

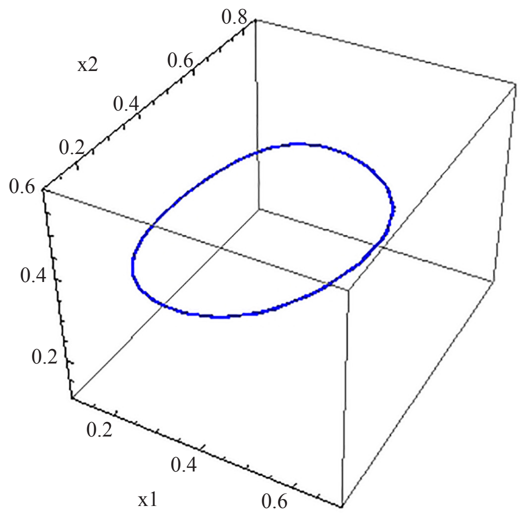

Figure 2. The periodic solution of the system (1) with the regulatory matrix (4).

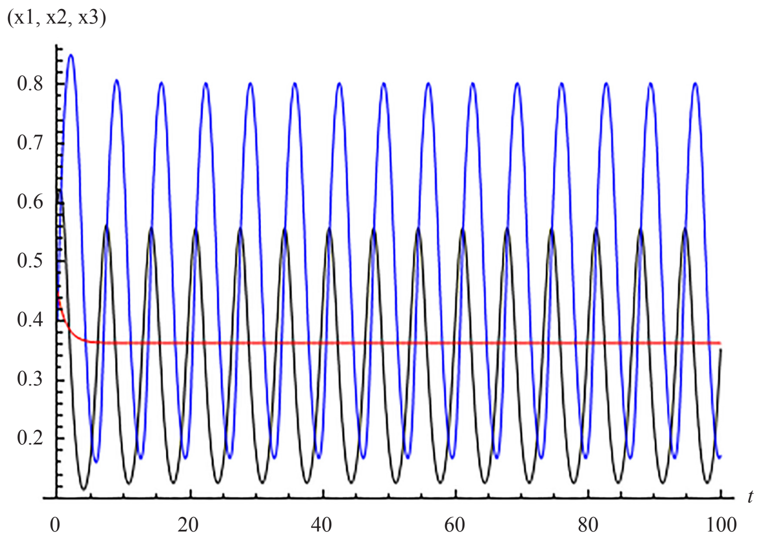

Figure 3. The graphs of $x_i(t)$, $i = 1, 2, 3$ of the system (1) with the regulatory matrix (4).

Similar systems are considered in paper [14].

The dynamics of Lyapunov exponents ($LE_1 = 0$, $LE_2 = -0.3166$; $LE_3 = -0.7227$) are shown in Figure 4. The following set of LEs characterizes the stable limit cycle.

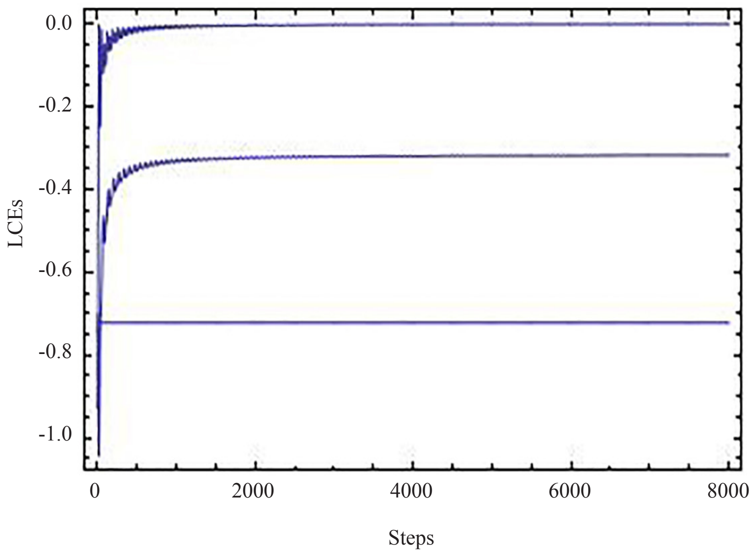

Figure 4. The dynamics of Lyapunov exponents.

The sum of Lyapunov exponents of the system (1) with the regulatory matrix (4) is negative that is why it is a dissipative system.

2.2 *An example of the system (2) with a chaotic solution*

Under chaos in ancient Greek mythology understood the pre-life confusion. Greek "chaos" is the infinite first everyday mass, which subsequently gave rise to all the existing. Physicists call this science-"nonlinear dynamics", mathematicians-"chaos theory", all the rest-"nonlinear science".

Chaos is a multifaceted phenomenon that is not easily classified or identified. There is no universally accepted definition for chaos, but the following characteristics are nearly always displayed by the solutions of chaotic systems [15].

There are several characteristics that identify the behavior of chaotic systems [16]. Usually to identify a chaotic system scientists use the method of Lyapunov exponents [16]. A 3D dynamical system is chaotic if it has one positive Lyapunov exponent (LE in short) [17]. Also, a system is said to be dissipative if the sum of all Lyapunov exponents of the system (1) is negative [18].

Consider the system (1), where $v_1 = 0.65$, $v_2 = 0.42$, $v_3 = 0.1$, $\mu_1 = \mu_2 = 7$, $\mu_3 = 13$, $\theta_1 = 0.5$, $\theta_2 = 0.3$, $\theta_3 = 0.7$.

The regulatory matrix of the system (1) is

$$W = \begin{pmatrix} 0 & 1 & -5.63 \\ 1 & 0 & 0.133 \\ 1 & 0.02 & 0.03 \end{pmatrix} \tag{5}$$

The initial conditions are

$$x_1(1) = 0.2;\ x_2(1) = 1.3;\ x_3(1) = 0.4. \tag{6}$$

There is one critical point. The characteristic equation for critical point (0.370688; 1.59227; 0.223125) is $-\lambda^3 + A\lambda^2 + B\lambda + C = 0$, where $A = -1.16149$; $B = -0.428566$; $C = -0.689604$. Solving the equation, we have $\lambda_1 = -1.257$; $\lambda_{2,3} = 0.0477516 \pm 0.739143i$. The type of critical point is an unstable saddle-focus. The nullclines of the system (1) with the regulatory matrix (4) and the chaotic attractor of the system (1) with the regulatory matrix (4) are depicted in Figure 5 and Figure 6. The graphs of $x_i(t)$, $i = 1, 2, 3$ of the system (1) with the regulatory matrix (4) are shown in Figure 7.

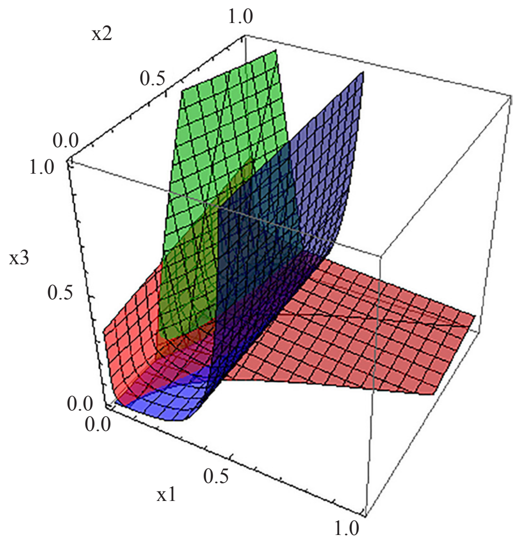

Figure 5. The nullclines of the system (1) with the regulatory matrix (4).

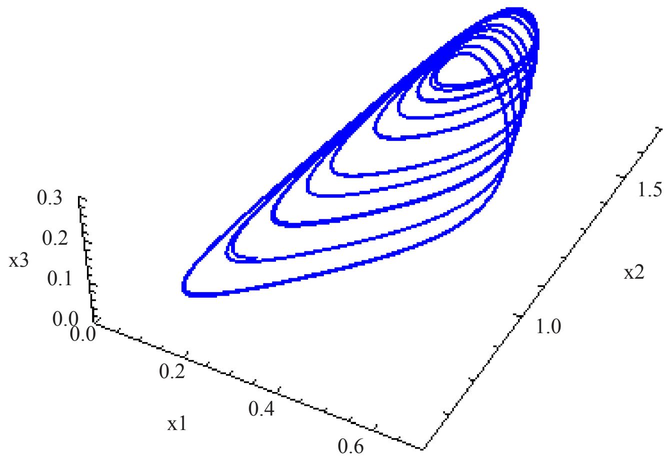

Figure 6. The chaotic attractor of the system (1) with the regulatory matrix (4).

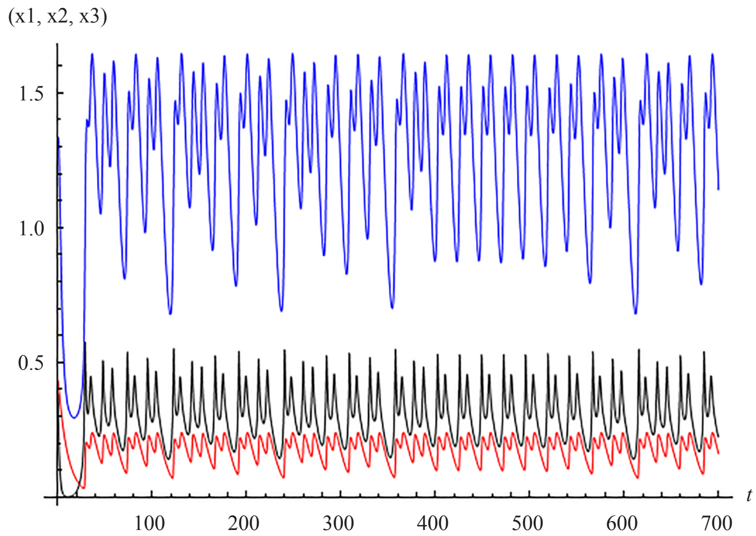

Figure 7. The graphs of $x_i(t)$, i = 1, 2, 3 of the system (1) with the regulatory matrix (4).

Similar systems were considered in papers [14], [19] and [20].

The dynamics of Lyapunov exponents ($LE_1 = 0.03$, $LE_2 = 0$; $LE_3 = -1.16$) are shown in Figure 8.

At the end of the 70s of the last century, the Kaplan-Yorke formula was proposed to estimate the fractal size-in terms of Lyapunov exponents [12].

Let calculate the Kaplan-Yorke dimension using the formula

$$D_{KY} = j + \frac{1}{|L_{j+1}|} \sum_{j=1}^{j} L_j$$

with j representing the index such that

$$\sum_{j=1}^{j} L_j > 0,\ \sum_{j=1}^{j+1} L_j < 0$$

Such formula is considered in papers [12, 21].

Kaplan-Yorke dimension for the system (1) with the regulatory matrix (5) is D_{KY} = 2.03. The sum of Lyapunov exponents of the system (1) with the regulatory matrix (5) is negative that is why it is a dissipative system. The dynamics of Lyapunov exponents are shown in Figure 8.

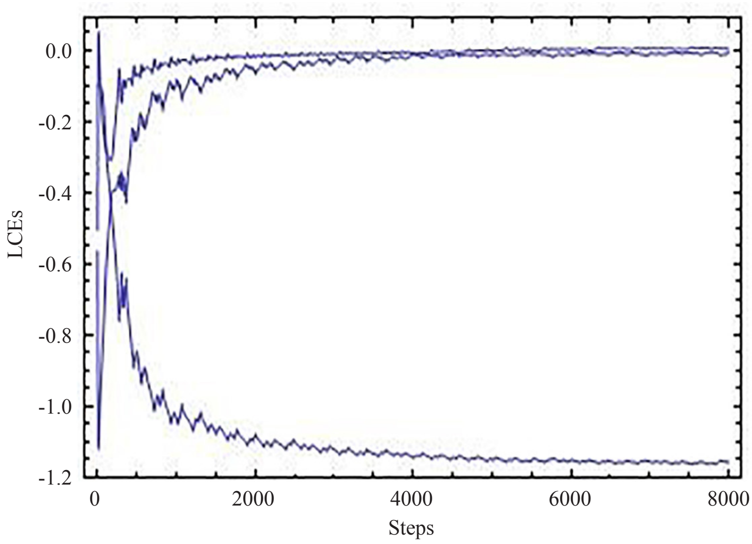

Figure 8. The dynamics of Lyapunov exponents.

3. Artificial neural network

Consider the three-dimensional system

$$\begin{cases} x_1' = \tanh(w_{11}x_1 + w_{12}x_2 + w_{13}x_3) - b_1x_1, \\ x_2' = \tanh(w_{21}x_1 + w_{22}x_2 + w_{23}x_3) - b_2x_2, \\ x_3' = \tanh(w_{31}x_1 + w_{32}x_2 + w_{33}x_3) - b_3x_3. \end{cases} \tag{7}$$

The system (7) is considered in papers [22, 23].

The nullclines are given as

$$\begin{cases} b_1x_1 = \tanh(w_{11}x_1 + w_{12}x_2 + w_{13}x_3), \\ b_2x_2 = \tanh(w_{21}x_1 + w_{22}x_2 + w_{23}x_3), \\ b_3x_3 = \tanh(w_{31}x_1 + w_{32}x_2 + w_{33}x_3). \end{cases} \tag{8}$$

All critical points are located in the open parallelepiped

$$\left\{(x_1, x_2, x_3) : -\frac{1}{b_1} < x_1 < \frac{1}{b_1},\ -\frac{1}{b_2} < x_2 < \frac{1}{b_2},\ -\frac{1}{b_3} < x_3 < \frac{1}{b_3}\right\}. \tag{9}$$

3.1 *Examples of the system (7) with regulatory matrices (4) and (5)*

Consider the coefficient matrix (4) and $b_1 = b_2 = b_3 = 1$.

There are three critical points. First critical point is (0; 0; 0). The characteristic numbers are $\lambda_1 = 0.2$ and $\lambda_{2,3} = 1.5 \pm 2.45i$. The type of critical point is an unstable focus-node. Second critical point is (0; 0; 0.66). The characteristic numbers are $\lambda_1 = -0.32$ and $\lambda_{2,3} = 1.5 \pm 2.45i$. The type of critical point is an unstable saddle-focus. Third critical point is (0; 0; -0.66). The characteristic numbers are $\lambda_1 = -0.32$ and $\lambda_{2,3} = 1.5 \pm 2.45i$. The type of critical point is an unstable saddle-focus. The nullclines of the system (7) with the regulatory matrix (4) and three periodic solutions of the system (7) with the regulatory matrix (4) are shown in Figure 9 and Figure 10. The graphs of $x_i(t)$, $i = 1, 2, 3$ of the system (7) with the regulatory matrix (4) are depicted in Figure 11.

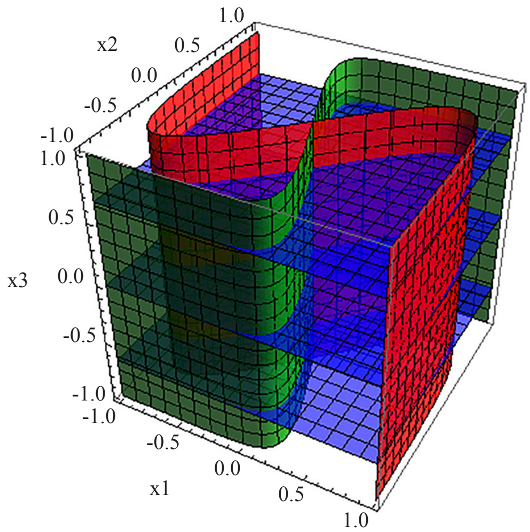

Figure 9. The nullclines of the system (7) with the regulatory matrix (4).

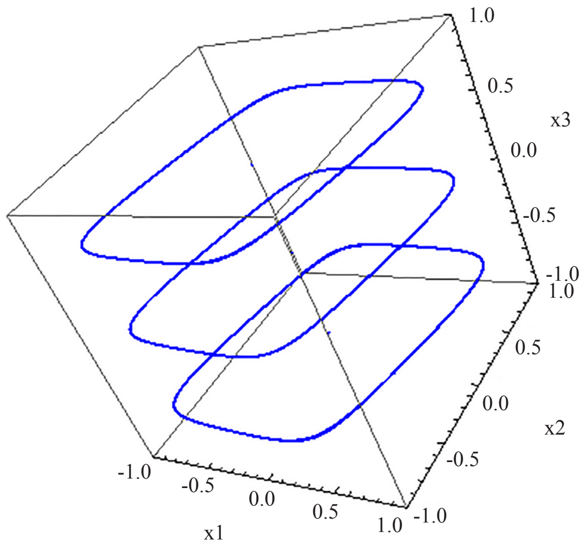

Figure 10. Three periodic solutions of the system (7) with the regulatory matrix (4).

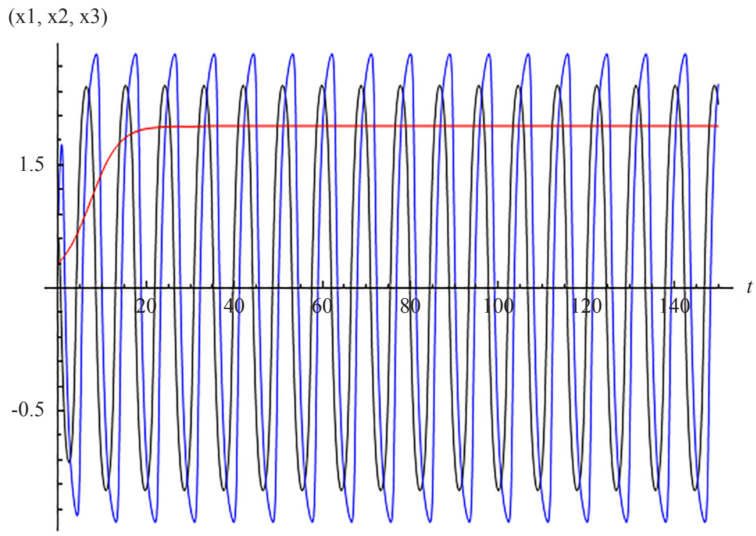

Figure 11. The graphs of $x_i(t)$, i = 1, 2, 3 of the system (7) with the regulatory matrix (4).

Similar systems are considered in paper [22].

The dynamics of Lyapunov exponents (LE_1 = 0, LE_2 = -0.32; LE_3 = -0.78) are shown in Figure 12. The following set of LEs characterizes the stable limit cycle.

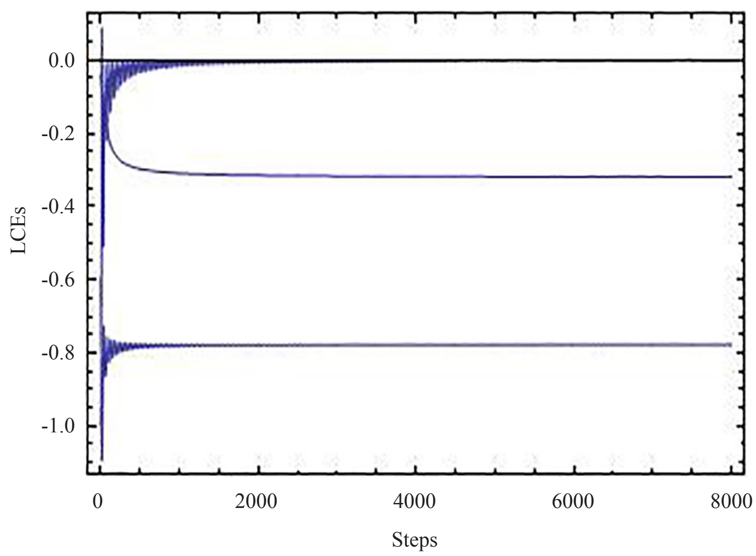

Figure 12. The dynamics of Lyapunov exponents.

The sum of Lyapunov exponents of the system (1) with the regulatory matrix (5) is negative that is why it is a dissipative system.

Consider the coefficient matrix (5) and b_1 = 0.65, b_2 = 0.42, b_3 = 0.1. The coefficients of regulatory matrix and parameters are the same. The initial conditions are (6). There is no chaotic solution. The nullclines of the system (7) with the regulatory matrix (5) and the solution of the system (7) with the regulatory matrix (5) and the initial conditions (6) are shown in Figure 13 and Figure 14. The graphs of $x_i(t)$, i = 1, 2, 3 of the system (7) with the regulatory matrix (5) are depicted in Figure 15.

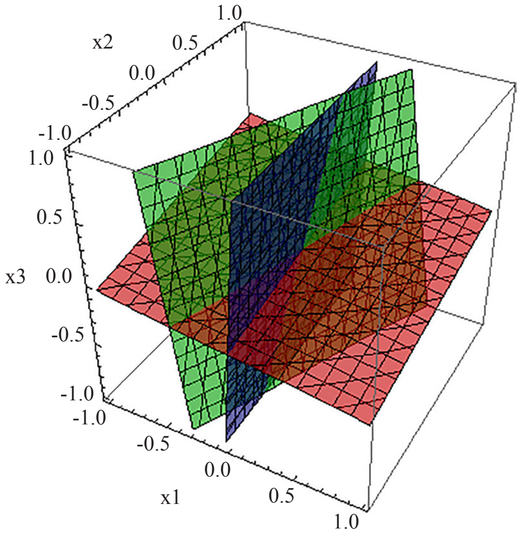

Figure 13. The nullclines of the system (7) with the regulatory matrix (5).

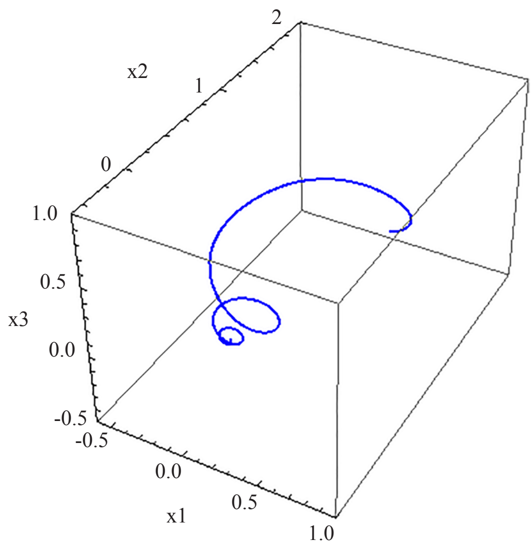

Figure 14. The solution of the system (7) with the regulatory matrix (5) and the initial conditions (6).

The dynamics of Lyapunov exponents (LE_1 = -0.32, LE_2 = -0.33; LE_3 = -0.49) are shown in Figure 16. The following set of LEs characterizes the stable fixed point.

The sum of Lyapunov exponents of the system (1) with the regulatory matrix (5) is negative that is why it is a dissipative system.

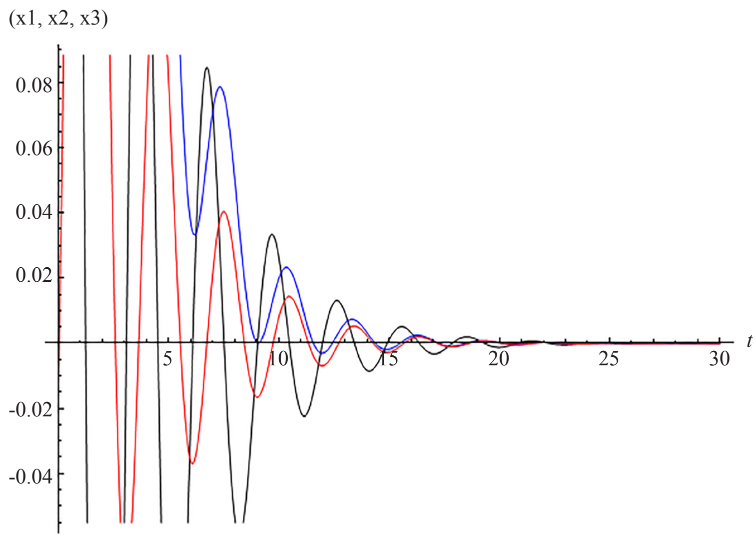

Figure 15. The graphs of $x_i(t)$, i = 1, 2, 3 of the system (7) with the regulatory matrix (5).

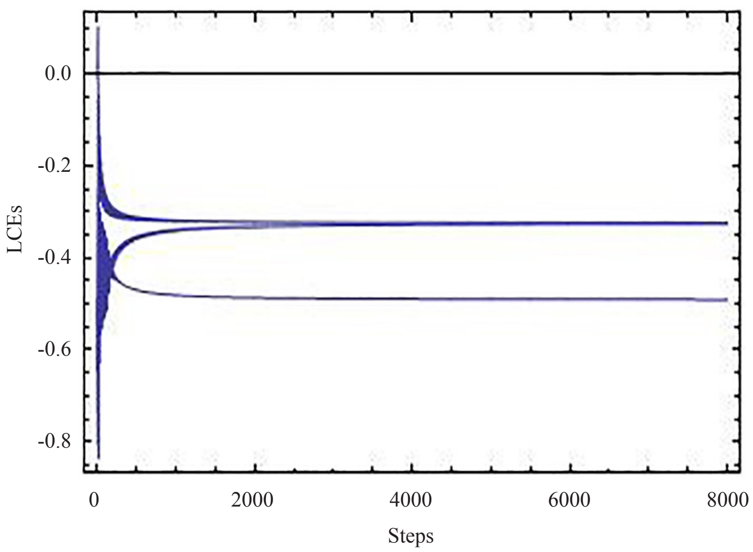

Figure 16. The dynamics of Lyapunov exponents.

4. Conclusion

The article deals with models of three-dimensional genetic and neural networks with a certain set of parameters and two different regulatory matrices. In a genetic system with a matrix (4), the existence of a periodic solution is shown. For a neural system with the same matrix, the existence of three periodic solutions is shown. In a genetic system with a matrix (5), a solution with chaotic behavior is indicated. This is evidenced by the Lyapunov curves, the three-dimensional graphics of the solution and the graphs of the three components of the solution. At the same time, in a neural system with the same regulatory matrix, this solution does not detect chaotic behavior. This observation is in line with statement [23, section 6.10.1] that the minimum dimension of systems of the form (7) in which chaos is possible is four.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Assieh S, Réka A. Boolean modeling of biological regulatory networks: A methodology tutorial. *Methods*. 2013; 62(1): 3-12. Available from: https://doi.org/10.1016/j.ymeth.2012.10.012.

[2] Spirov A, Holloway D. Using evolutionary computations to understand the design and evolution of gene and cell regulatory networks. *Methods*. 2013; 62(1): 3955. Available from: https://doi.org/10.1016/j.ymeth.2013.05.013.

[3] Sadyrbaev F, Ogorelova D, Samuilik I. A nullclines approach to the study of 2D artificial network. *Contemporary Mathematics.* 2019; 1(1): 1-11. Available from: https://doi.org/10.37256/cm.11201976.1-11.

[4] Alakwaa F. Modeling of gene regulatory networks: A literature review. *Journal of Computational Systems Biology*. 2014; 1(1): 102. Available from: https://doi.org/10.15744/2455-7625.1.102.

[5] Samuilik I, Sadyrbaev F. Genetic engineering-construction of a network of arbitrary dimension with periodic attractor. *Vibroengineering PROCEDIA*. 2022; 46: 67-72. Available from: https://doi.org/10.21595/vp.2022.22992.

[6] Kozlovska O, Sadyrbaev F. Models of genetic networks with given properties. *WSEAS Transactions on Computer Research*. 2022; 10: 43-49. Available from: https://doi.org/10.37394/232018.2022.10.6.

[7] Sadyrbaev F. Modelling the evolution of complex networks arising in applications. *Human-Assisted Intelligent Computing: Modelling, simulations and applications*. IOP Publishing, Bristol, United Kingdom; 2022. Available from: https://iopscience.iop.org.

[8] Samuilik I, Sadyrbaev F. On a dynamical model of genetic networks. *WSEAS Transactions on Business and Economics*. 2023; 20: 104-112. Available from: https://doi.org/10.37394/23207.2023.20.11.

[9] Zigunovs M. The Alzheimer's disease impact on artificial neural networks*. Environment. Technologies. Resources. Proceedings of the International Scientific and Practical Conference*. 2021; 2: 205-209. Available from: https://doi.org/10.17770/etr2021vol2.6632.

[10] Amato F, López A, Peña-Méndez EM, Vaňhara P, Hampl A, Havel J. Artificial neural networks in medical diagnosis. *Journal of Applied Biomedicine*. 2013; 11(2): 47-58. Available from: https://doi.org/10.2478/v10136-012-0031-x.

[11] Li EY. Artificial neural networks and their business applications. *Information & Management*. 1994; 27(5): 303-313. Available from: https://doi.org/10.1016/0378-7206(94)90024-8.

[12] Samuilik I. On a six-dimensional artificial neural network model. *WSEAS Transactions on Systems and Control*. 2023; 18: 58-64. Available from: https://doi.org/10.37394/23203.2022.18.6.

[13] Canepa E, Claudel C. A dual model/artificial neural network framework for privacy analysis in traffic monitoring systems. *Transportation Research Part C: Emerging Technologies*. 2019; 105: 126-144. Available from: https://doi.org/10.1016/j.trc.2019.05.031.

[14] Samuilik I, Sadyrbaev F. On trajectories of a system modeling evolution of genetic networks. *Mathematical Biosciences and Engineering*. 2023; 20(2): 2232-2242. Available from: https://doi.org/10.3934/mbe.2023104.

[15] Lynch S. *Dynamical systems with applications using mathematica*. Publisher: Birkhauser; 2017. Available from: https://doi.org/10.1007/978-0-8176-4586-1.

[16] Sukono, Yuningsih SH, Rusyaman E, Vaidyanathan S, Sambas A. Investigation of chaos behavior and integral sliding mode control on financial risk model. *AIMS Mathematics*. 2022; 7(10): 18377-18392. Available from: https://doi.org/10.3934/math.20221012.

[17] Gritli H, Khraief N, Belghith S. Further investigation of the period-three route to chaos in the passive compass-gait biped model. In: Vaidyanathan AAS. (ed.) *Handbook of Research on Advanced Intelligent Control Engineering and Automation*. Publisher: IGI Global. Hershey, PA: Engineering Science; 2015. p.279-300. Available from: https://doi.org/10.4018/978-1-4666-7248-2.ch010.

[18] Sayed WS, Radwan AG, Fahmy HAH. Chaos and bifurcation in controllable jerk-based self-excited attractors. *Nonlinear Dynamical Systems with Self-Excited and Hidden Attractors*. Springer; 2018. p.45-70. Available from: https://doi.org/10.1007/978-3-319-71243-7_2.

[19] Das A, Roy AB, Das P. Chaos in a three dimensional general model of neural network. *International Journal of Bifurcation and Chaos*. 2002; 12: 2271-2281.

[20] Das A, Roy AB, Das P. Chaos in a three dimensional general model of neural network. *International Journal of*

Bifurcation and Chaos. 2002; 12(10): 2271-2281. Available from: https://doi.org/10.1142/S0218127402005820.
[21] Samuilik I. Lyapunov exponents and kaplan-yorke dimension for five-dimensional system. *WSEAS Transactions on Systems*. 2022; 21: 268-275. Available from: https://doi.org/10.37394/23202.2022.21.29.
[22] Ogorelova D, Sadyrbaev F. On a three-dimensional neural network model. *Vibroengineering PROCEDIA*. 2022; 47: 69-73. Available from: https://doi.org/10.21595/vp.2022.23059.
[23] Sprott JC. *Elegant chaos: Algebraically simple chaotic flows*. Singapore: World Scientific Publishing; 2010.